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A Primal-Dual IPM Algorithm for LO Problem Based on a New Kernel Function with a Logarithmic Barrier Term

Abderrahim Guemmaz [1*], Bachir Bounibane [2] and El Amir Djeffal [2]

[1] *Department of Mathematics, Institute of Science, University Center of Barika, Algeria.*
[2] *Department of Mathematics, University of Batna 2, Batna, Algeria.*

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Abstract: In this paper, we consider a primal-dual Interior Point Method (IPM) for the linear optimization(LO) problem, based on a new kernel function with a logarithmic barrier term, which plays an important role for developing a new design of primal-dual IPM algorithms. New search directions and proximity functions are proposed based on this kernel function. We proved that our algorithm has $\mathbf{O}\left(qSn^{\frac{Sq+1}{2Sq}}\log\left(\frac{n}{\epsilon}\right)\right)$ iteration bound for large-update methods.

Keywords: *primal-dual interior point algorithm; kernel function; linear optimization problem; iteration bound; complexity.*

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1 Introduction

In this paper we deal with primal-dual IPMs for solving the standard linear optimization (LO) problem

$$(P) \qquad \min\left\{c^T x : Ax = b, x \geq 0\right\},$$

and the dual problem of (P) is given by

$$(D) \qquad \max\left\{b^T y + s = c, s \geq 0\right\},$$

where $A \in \mathbb{R}^{m\times n}, x, s, c \in \mathbb{R}^n$, and $y, b \in \mathbb{R}^m$.

* Corresponding author: mailto:abderrahim.guemmaz@cu-barika.dz

In 1984, Karmarkar [8] proposed a new polynomial-time method for solving linear programs. This method and its variants that were developed subsequently are now called IPMs, and they have become the most effective methods for solving LO problems. The new efficient algorithms of the interior-point methods (IPMs) have generated increased interest both in the application and the research of LO. In this paper, we deal with the so-called primal-dual IPMs. It is generally agreed that these IPMs are most efficient from a computational point of view [7]. Many researchers have designed different types of primal-dual interior-point methods. Among them, IPMs based on kernel functions have been designed. Several kernel functions have been introduced, including the so-called self-regular kernel functions [2,4] and the non-self-regular kernel functions [2,11]. In principle, a kernel function offers a search direction and hence the development of a primal-dual interior point method. Until now, all primal-dual IPMs have used the Newton direction as the search direction [6]; this direction is closely related to the well-known primal-dual logarithmic barrier function. In this paper, we consider the new kernel function with a logarithmic Barrier Term (1.1) from [11] as follows:

$$\psi_S(t) = \frac{\left(t^2-1\right)}{2} - \frac{\log(t)}{2} - \frac{1}{2S}\sum_{j=1}^{S}\frac{t^{1-jq}-1}{1-jq}, q>1, S\in\mathbb{N}\backslash\{0\}. \tag{1}$$

We will formulate an interior-point algorithm for LO by using a new proximity function and give its complexity analysis, and then we will show that the iteration bounds are $\mathbf{O}\left(qSn^{\frac{Sq+1}{2Sq}}\log\left(\frac{n}{\epsilon}\right)\right)$ and $\mathbf{O}\left(q^2S^2\sqrt{n}\log\left(\frac{n}{\epsilon}\right)\right)$ for large and small-update methods, respectively.

The remainder of this paper is organized as follows. First, in Section (2), we define the central path and the new search direction determined by Kernel Functions for LO, then we present the generic primal-dual IPM algorithm. The new kernel function and its properties are presented in Section (3). In Section (4), we analyse the algorithm and derive the complexity bound for LO. Finally, some concluding remarks follow in Section (5).

Some notations used throughout the paper are as follows. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the inner product $\langle .,.\rangle$, and $\|.\|$ denote the 2-norm. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors, respectively. For $x,s\in\mathbb{R}^n$, x_{min} and xs denote the smallest component of the vector x, and the componentwise product of the vector x and s, respectively. We denote by $X = diag(x)$ the $n\times n$ diagonal matrix with the components of vector $x\in\mathbb{R}^n$ being the diagonal entries, e denotes the n-dimensional vector, where each coordinate takes the value 1. For two real-valued functions $f(x), g(x) : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++},\ f(x) = O(g(x))$ if $f(x) \le cg(x)$ for some positive c, and $f(x) = \Theta(g(x))$ if $c_1 g(x) \le f(x) \le c_2 g(x)$ for some positive constants c_1 and c_2.

2 Preliminaries

It is well known that the optimality condition for (P) and (D) is equivalent to solving the following nonlinear system:

$$\begin{cases} Ax = b, & x\ge 0,\\ A^Ty + s = c, & s\ge 0,\\ xs = 0. \end{cases} \tag{2}$$

The basic idea of primal-dual IPMs for LO problems is to replace the third equation in (2), which is known as a complementarity condition for (P) and (D), by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus, the system (2) becomes

$$\begin{cases} Ax = b, & x \geq 0, \\ A^T y + s = c, & s \geq 0, \\ xs = \mu e. \end{cases} \tag{3}$$

Due to the last equation, any solution (x, y, s) of (3) will satisfy $x > 0$ and $s > 0$. So, a solution exists only if (P) and (D) satisfy the interior-point condition (IPC) [5], i.e., there exists (x^0, y^0, s^0) such that

$$\begin{cases} Ax^0 = b, & x^0 > 0, \\ A^T y^0 + s^0 = c, & s^0 > 0. \end{cases}$$

So, if the IPC is satisfied, the system (3) has only one solution $(x(\mu), y(\mu), s(\mu))$ for every $\mu > 0$ (see Lemma 4.3 in [13]), $x(\mu)$ is called the μ-center of (P), and $(y(\mu), s(\mu))$ is the μ-center of (D). The set of μ-centers is called the central path of (P) and (D). If $\mu \longrightarrow 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D) [5].

Let $\mu > 0$ be fixed. A direct application of the Newton method to (3) provides the following system for $\Delta x, \Delta y$ and Δs:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s = 0, \\ x\Delta s + s\Delta x = \mu e - sx. \end{cases} \tag{4}$$

Since A has full row rank, the system (4) has a unique solution $(\Delta x, \Delta y, \Delta s)$ which is called the search direction (see [5, 9]). By taking a step along the search direction $(\Delta x, \Delta y, \Delta s)$, one constructs a new positive iterate (x_+, y_+, s_+) with

$$x_+ := x + \alpha\Delta x, y_+ := y + \alpha\Delta y, \;\; s_+ := s + \alpha\Delta s,$$

where α satisfies $0 < \alpha \leq 1$.

Now, we introduce the scaled vector v and the scaled search directions dx and ds as follows:

$$v := \sqrt{\frac{xs}{\mu}}, \; d_x := \frac{v\Delta x}{x}, \; d_s := \frac{v\Delta s}{s}. \tag{5}$$

The system (4) can be rewritten as follows:

$$\begin{cases} \overline{A} d_x = 0, \\ \overline{A}^T \Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v, \end{cases} \tag{6}$$

where $\overline{A} := \frac{1}{\mu} A V^{-1} X, V := diag\,(v)$ and $X := diag\,(x)$.

Note that

$$d_x = d_s = 0 \Leftrightarrow v^{-1} - v = 0 \Leftrightarrow v = e \Longleftrightarrow x = x(\mu), s = s(\mu).$$

A useful observation is that the right-hand side of the third equation in (6) equals to the minus gradient of the following proximity function:

$$\Phi(v) = \Phi(x, s; \mu) = \sum_{i=1}^{n} \psi(v_i) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right), v_i > 0.$$

Here, ψ is the so-called kernel function of Φ. And therefore, $d_x + d_s = -\nabla\Phi(v)$. We can rewrite the system (6) as

$$\begin{cases} \overline{A} d_x = 0, \\ \overline{A}^T \Delta y + d_s = 0, \\ d_x + d_s = -\nabla\Phi(v). \end{cases} \tag{7}$$

It is easy to notice that $\nabla\Phi(v) = 0$, therefore $\Phi(v)$ reaches its minimum value at $v = e$, with $\Phi(v) = 0$.

In order to measure the distance between the μ-center and the current iterate, we resort to using $\Phi(v)$, and this is for a given $\tau > 0$.

Now, we introduce a norm-based proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ in accordance with

$$\delta(v) = \frac{1}{2} ||\nabla\Phi(v)|| = \frac{1}{2} ||d_x + d_s||, \tag{8}$$

in terms of $\psi(v_i)$. Then we have $\psi(v_i) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e$.

Using (5) and (8), we can write the system (4) in the form of a modified Newton system. We get the following:

$$\begin{cases} A\Delta x = 0, \\ A^T \Delta y + \Delta s = 0, \\ x\Delta s + s\Delta x = -\mu v \nabla\Phi(v). \end{cases} \tag{9}$$

In this paper, we replace $\psi(t)$ by a new kernel function $\psi_S(t)$, and $\Phi(v)$ by a new barrier function $\Phi_S(v)$, which will be defined in Section (3).

The new interior-point algorithm works as follows. Assume that we are given a strictly feasible point $(x; y; s)$ which is in a τ-neighbourhood of the given μ-centre. Then we decrease μ to $\mu_+ = (1 - \theta)\mu$ for some fixed $\theta \in (0, 1)$, and then solve the Newton system (4) to obtain the unique search direction. The positivity of a new iterate is ensured by an appropriate choice of the step size α which is defined by some line search rule. This procedure is repeated until we find a new iterate (x_+, y_+, s_+) that is in a τ-neighbourhood of the μ_+-centre, and then we let $\mu := \mu_+$ and $(x; y; s) := (x_+, y_+, s_+)$. Then μ is again reduced by the factor $(1 - \theta)$ and we solve the Newton system targeting at the new μ_+-centre, and so on. This process is repeated until μ is small enough, say until $n\mu < \epsilon$. The generic form of the algorithm is shown in Fig.1.

Generic Primal-Dual Algorithm for LO

Input:
a proximity function $\Phi_s(v)$; a threshold parameter $\tau > 0$;
an accuracy parameter $\epsilon > 0$; a fixed barrier update parameter $\theta, 0 < \theta < 1$;
(x^0, s^0) and $\mu^0 := 1$ such that $\Phi_s(x^0, s^0, \mu^0) \le \tau$.
begin
 $x := x^0; s := s^0; \mu := \mu^0;$
  while $n\mu \ge \epsilon$ **do**
  begin (outer iteration)
   $\mu := (1-\theta)\mu; v := \sqrt{\frac{xs}{\mu}};$
   while $\Phi_s(v) > \tau$ **do**
    begin (inner iteration)
     Solve the system (9) for $(\Delta x, \Delta y, \Delta s)$;
     Determine a step size α;
     $x := x + \alpha\Delta x; y := y + \alpha\Delta y; s := s + \alpha\Delta s; v := \sqrt{\frac{xs}{\mu}};$
   end (inner iteration)
  end (outer iteration)
end

Fig. 1 Generic algorithm.

We want to "optimize" the algorithm by minimizing the number of iterates in the algorithm. To do this, we must carefully choose the parameters τ, θ, and the step size α. Choosing the barrier update parameter θ is very important in application and theory. If θ is a constant number which is independent of the dimension n of the problem, i.e., $\theta = \mathbf{\Theta}(1)$, then the algorithm is called a large update method. If θ depends on the dimension n of the problem, then we call the algorithm a small update method. In this case, θ is usually chosen as follows: $\theta = \mathbf{\Theta}\left(\frac{1}{\sqrt{n}}\right)$.

Choosing the step size, $\alpha > 0$, is another key step in obtaining good convergence properties of the algorithm. It must be set in such a way that the closeness of the iterates to the current μ-center improves by a sufficient amount.

In this paper, we define a new kernel function and propose primal-dual interior point methods which improve all the results of the complexity bound for large-update methods based on a logarithmic kernel function for LO. More precisely, based on the proposed kernel function, we prove that the corresponding algorithm has $\mathbf{O}\left(qSn^{\frac{Sq+1}{2Sq}}\log\left(\frac{n}{\epsilon}\right)\right)$complexity bound for the large-update method, and $\mathbf{O}\left(q^2S^2\sqrt{n}\log\left(\frac{n}{\epsilon}\right)\right)$ for the small-update method. Another interesting choice is q dependences with n and S, which minimizes the iteration complexity bound. In fact, if we take $q = \frac{\log n}{2S}$, we obtain the best known complexity bound for large-update methods, namely, $\mathbf{O}\left(\sqrt{n}\log n \log\frac{n}{\varepsilon}\right)$. This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [10].

3 The Properties of the New Kernel Function

We will now address a new kernel function with its properties being provided. Let the new univariate function be defined in [11].

$$\psi_S(t) = \frac{\left(t^2-1\right)}{2} - \frac{\log(t)}{2} - \frac{1}{2S}\sum_{j=1}^{S}\frac{t^{1-jq}-1}{1-jq},\ q>1,\ S\in\mathbb{N}\backslash\{0\}.$$

It is easy to observe that as $t\to 0$ or $t\to\infty$, then $\psi(t)\to\infty$. So, $\psi_S(t)$ is without a doubt a kernel function.

We will need the first three derivatives of $\psi_S(t)$, we provide them as follows:

$$\psi_S^{'}(t) = t - \frac{1}{2t} - \frac{1}{2S}\sum_{j=1}^{S} t^{-jq}, \tag{10}$$

$$\psi_S^{''}(t) = 1 + \frac{1}{2t^2} + \frac{1}{2S}\sum_{j=1}^{S} qjt^{-jq-1}, \tag{11}$$

$$\psi_S^{'''}(t) = -\frac{1}{t^3} - \frac{1}{2S}\sum_{j=1}^{S} jq(jq+1)t^{-jq-2}. \tag{12}$$

If $S=1$, we obtain the kernel function (12) given by Bouaafia et al. [10].

The following lemma establishes the efficiency of the new kernel function (1).

Lemma 3.1 *Let $\psi_S(t)$ be as defined in* (1) *and $t>0$. Then*

$$\psi_S^{''}(t) > 1, \tag{13}$$

$$\psi_S^{'''}(t) < 0, \tag{14}$$

$$t\psi_S^{''}(t) - \psi_S^{'}(t) > 0, \tag{15}$$

$$t\psi_S^{''}(t) + \psi_S^{'}(t) > 0. \tag{16}$$

The last property (16) in Lemma 3.1 is equivalent to the convexity of composed functions $t\to\psi_S(e^t)$ and this holds if and only if $\psi_S(\sqrt{t_1t_2}) \le \frac{1}{2}\left(\psi_S(t_1)+\psi_S(t_2)\right)$ for any $t_1,t_2\ge 0$. This property is well-known in the literature, and numerous researchers have demonstrated it (see [3, 12]).

We provide some technical findings of the new kernel function in preparation for further.

Lemma 3.2 *For $\psi_S(t)$, we get*

$$\frac{1}{2}(t-1)^2 \le \psi_S(t) \le \frac{1}{2}\left[\psi_S^{'}(t)\right]^2,\ t>0. \tag{17}$$

$$\psi_S(t) \le \left[\frac{6+q(S+1)}{8}\right](t-1)^2,\ t>1. \tag{18}$$

Proof. For (17), use (13). For (18), use Taylor's Theorem.

Let $\sigma : [0,\infty[\ \to [1,+\infty[$ be the inverse function of $\psi_S(t)$ for $t\ge 1$ and $\rho : [0,\infty[\ \to\ ]0,1]$ be the inverse function of $-\frac{1}{2}\psi_S^{'}(t)$ for all $t\in\ ]0,1]$. Then we have the following lemma.

Lemma 3.3 *[Lemma 3.3 from [11]] For $\psi_S(t)$, we have*

$$1+\sqrt{\frac{8s}{6+q(S+1)}} \leq \sigma(s) \leq 1+\sqrt{2s},\ s \geq 0. \tag{19}$$

$$\rho(z) > \left[\frac{1}{4z+2}\right]^{\frac{1}{Sq}},\ z > 0. \tag{20}$$

Lemma 3.4 *Let $\sigma : [0,\infty[\to [1,+\infty[$ be the inverse of $\psi_S(t)$. We have*

$$\Phi_S(\beta v) \leq n\psi_S\left(\beta\sigma\left(\frac{\Phi_S(v)}{n}\right)\right),\ v \in \mathbb{R}^*,\ \beta \geq 1.$$

Proof. Using (14) and (15), and Lemma 2.4 from [1], we can obtain the result. This completes the proof.

Lemma 3.5 *[Lemma 3.5 from [11]] Let $0 \leq \theta < 1$, $v_+ = \frac{v}{\sqrt{1-\theta}}$. If $\Phi_S(v) \leq \tau$, then we have*

$$\Phi_S(v_+) \leq \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)}.$$

Denote

$$\left(\Phi_S\right)_0 = \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)} = L\left(n,\theta,\tau\right).$$

So, during the algorithm's execution, $\left(\Phi_S\right)_0$ is the upper bound of $\Phi_S(v_+)$.

4 Complexity Analysis

In the next subsection, we compute a default step size α and the resulting decrease in the barrier function.

4.1 An estimation of the step size

We devoted this section to calculating a default step size α and the consequent decrease in the barrier function. And after the damping step, we obtain

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y,\ s_+ = s + \alpha\Delta s.$$

By using (5), we get that

$$x_+ = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}\left(v + \alpha d_x\right),$$
$$s_+ = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}\left(v + \alpha d_s\right).$$

Hence, $v_+ = \sqrt{\dfrac{x_+s_+}{\mu}} = \sqrt{\left(v+\alpha d_x\right)\left(v+\alpha d_s\right)}$. Define for $\alpha > 0$,

$$f(\alpha) = \Phi_S(v_+) - \Phi_S(v). \tag{21}$$

Therefore, $f(\alpha)$ represents the difference in proximities between a new iterate and a current iterate for a given value of μ . By (5), we can get

$$\Phi_S(v_+) = \Phi_S\left(\sqrt{(v+\alpha d_x)\,(v+\alpha d_s)}\right) \le \frac{1}{2}\left(\Phi_S((v+\alpha d_x)) + \Phi_S((v+\alpha d_s))\right).$$

Thus, we have $f(\alpha) \le f_1(\alpha)$ such that

$$f_1(\alpha) = \frac{1}{2}\left(\Phi_S((v+\alpha d_x)) + \Phi_S((v+\alpha d_s))\right) - \Phi_S(v). \tag{22}$$

Clearly, $f(0) = f_1(0) = 0$. We calculate $f_1^{'}(\alpha)$ and $f_1^{''}(\alpha)$, we find

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi_S^{'}(v_i+\alpha d_{x_i})d_{x_i} + \psi_S^{'}(v_i+\alpha d_{s_i})d_{s_i}\right),$$
$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi_S^{''}(v_i+\alpha d_{x_i})d_{x_i}^2 + \psi_S^{''}(v_i+\alpha d_{s_i})d_{s_i}^2\right).$$

By using (5) and (8), we conclude that

$$f_1^{'}(0) = \frac{1}{2} < \nabla\Phi_S(v), (d_x+d_s) >= -\frac{1}{2} < \nabla\Phi_S(v), \nabla\Phi_S(v) >= -2\delta(v)^2.$$

We denote $v_1 = min(v)$, $\delta = \delta(v)$, $\Phi_S = \Phi_S(v)$.

Lemma 4.1 *Let $\delta(v)$ be defined in* (8). *Then*

$$\delta(v) \ge \sqrt{\frac{\Phi_S(v)}{2}}. \tag{23}$$

Proof. Using (17), we have

$$\Phi_S(v) = \sum_{i=1}^{n}\psi_S(v_i) \le \sum_{i=1}^{n}\frac{1}{2}\left[\psi_S^{'}(v_i)\right]^2 = \frac{1}{2}||\nabla\Phi_S(v)||^2 = 2\delta(v)^2.$$

Hence, $\delta(v) \ge \sqrt{\frac{1}{2}\Phi_S(v)}$. This completes the proof.

Remark 4.1 Throughout the paper, we assume that $\Phi_S(v) \ge \tau \ge 1$, and we have $\delta(v) \ge \frac{1}{2}$.

According to Lemmas 4.1-4.4 in [1], we get the following Lemmas 4.2 and 4.5 since $\psi_S(t)$ is a kernel function, and $\psi_S^{''}(t)$ decreases monotonically.

Lemma 4.2 *[Bai et al. [1]] Let $f_1(\alpha)$ be as defined in* (21) *and $\delta(v)$ be as defined in* (8). *Then we have $f_1^{''}(\alpha) \le 2\delta^2\psi_S^{''}(v_{min} - 2\alpha\delta)$. Because of the convexity of $f_1(\alpha)$, we will have $f_1^{'}(\alpha) \le 0$ for any α less than or equal to the minimum value of $f_1(\alpha)$, and vice versa.*

The following three Lemmas result from the preceding Lemma.

Lemma 4.3 *[Bai et al. [1]]* $f_1^{'}(\alpha) \leq 0$ *certainly holds if* α *satisfies the inequality*

$$\psi_S^{'}(v_{min}) - \psi_S^{'}(v_{min} - 2\alpha\delta) \leq 2\delta. \tag{24}$$

Lemma 4.4 *[Bai et al. [1]] The largest step size* $\overline{\alpha}$ *satisfying* (24) *is given by*

$$\overline{\alpha} = \frac{\rho(\delta) - \rho(2\delta)}{2\delta}.$$

Lemma 4.5 *[Bai et al. [1]] Let* $\overline{\alpha}$ *be as defined in Lemma* 4.4. *Then*

$$\overline{\alpha} \geq \frac{1}{\psi_S^{''}(\rho(2\delta))}.$$

We are able to demonstrate the following Lemma.

Lemma 4.6 *[Lemma 4.6 from [11]] Let* ρ *and* $\overline{\alpha}$ *be as determined in Lemma* 4.5. *If* $\Phi_S(v) \geq \tau \geq 1$, *then we have*

$$\overline{\alpha} \geq \frac{2S}{2S + S\,(4\delta+2)^{\frac{2}{Sq}} + q\sum_{j=1}^{S} j\,(4\delta+2)^{\frac{jq+1}{Sq}}}.$$

Denoting

$$\widetilde{\alpha} = \frac{2S}{2S + S\,(4\delta+2)^{\frac{2}{Sq}} + q\sum_{j=1}^{S} j\,(4\delta+2)^{\frac{jq+1}{Sq}}}, \tag{25}$$

we have $\widetilde{\alpha}$ is the default step size, and $\widetilde{\alpha} \leq \overline{\alpha}$.

Lemma 4.7 *[Lemma 3.12 from [3]] Let* h *be a convex and twice differentiable function with* $h(0) = 0$, $h^{'}(0) < 0$, *which reaches its minimum at* $t^* > 0$. *If* $h^{''}$ *is increasing for* $t \in [0, t^*]$, *then*

$$h(t) \leq \frac{t h^{'}(0)}{2},\ 0 \leq t \leq t^*.$$

The following result is of great importance.

Lemma 4.8 *[Lemma 4.5 from [1]] If the step size* α *satisfies* $\alpha \leq \overline{\alpha}$, *then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Lemma 4.9 *Let* $\Phi_S(v) \geq 1$ *and let* $\widetilde{\alpha}$ *be the default step size as defined in* (25). *Then we have*

$$f(\widetilde{\alpha}) \leq -\frac{2S}{8\sqrt{2}(S+8)(1+4qS)}\left[\Phi_S(v)\right]^{\frac{Sq-1}{2Sq}}. \tag{26}$$

Proof. Since $\Phi_S(v) \geq 1$, from (23), we have

$$\delta \geq \sqrt{\frac{1}{2}\Phi_S(v)} \geq \sqrt{\frac{1}{2}}.$$

Due to Lemma 4.8, with $\alpha = \widetilde{\alpha}$ and (25), this completes the proof.

4.2 Iteration bound

Following the updating of μ to $(1-\theta)\mu$, we obtain

$$\Phi_S(v_+) \leq (\Phi_S)_0 = \frac{\theta n + 2\tau + 2\sqrt{2\tau n}}{2(1-\theta)} = L(n, \theta, \tau).$$

After μ-update to $(1-\theta)\mu$, it is necessary to count how many inner iterations are required to come back to the situation where $\Phi_S(v_+) \leq \tau$. We declare the value of $\Phi_S(v)$ after the updating of μ as $(\Phi_S)_0$ and we denote by $(\Phi_S)_k$, $k = 1, 2, ..., K$, the subsequent values in the same outer iteration such that K represents the total number of inner iterations per the outer iteration.

Lemma 4.10 *[Lemma 14 from [3]] Let $t_0, t_1, ..., t_k$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, k = 0, 1, ..., K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$, then $K \leq \left[\frac{t_0^\gamma}{\beta\gamma}\right]$.

Thus, it follows that

$$(\Phi_S)_{k+1} \leq (\Phi_S)_k - k(\Phi_S)^{1-\gamma}, k = 0, 1, ..., K-1,$$

with

$$\kappa = \frac{2S}{8\sqrt{2}(S+8)(1+4qS)}, \gamma = 1 - \frac{Sq-1}{2Sq} = \frac{Sq+1}{2Sq}.$$

Lemma 4.11 *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq \frac{8\sqrt{2}q(S+8)(1+4qS)}{1+Sq} \left[(\Phi_S)_0\right]^{\frac{Sq+1}{2Sq}}.$$

Proof. By Lemma 1.3.2 from [3], we have
$K \leq \frac{\left[(\Phi_S)_0\right]^\gamma}{\kappa\gamma} = \frac{8\sqrt{2}q(S+8)(1+4qS)}{Sq+1}\left[(\Phi_S)_0\right]^{\frac{Sq+1}{2Sq}}$. This completes the proof.
Now, we estimate the total number of iterations of our algorithm.

We recall that the number of outer iterations is limited from above by $\frac{\log\left(\frac{n}{\epsilon}\right)}{\theta}$ (see Lemma II.17, page 116 in [5]). We can establish an upper bound on the total number of iterations by multiplying the number of outer iterations by the number of inner iterations such as

$$\frac{8\sqrt{2}q(S+8)(1+4qS)}{Sq+1}\left[(\Phi_S)_0\right]^{\frac{Sq+1}{2Sq}} \frac{\log\left(\frac{n}{\epsilon}\right)}{\theta}. \quad (27)$$

In the methods of large-update with $\tau = \mathbf{O}(n)$ and $\theta = \Theta(1)$, we have

$$\mathbf{O}\left(qSn^{\frac{Sq+1}{2Sq}} \log\left(\frac{n}{\epsilon}\right)\right) \text{ iterations complexity.}$$

This is the best well-known complexity result for large-update methods.

In the methods of small-update, the replacement of $\tau = \mathbf{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$ in (27) does not provide the best possible bound. The best bound is obtained as follows. By (18), with $\psi_S(t) \le \left[\frac{6+q(S+1)}{8}\right](t-1)^2$, $t > 1$, we have

$$\begin{aligned}\Phi_S(V_+) &\le n\psi_S\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi_S(V)}{n}\right)\right)\\ &\le n\left[\frac{6+q(S+1)}{8}\right]\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Phi_S(V)}{n}\right)-1\right)^2\\ &= \frac{n\,(6+q(S+1))}{8(1-\theta)}\left(\sigma\left(\frac{\Phi_S(V)}{n}\right)-\sqrt{1-\theta}\right)^2.\end{aligned}$$

Using (19), we have

$$\begin{aligned}\frac{n\,(6+q(S+1))}{8(1-\theta)}\left(\sigma\left(\frac{\Phi_S(V)}{n}\right)-\sqrt{1-\theta}\right)^2 &\le \frac{n\,(6+q(S+1))}{8(1-\theta)}\left(\left(1+\sqrt{2\frac{\Phi_S(V)}{n}}\right)-\sqrt{1-\theta}\right)^2\\ &= \frac{n\,(6+q(S+1))}{8(1-\theta)}\left(\left(1-\sqrt{1-\theta}\right)+\sqrt{2\frac{\Phi_S(V)}{n}}\right)^2\\ &\le \frac{n\,(6+q(S+1))}{8(1-\theta)}\left(\theta+\sqrt{2\frac{\tau}{n}}\right)^2\\ &= \frac{(6+q(S+1))}{8(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 = (\Phi_S)_0\,,\end{aligned}$$

where we also use $1-\sqrt{1-\theta} = \frac{\theta}{1+\theta} \le \theta$ and $\Phi_S(v) \le \tau$, utilizing this upper bound for $(\Phi_S)_0$, we obtain the following iteration bound:

$$\frac{8\sqrt{2}q(S+8)\,(1+4qS)}{Sq+1}\left[(\Phi_S)_0\right]^{\frac{Sq+1}{2Sq}}\frac{\log\left(\frac{n}{\epsilon}\right)}{\theta}.$$

Note now that $(\Phi_S)_0 = \mathbf{O}(qS)$, and the iteration bound is given as follows:

$$\mathbf{O}\left(q^2S^2\sqrt{n}\log\left(\frac{n}{\epsilon}\right)\right) \text{ iterations complexity.}$$

5 Conclusion

In this work, we have improved the algorithmic complexity of IPM methods for LO problems by a new kernel function. More specifically, we have proved the large-update and small-update versions of the primal-dual algorithm based on a new kernel function with a logarithmic barrier term defined by (1). This new kernel function has never been mentioned before, and the resulting analysis is also different from others. Moreover, we intend to extend this work in the future to semi-definite linear complementarity problems (SDLCPs) based on this kernel function.

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