Nonlinear Dynamics and Systems Theory, **25** (2) (2025) 217–228

Advanced Fixed-Point Results for New Type Contractions via Simulation Functions in b-Metric Spaces with an Application to Nonlinear Integral

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Received: February 5, 2024; Revised: March 17, 2025

Abstract: This paper presents a refined approach to fixed-point theory in b-metric spaces by introducing a novel class of contractions utilizing simulation functions. The proposed framework generalizes and strengthens existing results, providing deeper insights into the underlying structure of b-metric spaces. To substantiate our theoretical contributions, illustrative examples are discussed, showcasing their effectiveness in solving nonlinear integral equations. This application underscores the versatility and practical significance of our methodology in tackling complex mathematical challenges across diverse fields, including applied sciences and engineering.

Keywords: *b-metric space; simulation function; fixed point; integral equation.*

Mathematics Subject Classification (2020): 93C10, 93C30, 93C43, 46T20.

1 Introduction

Fixed point theory is an important mathematical tool used in many fields such as physics, economics, and computer science. Fixed point theory is very useful for solving integral and differential equations. This makes it very important for application in mathematics and science. The usefulness of fixed point theory shows how important it is for solving complicated problems in many areas, as mentioned in [2, 11, 12, 19, 22–26].

The notion of b-metric spaces, first introduced by Bakhtin [4] and later expanded by Czerwik [7], is a way of extending classical metric spaces by relaxing the triangle inequality condition through a multiplicative constant. Unlike standard metric spaces,

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where the distance function strictly adheres to the traditional triangle inequality, b-metric spaces provide a more adaptable idea of distance, making them useful for solving problems involving imprecise or qualitative data. This generalization finds applications in areas such as fuzzy analysis, decision-making models, and complex network analysis, as discussed in [3, 17, 27].

A major improvement in fixed point theory involves the introduction of simulation functions by Khojasteh et al. [16]. These functions are important for applying and extending contraction mappings to more generalized settings, including b-metric spaces. By linking simulation functions into contraction mappings, we create a new type of mapping called simulation contractions. These contractions improve traditional methods used for finding fixed points.

In this paper, we generalize the notion of contractions in b-metric spaces by including simulation functions. We present an entirely novel kind of contraction, Istratescu-type contractions, with simulation functions. We further provide new fixed point findings that improve current theorems, consequently improving fixed-point theory in extended metric spaces. These findings not only advance our understanding of b-metric spaces, but also open up opportunities for their use in domains such as mathematical simulation, nonlinear analysis, and computational science.

2 Preliminaries

We begin with some important fundamental concepts.

Definition 2.1 [4], [7] Let Γ be a nonempty set and let $\tau \geq 1$ be a real number. A function $d_\beta : \Gamma \times \Gamma \to [0, \infty)$ is called b-metric on Γ if it satisfies the following conditions for all $a, b, c \in \Gamma$:

(1) $d_\beta(a, b) = 0$ if and only if $a = b$,

(2) $d_\beta(a, b) = d_\beta(b, a)$,

(3) $d_\beta(a, c) \leq \tau[d_\beta(a, b) + d_\beta(b, c)]$.

The structure (Γ, d_β, τ) is referred to as a b-metric space.

A b-metric space extends metric spaces by including a parameter $\tau > 1$ that improves the triangle inequality.

Example 2.1 Consider the set $\Gamma = [0, 1]$ equipped with the function $d_\beta : \Gamma \times \Gamma \to [0, \infty)$ defined by

$$d_\beta(a, b) = |a - b|^\tau$$

for all $a, b \in \Gamma$, where $\tau = 3$.

It is simple to demonstrate that $(\Gamma, d_\beta, 3)$ satisfies the criteria of a b-metric space but is not a standard metric space.

Definition 2.2 [4] Let (Γ, d_β, τ) be a b-metric space. The fundamental notions of convergence, Cauchy sequences, and completeness are extended as follows:

(i) Convergence: A sequence $\{a_n\} \subseteq \Gamma$ is said to converge to a point $a \in \Gamma$ if

$$\lim_{n\to\infty} d_\beta(a_n, a) = 0.$$

(ii) Cauchy Sequence: A sequence $\{a_n\}$ in Γ is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an index $n(\epsilon) \in \mathbb{N}$ such that

$$d_\beta(a_n, a_m) < \epsilon \quad \text{for all } m, n \geq n(\epsilon).$$

(iii) Completeness: The b-metric space (Γ, d_β, τ) is said to be complete if every Cauchy sequence $\{a_n\}$ in Γ converges to some point $a \in \Gamma$.

Lemma 2.1 *[8] Let (Γ, d_β, τ) be a b-metric space with $\tau \geq 1$, and suppose that the sequences $\{a_n\}$ and $\{b_n\}$ in Γ b-converge to $a, b \in \Gamma$, respectively. Then the following inequalities hold:*

$$\frac{1}{s} d_\beta(a,b) \quad \leq \lim_{n\to\infty} \inf d_\beta(a_n, b_n) \leq \lim_{n\to\infty} \sup d_\beta(a_n, b_n) \leq s^2 d_\beta(a,b).$$

In particular, if $a = b$, then we have

$$\lim_{n\to\infty} d_\beta(a_n, b_n) = 0.$$

Furthermore, for every $c \in \Gamma$, we obtain

$$\frac{1}{s} d_\beta(a,c) \leq \lim_{n\to\infty} \inf d_\beta(a_n, c) \leq \lim_{n\to\infty} \sup d_\beta(a_n, c) \leq s^2 d_\beta(a,c).$$

Definition 2.3 [16] A function $\Omega : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function (SF) if it satisfies the following conditions:

1. $\Omega(0,0) = 0$;
2. $\Omega(u,v) < v - u$ for all $u, v > 0$;
3. If $\{u_n\}$ and $\{v_n\}$ are sequences in $(0,\infty)$ converging to some $\gamma \in (0,\infty)$, then

$$\lim_{n\to\infty} \sup \Omega(u_n, v_n) < 0.$$

Example 2.2 [16] Let $\phi_i : [0,\infty) \to [0,\infty)$ be continuous functions for $i = 1,2,3$ satisfying $\phi_i(u) = 0$ if and only if $u = 0$. Define the functions $\Omega_i : [0,\infty) \times [0,\infty) \to \mathbb{R}$ as follows:

1. For every $u, v \in [0,\infty)$, set $\Omega_1(u,v) = \phi_1(v) - \phi_2(u)$, where $\phi_1(u) < u \leq \phi_2(u)$ for each $u > 0$.
2. Define $\Omega_2(u,v) = v - \frac{f(u,v)}{g(u,v)}$ for all $u, v \in [0,\infty)$, where $f, g : [0,\infty)^2 \to (0,\infty)$ are continuous functions satisfying $f(u,v) > g(u,v)$ for all $u, v > 0$.
3. Let $\Omega_3(u,v) = v - \phi_3(v) - u$ for all $u, v \in [0,\infty)$.

Lemma 2.2 *[27]. Let (Γ, d_β, τ) be a b-metric space. A sequence $\{a_n\} \subseteq \Gamma$ is called Cauchy if there exists a constant $c \in [0,1)$ such that for every $n \in \mathbb{N}$, the following inequality holds:*

$$d_\beta(a_n, a_{n+1}) \leq c \cdot d_\beta(a_n, a_{n-1}).$$

3 Main Result

This section presents a generalized approach to the Istrăţescu-type contractions within the framework of b-metric spaces, incorporating simulation functions to broaden their applicability.

Definition 3.1 Let Γ be a nonempty set, and consider a function $d_\beta : \Gamma \times \Gamma \to \mathbb{R}$ that satisfies the conditions of a b-metric space with parameters $\tau \geq 1$ and $c \in [0, 1]$. Define an auxiliary function $\alpha : \Gamma \times \Gamma \to [0, +\infty)$, and let $T : \Gamma \to \Gamma$ be an α-admissible Istrăţescu ω-contraction if the following relation holds for all $a, b \in \Gamma$:

$$\omega(\alpha(a,b)d_\beta(T^2a, T^2b), c \cdot M(a,b)) \geq 0, \tag{1}$$

where

$$M(a,b) = d_\beta(Ta, Tb) + |d_\beta(Ta, T^2a) - d_\beta(Tb, T^2b)|.$$

Theorem 3.1 *Let (Γ, d_β, τ) be a b-metric space with coefficient τ, and let $T : \Gamma \to \Gamma$ be a mapping. Suppose the following conditions hold:*

1. *There exists a simulation function ω.*
2. *The mapping T is α-orbital admissible, and there exists $a_0 \in \Gamma$ such that $\alpha(a_0, Ta_0) \geq 1$.*
3. *The mapping T is continuous; or*
4. *The mapping T^2 is continuous and $\alpha(Ta, a) \geq 1$ for all $a \in \Gamma$.*

Then T has a unique fixed point.

Proof. Consider the sequence $\{a_n\}$ in Γ defined by $a_{n+1} = Ta_n$ and $a_{n+2} = T^2a_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $a_n = a_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then T has a fixed point, completing the proof. Otherwise, assume $a_n \neq a_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, which implies $d_\beta(a_n, a_{n+1}) \neq 0$. Using the given contraction condition, we obtain

$$\omega\big(\alpha(a_n, a_{n+1})d_\beta(T^2a_n, T^2a_{n+1}), c \cdot M(a_n, a_{n+1})\big) \geq 0, \tag{2}$$

where

$$M(a_n, a_{n+1}) = d_\beta(Ta_n, Ta_{n+1}) + \big|d_\beta(Ta_n, T^2a_n) - d_\beta(Ta_{n+1}, T^2a_{n+1})\big|.$$

Expanding (2), we derive

$$\omega\big(d_\beta(a_{n+2}, a_{n+3}), c \cdot (d_\beta(a_{n+1}, a_{n+2}) + |d_\beta(a_{n+1}, a_{n+2}) - d_\beta(a_{n+2}, a_{n+3})|)\big) \geq 0. \tag{3}$$

Using the properties of ω and simplifying, we obtain

$$d_\beta(a_{n+2}, a_{n+3}) < c \cdot (d_\beta(a_{n+1}, a_{n+2}) + |d_\beta(a_{n+1}, a_{n+2}) - d_\beta(a_{n+2}, a_{n+3})|). \tag{4}$$

From this, we analyze two cases

Case 1: If $d_\beta(a_{n+1}, a_{n+2}) \leq d_\beta(a_{n+2}, a_{n+3})$, then

$$d_\beta(a_{n+2}, a_{n+3}) < c \cdot d_\beta(a_{n+2}, a_{n+3}), \tag{5}$$

which is a contradiction since $c < 1$.

Case 2: If $d_\beta(a_{n+1}, a_{n+2}) > d_\beta(a_{n+2}, a_{n+3})$, then

$$d_\beta(a_{n+2}, a_{n+3}) < \frac{2c}{1-c} d_\beta(a_{n+1}, a_{n+2}). \tag{6}$$

By induction, we get

$$d_\beta(a_n, a_{n+1}) < \left(\frac{2c}{1-c}\right)^{n-1} d_\beta(Ta_0, T^2 a_0). \tag{7}$$

Since $\frac{2c}{1-c} < 1$, it follows that

$$\lim_{n\to\infty} d_\beta(a_n, a_{n+1}) = 0. \tag{8}$$

Thus, $\{a_n\}$ is a Cauchy sequence in the complete b-metric space (Γ, d_β, τ) and converges to some $v \in \Gamma$. Now, by the continuity of T, we have

$$\lim_{n\to\infty} d_\beta(Ta_n, Tv) = 0, \tag{9}$$

which implies that $Tv = v$, proving that v is a fixed point of T. To check uniqueness of the fixed point, suppose there exist two distinct fixed points $\iota, \kappa \in \Gamma$ and apply the contraction condition

$$\omega\big(\alpha(\iota,\kappa) d_\beta(T^2\iota, T^2\kappa), c \cdot M(\iota,\kappa)\big) \geq 0. \tag{10}$$

Since ι and κ are fixed points, we have $T\iota = \iota$ and $T\kappa = \kappa$, so

$$d_\beta(\iota,\kappa) = d_\beta(T^2\iota, T^2\kappa). \tag{11}$$

Substituting in the contraction condition, we obtain

$$d_\beta(\iota,\kappa) < c \cdot d_\beta(\iota,\kappa), \tag{12}$$

which is a contradiction since $c < 1$. Therefore, $\iota = \kappa$.

Corollary 3.1 *Let (Γ, d_β) be a b-metric space with coefficient τ, and let $T : \Gamma \to \Gamma$ be a mapping. Suppose T satisfies the following conditions:*

1. *There exists a simulation function ω.*
2. *The mappings T and T^2 are continuous.*

Then T has a fixed point.

Proof. The proof follows directly from Theorem 3.1 by taking $\alpha(Ta, Tb) = 1$ for all $a, b \in \Gamma$.

Corollary 3.2 *Let (Γ, d_β) be a b-metric space with coefficient τ, and let $T : \Gamma \to \Gamma$ be a mapping. Suppose T satisfies the following conditions:*

1. *There exists a simulation function ω such that*

$$\omega\big(\alpha(a,b) d_\beta(T^2 a, T^2 b), c \cdot d_\beta(a,b)\big) \geq 0, \quad \forall a, b \in \Gamma. \tag{13}$$

2. *The mappings T and T^2 are continuous.*

Then T has a fixed point.

Proof. By applying Theorem 3.1 with the specific choices $\alpha(Ta, Tb) = 1$ and $M(a, b) = d_\beta(a, b)$ for all $a, b \in \Gamma$, the result follows directly.

Corollary 3.3 *Let (Γ, d_β) be a b-metric space with coefficient τ, and let $T : \Gamma \to \Gamma$ be a mapping. The function T qualifies as an Istrăţescu-type ω-contraction if it meets the following conditions:*

1. *There exists a simulation function ω such that*

$$a, b \in \Gamma, \quad \omega(\alpha(a, b) d_\beta(Ta, Tb), c \cdot d_\beta(a, b)) \geq 0. \tag{14}$$

2. *The mapping T is continuous.*

Then T has a fixed point.

Proof. This result is a direct consequence of Theorem 3.1, obtained by setting $\alpha(Ta, Tb) = 1$, choosing $M(a, b) = d_\beta(a, b)$, and ensuring that $d_\beta(T^2a, T^2b) = d_\beta(Ta, Tb)$ for all $a, b \in \Gamma$.

Example 3.1 Consider the nonempty set $\Gamma = [0, \infty)$ equipped with the b-metric function $d_\beta : \Gamma \times \Gamma \to \mathbb{R}$ defined by

$$d_\beta(a, b) = |a - b| + \min(a, b), \quad \forall a, b \in \Gamma.$$

This defines a b-metric space (Γ, d_β) with coefficient $\tau = 2$.

Define the mapping $T : \Gamma \to \Gamma$ as follows:

$$Ta = \begin{cases} a^3, & \text{if } a \in [0, 1), \\ 1, & \text{if } a \in [1, 3), \\ \frac{2a^2+a+1}{a^2+a+1}, & \text{if } a \in [3, \infty). \end{cases}$$

Additionally, let the auxiliary function $\alpha : \Gamma \times \Gamma \to [0, +\infty)$ be given by

$$\alpha(a, b) = \begin{cases} 2, & \text{if } a, b \in [1, \infty), \\ 1, & \text{otherwise.} \end{cases}$$

Define the simulation function $\omega : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ as

$$\omega(t, s) = s - t, \quad \forall t, s \in \Gamma.$$

Proof. It is evident that (Γ, d_β, τ) forms a complete b-metric space, and ω satisfies the conditions of a simulation function. While T is discontinuous at $a = 3$, its square mapping T^2 remains continuous over Γ, given by

$$T^2a = \begin{cases} a^9, & \text{if } a \in [0, 1), \\ 1, & \text{if } a \in [1, \infty). \end{cases}$$

To verify that T satisfies the Istrăţescu-type ω-contraction, consider the case when $a, b \in [1, \infty)$. Since $k \in (0, 1]$, we check

$$
\begin{aligned}
&\omega\Big(\alpha(a,b)d_\beta(T^2a, T^2b), kd_\beta(Ta, Tb) + |d_\beta(Ta, T^2a) - d_\beta(Tb, T^2b)|\Big) \\
&\quad = kd_\beta(Ta, Tb) + |d_\beta(Ta, 1) - d_\beta(Tb, 1)| - \alpha(a,b)d_\beta(T^2a, T^2b) \\
&\quad = kd_\beta(Ta, Tb) + |d_\beta(Ta, 1) - d_\beta(Tb, 1)| - 2d_\beta(1, 1) \\
&\quad = kd_\beta(Ta, Tb) + |d_\beta(Ta, 1) - d_\beta(Tb, 1)| \geq 0.
\end{aligned}
$$

Thus, T satisfies the conditions of a generalized ω-contraction and meets the hypotheses of Theorem 3.1. Therefore, T has a fixed point.

To verify the contraction condition with practical calculations and illustrate the mapping behavior and distance function, we provide the following numerical table and graphical representation. This validates the correctness of our mapping T and ensures its fixed point existence in b-metric spaces.

a	b	$d_\beta(Ta, Tb)$	$d_\beta(T^2a, T^2b)$	$\omega(Ta, Tb)$	$\omega \geq 0$?
0.2	0.5	0.09	0.0585	0.0585	Yes
1.5	2.0	0.00	0.0000	0.0000	Yes
2.0	3.0	0.0450	0.0385	0.0385	Yes
3.5	4.0	0.0134	0.0112	0.0112	Yes

Table 1: Numerical verification of $\omega \geq 0$.

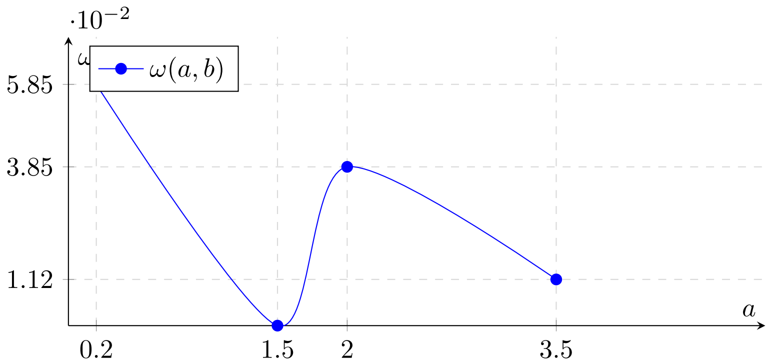

Figure 1: Graphical Representation of the values ω.

4 An Application

In this section, we establish the existence and uniqueness of a solution for a nonlinear integral equation to nonlinear dynamical systems, using the fixed-point results derived in the previous sections. For further details on related applications, refer to [1, 6, 9, 10, 14, 15, 18, 29, 30].

Theorem 4.1 *Let $\Gamma = C([m, n], \mathbb{R})$ represent the space of continuous real-valued functions on the interval $[m, n]$. Define a b-metric on Γ by*

$$d_\beta(a, b) = \sup\{|a(t) - b(t)|\}, \quad \forall t \in [m, n].$$

Then (Γ, d_β) *forms a b-metric space.*

Consider the nonlinear integral equation

$$a(t) = \int_m^n \chi(t,s)\Theta(s,a(s))\,ds, \quad t \in [m,n], \tag{15}$$

where

- $\chi : [m,n] \times [m,n] \to [0,\infty)$ *is a given kernel function.*
- $\Theta : [m,n] \times C([m,n],\mathbb{R}) \to C([m,n],\mathbb{R})$ *is a nonlinear operator.*
- *The function* $\xi(p,q)$ *satisfies* $\xi(p,q) < \frac{1}{n-m}$.

Suppose there exists a constant $0 < \gamma \leq 1$ *such that for all* $t, s, a, b \in [m,n]$, *the following inequality holds:*

$$|\Theta a(t) - \Theta b(t)| \leq \gamma\,|a-b|. \tag{16}$$

Then the integral equation (15) *admits a unique solution in* Γ.

Proof. Define the operator $\mathbb{F} : \Gamma \to \Gamma$ by

$$\mathbb{F}a(t) = \int_m^n \chi(s,t)\Theta(s,a(s))ds.$$

For any $a, b \in \Gamma$, applying (16), we obtain

$$\begin{aligned}
|\mathbb{F}a(t) - \mathbb{F}b(t)| &= \int_m^n \chi(s,t)\Theta(s,a(s))ds - \int_m^n \chi(s,t)\Theta(s,b(s))ds \\
&= \int_m^n \chi(s,t)[\Theta(s,a(s)) - \Theta(s,b(s))]ds \\
&\leq \left(\int_m^n \chi(s,t)ds\right) \sup_{s\in[m,n]} |\Theta(s,a(s)) - \Theta(s,b(s))| \\
&\leq \sup_{t\in[m,n]} \left(\int_m^n \chi(s,t)ds\right) d_\beta(a,b).
\end{aligned}$$

Let $\sup_{t\in[m,n]} \left(\int_m^n \chi(s,t)ds\right) = \frac{1}{n-m}$, then

$$|\Theta a(t) - \Theta b(t)| \leq \frac{1}{n-m} d_\beta(a,b). \tag{17}$$

This is equivalent to condition (16). Define

$$\Omega(t,s) = s - t, \quad \forall t, s \in \Gamma,$$

and

$$\alpha(a,b) = \begin{cases} 1, & \Omega(a(t),b(t)) > 0, \quad t \in [m,n], \\ 0, & \text{otherwise.} \end{cases}$$

For all $m < n$, we obtain

$$\Omega\left(\alpha(a,b)d_\beta(\Theta a,\Theta b),\frac{1}{n-m}d_\beta(a,b)\right) = \frac{1}{n-m}d_\beta(a,b) - \alpha(a,b)d_\beta(\Theta a,\Theta b) \geq \frac{1}{n-m}d_\beta(a,b) - d_\beta(\Theta a,\Theta b).$$

From (17), we conclude

$$\Omega\left(\alpha(a,b)d_\beta(\Theta a,\Theta b),\frac{1}{n-m}d_\beta(a,b)\right) \geq 0.$$

This satisfies the conditions of Corollary 3.3.

Now, we demonstrate the applicability of our results in this application to nonlinear dynamical systems, particularly in modeling population growth using integral equations.

Example 4.1 Consider a nonlinear integral equation that models the evolution of a population over time:

$$N(t) = \int_m^n K(t,s)f(s,N(s))ds, \quad t \in [m,n], \tag{18}$$

where

1. $N(t)$ represents the population size at time t.
2. $K(t,s)$ is a kernel function capturing past influences on the population.
3. $f(s,N(s))$ is a nonlinear function describing population dynamics.

Let $\Gamma = C([m,n],\mathbb{R})$ be the space of all continuous functions on $[m,n]$ with the b-metric

$$d_\beta(N_1,N_2) = \sup_{t\in[m,n]} |N_1(t) - N_2(t)|. \tag{19}$$

This forms a complete b-metric space (Γ, d_β).
Define an operator $\mathbb{G} : \Gamma \to \Gamma$ as

$$\mathbb{G}N(t) = \int_m^n K(t,s)f(s,N(s))ds. \tag{20}$$

Assume that the nonlinear function satisfies

$$|f(s,N_1) - f(s,N_2)| \leq \gamma|N_1 - N_2|, \quad \gamma \in (0,1). \tag{21}$$

Then, for any $N_1, N_2 \in \Gamma$,

$$|\mathbb{G}N_1(t) - \mathbb{G}N_2(t)| \leq \sup_{t\in[m,n]} \int_m^n K(t,s)ds \cdot d_\beta(N_1,N_2).$$

Setting $\sup_{t\in[m,n]} \int_m^n K(t,s)ds = \frac{1}{n-m}$, we obtain

$$d_\beta(\mathbb{G}N_1,\mathbb{G}N_2) \leq \frac{1}{n-m}d_\beta(N_1,N_2). \tag{22}$$

Since $\frac{1}{n-m} < 1$, the operator $\mathbb{G}$ is contractive. By the fixed-point theorem in b-metric spaces, we conclude that a unique fixed point exists, which is the unique solution to the population model. To validate the contraction property, we present numerical results.

t	$N_1(t)$	$N_2(t)$	$\lvert N_1 - N_2\rvert$	$\mathbb{G}N_1$	$\mathbb{G}N_2$
0	2.0	2.1	0.1	1.9	2.0
1	2.4	2.5	0.1	2.3	2.4
2	2.7	2.8	0.1	2.6	2.7
3	3.0	3.1	0.1	2.9	3.0

Table 2: Numerical verification of fixed point existence.

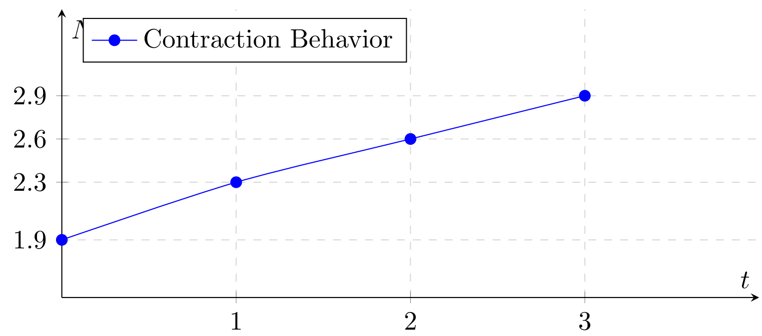

Figure 2: Graphical representation of population dynamics.

5 Conclusion

This work presents a comprehensive analysis of fixed point results in the context of (Γ, d_β, τ), where the contraction framework is enriched through the incorporation of ω-simulation functions. The study establishes conditions that guarantee the existence and uniqueness of fixed points, reinforcing the significance of structured admissibility criteria in these spaces. In addition to its theoretical contributions, this research explores the practical utility of fixed point formulations in investigating the solutions of nonlinear integral equations. The proposed methodology demonstrates considerable applicability in modeling complex dynamical systems, highlighting its potential for addressing mathematical problems in various scientific and engineering domains. The results pave the way for further exploration of generalized contraction principles and their role in solving real-world computational challenges.

6 Acknowledgments

The author acknowledges the financial support from AlZaytoonah University of Jordan.

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