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A Note on Linear Matrix Functions and Applications

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Abstract: This paper focuses on some algebraic characterizations between linear matrix functions (LMFs) and their domains defined over the field of complex numbers $\mathbb{C}$. We discuss the intersection, as well as the inclusion of two domains of some LMFs. By applying specific algebraic methods on ranks and ranges, we consider certain forms of LMFs, where the general solutions can be expressed via specific explicit LMFs to establish some relationships between their domains. As a consequence, we have obtained a well-known result of Lin and Wang.

Keywords: *linear matrix function; algebraic method; generalized inverse; general solutions; rank.*

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1 Introduction

In this work, we use the notation $\mathbb{C}^{n\times m}$ to represent the set of all $n \times m$ complex matrices. The symbols A^*, $\mathfrak{R}(A)$, $r(A)$ and I_n denote the conjugate transpose, the range, the rank of the matrix A and the identity matrix of order n, respectively. The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{n\times m}$ is defined as the unique $m \times n$ complex matrix denoted by A^+ satisfying the following four equations:

$$AA^+A = A,\ \ A^+AA^+ = A^+,\ \ \left(AA^+\right)^* = AA^+,\ \ \left(A^+A\right)^* = A^+A. \tag{1}$$

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Extensive studies and results regarding the Moore-Penrose inverse can be found in [3, 4, 10]. Additionally, we introduce two orthogonal projectors induced by $A \in \mathbb{C}^{m\times n}$, namely $F_A = I_n - A^+A$ and $E_A = I_m - AA^+$.

A linear matrix function

$$Y = f(X_1, X_2, ..., X_p),$$

where X_1, X_2, ..., X_p are the variables over the field of complex numbers $\mathbb{C}$ and Y is the matrix value associated with the matrix function corresponding to $X_1, X_2, \dots, X_p$. In addition, we define the domain of the function f mentioned above as

$$S = \{Y \mid Y = f(X_1, X_2, ..., X_p)\}.$$

The majority of problems with linear or nonlinear matrix functions should be understood in terms of their analytic or algebraic aspects and behaviors, and used to solve matrix function-related problems in both computational and pure mathematics. Further, matrix equations play an important role in nonlinear dynamics, control engineering, mathematical models, for a variety of reasons, including the analysis, modeling, and simulation of complex systems to linearize nonlinear systems for local analysis, determine stability through eigenvalue analysis, analyze normal modes in oscillatory systems, their use ranges from fundamental stability analysis to advanced control and bifurcation studies. For instance, Baddi et al. [1] studied the stabilization problem of inhomogeneous semilinear control systems; they established the existence and uniqueness of solutions of the system using the semigroup theory. By algebraic method, Tian and Yuan [17] studied and suggested connections between specific LMFs, then explored some specific subjects about the algebraic relationships between the reduced equations and solutions of a certain linear matrix problems. Guerarra [5] investigated the inclusion relationships between the set of persymmetric solutions and the set of minimal rank persymmetric solutions of the quaternion matrix equation $AXA^{(*)} = B$. Özgüler and Akar [9] provided equivalent conditions for the existence of a common solution to a pair of linear matrix equations over a principal ideal domain. Jiang et al. [6] studied the relationships between the set of solutions to $AXB = C$ and the set of solutions of its reduced equations. Therefore, all matrix functions possess a class of fundamental types called LMFs, and they can be defined consistently using matrix additions and multiplications. On the other hand, nonlinear matrix functions have been studied in many works, one may refer to [15, 16] and references therein.

Here, we just provide a common illustration of an LMF

$$f(X_1, X_2, \dots, X_p) = A + B_1X_1C_1 + B_2X_2C_2 + \cdots + B_pX_pC_p,$$

where $A \in \mathbb{C}^{m\times n}$, $B_i \in \mathbb{C}^{m\times l_i}$, $C_i \in \mathbb{C}^{n_i\times n}$ are given, and $X_i \in \mathbb{C}^{l_i\times n_i}$ are matrices with variable entries, where $i = 1, 2, \dots, p$. Hence, its domain is given as

$$S = \left\{Y = A + B_1X_1C_1 + B_2X_2C_2 + ... + B_pX_pC_p \mid X_i \in \mathbb{C}^{l_i\times n_i},\ i = 1, 2, \dots, p\right\}.$$

The rank of a matrix is one of the most basic quantities and useful methods and tools that are widely used in linear algebra, specifically in matrix theory. This finite nonnegative integer can be used to represent many properties of matrices such as singularity or nonsingularity of a matrix, identification of matrices, consistency of a matrix equation, etc. For further details, see [2, 10, 12, 13]. The rank of matrices or partitioned matrices was first studied by Matsaglia and Styan [8], where they provided various formulas that simplify complicated matrix expressions or equalities.

Based on the results of Tian and Yuan [17], this work aims to explore and suggest some basic aspects concerning the domains of some specific examples of LMF using the matrix rank method. Because of this fact, we will consider the following new domains of LMFs:

$$S_1 = \left\{A_1 + B_1X_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times n_1}\right\}, \tag{2}$$

$$S_2 = \left\{A_2 + B_2X_2C_2 + B_3X_3C_3 \mid X_2 \in \mathbb{C}^{p_2\times n_2},\ X_3 \in \mathbb{C}^{p_3\times n_3}\right\}, \tag{3}$$

where $A_1,\ A_2 \in \mathbb{C}^{l\times n}$, $B_i \in \mathbb{C}^{l\times p_i}$, $C_i \in \mathbb{C}^{n_i\times n}$, for $i = \overline{1,3}$, are given. This paper is organized as follows. In Section 2, we recall some results. In Section 3, we establish the necessary and sufficient conditions for the two relations $S_1 \cap S_2 \neq \varnothing$, $S_1 \subseteq S_2$ to hold. As a consequence, we give conditions for some matrix equations to have common solutions. We conclude our discussion in Section 4.

2 Preliminaries

To advance this objective, we require the following basic lemmas.

Lemma 1 *[8] Let $A \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{l\times k}$, and $C \in \mathbb{C}^{p\times n}$. Then*

$$r\begin{bmatrix}A & D\end{bmatrix} - r(E_AD) = r(A),\ r\begin{bmatrix}A & D\end{bmatrix} - r(E_DA) = r(D), \tag{4}$$

$$r\begin{bmatrix}A\\ C\end{bmatrix} - r(CF_A) = r(A),\ r\begin{bmatrix}A\\ C\end{bmatrix} - r(AF_C) = r(C), \tag{5}$$

$$r\begin{bmatrix}A & D\\ C & 0\end{bmatrix} - r(E_DAF_C) = r(D) + r(C), \tag{6}$$

from (4)-(6), it follows

$$r\begin{bmatrix}A & BF_P\\ E_QC & 0\end{bmatrix} = r\begin{bmatrix}A & B & 0\\ C & 0 & Q\\ 0 & P & 0\end{bmatrix} - r(P) - r(Q),$$

$$r\begin{bmatrix}E_{B_1}AF_{C_1} & E_{B_1}B\\ CF_{C_1} & 0\end{bmatrix} = r\begin{bmatrix}A & B & B_1\\ C & 0 & 0\\ C_1 & 0 & 0\end{bmatrix} - r(B_1) - r(C_1).$$

Lemma 2 *[10] Consider the matrix equation*

$$AXB = D, \tag{7}$$

where $A \in \mathbb{C}^{l\times n}$, $B \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{l\times q}$ are given, and $X \in \mathbb{C}^{n\times p}$ is an unknown matrix. Then the following are equivalent:
(i) Eq (7) is consistent.
(ii) $AA^+DB^+B = D$.
(iii) $r\begin{bmatrix} A & D \end{bmatrix} = r(A)$ and $r\begin{bmatrix}B\\ D\end{bmatrix} = r(B)$.
(iv) $\mathfrak{R}(D) \subseteq \mathfrak{R}(A)$ and $\mathfrak{R}(D^*) \subseteq \mathfrak{R}(B^*)$.

In this case, the general solution can be expressed as

$$X = A^{+}DB^{+} + F_A V + U E_B,$$

where V, U are arbitrary with appropriate sizes. In particular, Eq (7) holds for matrix $X \in \mathbb{C}^{n\times p}$ if and only if

$$\begin{bmatrix} A & D \end{bmatrix} = 0 \text{ or } \begin{bmatrix} B \\ D \end{bmatrix} = 0.$$

Lemma 3 *[9] The matrix equation*

$$A_1X_1B_1 + A_2X_2B_2 = D \tag{8}$$

is solvable for X_1 and X_2 of suitable sizes if and only if all the following equalities

$$r\begin{bmatrix} D & A_1 & A_2 \end{bmatrix} = r\begin{bmatrix} A_1 & A_2 \end{bmatrix},\ r\begin{bmatrix} D & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2),$$
$$r\begin{bmatrix} D & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1),\ r\begin{bmatrix} D \\ B_1 \\ B_2 \end{bmatrix} = r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

hold, or, equivalently,

$$E_A D = 0,\ E_{A_1} D F_{B_2} = 0,\ E_{A_2} D F_{B_1} = 0,\ D F_B = 0 \ \text{ hold},$$

where $A = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$ and $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$.

Lemma 4 *[12] Eq (8) holds for all X_1 and X_2 of suitable sizes if and only if any one of the following equalities*

$$\begin{bmatrix} D & A_1 & A_2 \end{bmatrix} = 0,\ \begin{bmatrix} D & A_1 \\ B_2 & 0 \end{bmatrix} = 0,\ \begin{bmatrix} D & A_2 \\ B_1 & 0 \end{bmatrix} = 0,\ \begin{bmatrix} D \\ B_1 \\ B_2 \end{bmatrix} = 0$$

holds.

Lemma 5 *[11] The matrix equation*

$$A_1X_1B_1 + A_2X_2B_2 + A_3X_3B_3 = C \tag{9}$$

is solvable for X_1, X_2 and X_3 of suitable sizes if and only if all the following equalities *hold*

$$r\begin{bmatrix}A_1 & A_2 & A_3 & C\end{bmatrix} = r\begin{bmatrix}A_1 & A_2 & A_3\end{bmatrix},\ r\begin{bmatrix}B_1\\ B_2\\ B_3\\ C\end{bmatrix} = r\begin{bmatrix}B_1\\ B_2\\ B_3\end{bmatrix},$$

$$r\begin{bmatrix}C & A_1 & A_2\\ B_3 & 0 & 0\end{bmatrix} = r\begin{bmatrix}A_1 & A_2\end{bmatrix} + r(B_3),\ r\begin{bmatrix}C & A_1 & A_3\\ B_2 & 0 & 0\end{bmatrix} = r\begin{bmatrix}A_1 & A_3\end{bmatrix} + r(B_2),$$

$$r\begin{bmatrix}C & A_2 & A_3\\ B_1 & 0 & 0\end{bmatrix} = r\begin{bmatrix}A_2 & A_3\end{bmatrix} + r(B_1),\ r\begin{bmatrix}C & A_3\\ B_1 & 0\\ B_2 & 0\end{bmatrix} = r\begin{bmatrix}B_1\\ B_2\end{bmatrix} + r(A_3),$$

$$r\begin{bmatrix}C & A_2\\ B_1 & 0\\ B_3 & 0\end{bmatrix} = r\begin{bmatrix}B_1\\ B_3\end{bmatrix} + r(A_2),\ r\begin{bmatrix}C & A_1\\ B_2 & 0\\ B_3 & 0\end{bmatrix} = r\begin{bmatrix}B_2\\ B_3\end{bmatrix} + r(A_1),$$

$$r\begin{bmatrix}C & 0 & A_1 & 0 & A_3\\ 0 & -C & 0 & A_2 & A_3\\ B_2 & 0 & 0 & 0 & 0\\ 0 & B_1 & 0 & 0 & 0\\ B_3 & B_3 & 0 & 0 & 0\end{bmatrix} = r\begin{bmatrix}B_2 & 0\\ 0 & B_1\\ B_3 & B_3\end{bmatrix} + r\begin{bmatrix}A_1 & 0 & A_3\\ 0 & A_2 & A_3\end{bmatrix}.$$

Lemma 6 *[18] Let $T \in \mathbb{C}^{l\times n}$, $N \in \mathbb{C}^{l\times p}$, $B \in \mathbb{C}^{p\times k}$ and $D \in \mathbb{C}^{n\times k}$ be given. Then the system of matrix equations $TX = N$ and $XB = D$ has a solution if and only if*

$$TT^{+}N = N,\ \ DB^{+}B = D \quad \text{and} \quad TD = NB.$$

In this case, the general solution can be written as

$$X = T^{+}N + F_T DB^{+} + F_T V E_B,$$

where $V \in C^{n\times p}$ is arbitrary.

Lemma 7 *[14] Define two domains as*

$$\Gamma_1 = \left\{D_1 + B_1X_1C_1 \mid X_1 \in \mathbb{C}^{s_1\times t_1}\right\} \text{ and } \Gamma_2 = \left\{D_2 + B_2X_2C_2 \mid X_2 \in \mathbb{C}^{s_2\times t_2}\right\},$$

where $D_i \in \mathbb{C}^{l\times n}$, $B_i \in \mathbb{C}^{l\times s_i}$, and $C_i \in \mathbb{C}^{t_i\times n}$ are given, and $X_i \in \mathbb{C}^{s_i\times t_i}$ are variable for $i = 1, 2$. Then
(a) $\Gamma_1 \cap \Gamma_2 \neq \varnothing$ if and only if all the following conditions hold:

$$\Re(D_1 - D_2) \subseteq \Re\begin{bmatrix}B_1 & B_2\end{bmatrix},\ \Re(D_1^* - D_2^*) \subseteq \Re\begin{bmatrix}C_1^* & C_2^*\end{bmatrix},$$

$$r\begin{bmatrix}D_1 - D_2 & B_1\\ C_2 & 0\end{bmatrix} = r(B_1) + r(C_2),\ r\begin{bmatrix}D_1 - D_2 & B_2\\ C_1 & 0\end{bmatrix} = r(B_2) + r(C_1).$$

(b) $\Gamma_1 \subseteq \Gamma_2$ if and only if

$$\Re\begin{bmatrix}D_1 - D_2 & B_1\end{bmatrix} \subseteq \Re(B_2) \text{ and } \Re\begin{bmatrix}D_1^* - D_2^* & C_1^*\end{bmatrix} \subseteq \Re(C_2^*).$$

(c) $\Gamma_1 = \Gamma_2$ if and only if

$$\Re(D_1 - D_2) \subseteq \Re(B_1) = \Re(B_2) \text{ and } \Re(D_1^* - D_2^*) \subseteq \Re(C_1^*) = \Re(C_2^*).$$

3 Relationship between Linear Matrix Functions

In this section, we consider two domains given in (2) and (3), we discuss the necessary and sufficient conditions for two relations $S_1 \cap S_2 \neq \varnothing$, $S_1 \subseteq S_2$ to hold. We also present connections between two domains of some well known linear matrix functions.

Theorem 8 *Let S_1 and S_2 be as given in (2) and (3), respectively. Then*
(a) $S_1 \cap S_2 \neq \varnothing$ if and only if all the following equalities hold:

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix}, \; r\begin{bmatrix} A_2 - A_1 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix},$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_2 \end{bmatrix} + r(C_3),$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_1 & B_3 \end{bmatrix} + r(C_2),$$

$$r\begin{bmatrix} A_2 - A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix} + r(C_1),$$

$$r\begin{bmatrix} A_2 - A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + r(B_3), \; r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix} + r(B_2),$$

$$r\begin{bmatrix} A_2 - A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} + r(B_1),$$

$$r\begin{bmatrix} A_2 - A_1 & 0 & B_1 & 0 & B_3 \\ 0 & A_1 - A_2 & 0 & B_2 & B_3 \\ C_2 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 \\ C_3 & C_3 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 & 0 \\ 0 & C_1 \\ C_3 & C_3 \end{bmatrix} + r\begin{bmatrix} B_1 & 0 & B_3 \\ 0 & B_2 & B_3 \end{bmatrix}.$$

(b) $S_1 \subseteq S_2$ if and only if any one of the following equalities holds:

$$\begin{gathered} r\begin{bmatrix} B_2 & B_3 \end{bmatrix} = l \quad or \; r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}, \\ or \; r\begin{bmatrix} A_2 - A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}, \end{gathered} \tag{10}$$

$$\begin{gathered} r(B_2) = l \quad or \; r\begin{bmatrix} A_2 - A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r(B_2) + r(C_3), \\ or \; r\begin{bmatrix} A_2 - A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r(C_3) + r(B_2) \quad or \; r(C_3) = n, \end{gathered} \tag{11}$$

$$\begin{gathered} r(B_3) = l \quad or \; r\begin{bmatrix} A_2 - A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{bmatrix} = r(B_3) + r(C_2), \\ or \; r\begin{bmatrix} A_2 - A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix} = r(C_2) + r(B_3) \quad or \; r(C_2) = n, \end{gathered} \tag{12}$$

$$r\begin{bmatrix} A_2-A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} \quad or\ r\begin{bmatrix} A_2-A_1 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} \quad or\ r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = n. \tag{13}$$

Proof. (a) The intersection $S_1 \cap S_2 \neq \varnothing$ is obviously equivalent to

$$A_1 + B_1X_1C_1 = A_2 + B_2X_2C_2 + B_3X_3C_3. \tag{14}$$

Eq (14) can be written as

$$B_1X_1C_1 - B_2X_2C_2 - B_3X_3C_3 = A_2 - A_1. \tag{15}$$

By applying Lemma 5 to the Eq (15), we get (a).

(b) Eq (14) can be written as

$$B_2X_2C_2 + B_3X_3C_3 = A_1 - A_2 + B_1X_1C_1. \tag{16}$$

From Lemma 3, Eq (16) holds for two matrices X_2 and X_3 if and only if all the following four conditions hold:

$$E_{[B_2,B_3]}(A_1 - A_2 + B_1X_1C_1) = 0, \tag{17}$$
$$E_{B_2}(A_1 - A_2 + B_1X_1C_1)F_{C_3} = 0, \tag{18}$$
$$E_{B_3}(A_1 - A_2 + B_1X_1C_1)F_{C_2} = 0, \tag{19}$$
$$((A_1 - A_2) + B_1X_1C_1)F_Z = 0, \tag{20}$$

where $Z = \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$. By Lemma 2, Eq (17) holds for all X_1 if and only if

$$E_{[B_2,B_3]} = 0 \ \text{ or } \begin{bmatrix} E_{[B_2,B_3]}B_1 & E_{[B_2,B_3]}(A_2-A_1) \end{bmatrix} = 0 \ \text{ or } \begin{bmatrix} C_1 \\ E_{[B_2,B_3]}(A_2-A_1) \end{bmatrix} = 0,$$

which are equivalent, respectively, to

$$r\begin{bmatrix} B_2 & B_3 \end{bmatrix} = l \ \text{ or } r\begin{bmatrix} A_2-A_1 & B_1 & B_2 & B_3 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix},$$
$$\text{or } r\begin{bmatrix} A_2-A_1 & B_2 & B_3 \\ C_1 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} B_2 & B_3 \end{bmatrix}.$$

This proves (10). Eq (18) holds for all X_1 if and only if

$$E_{B_2} = 0 \ \text{ or } \begin{bmatrix} E_{B_2}B_1 & E_{B_2}(A_2-A_1)F_{C_3} \end{bmatrix} = 0 \ \text{ or } \begin{bmatrix} C_1F_{C_3} \\ E_{B_2}(A_2-A_1)F_{C_3} \end{bmatrix} \ \text{ or } F_{C_3} = 0,$$

which, in consequence, is equivalent to

$$r\,(B_2) = l \ \text{ or } r\begin{bmatrix} A_2-A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix} = r\,(B_2) + r\,(C_3),$$
$$\text{or } r\begin{bmatrix} A_2-A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix} = r\,(C_3) + r\,(B_2) \ \text{ or } r\,(C_3) = n.$$

Then (11) holds. Similarly, Eq (19) holds for all X_1 if and only if

$$r\left(B_3\right)=l \text{ or } r\left[\begin{array}{ccc} A_2-A_1 & B_1 & B_3 \\ C_2 & 0 & 0 \end{array}\right]=r\left(B_3\right)+r\left(C_2\right),$$

$$\text{or } r\left[\begin{array}{cc} A_2-A_1 & B_3 \\ C_1 & 0 \\ C_2 & 0 \end{array}\right]=r\left(C_2\right)+r\left(B_3\right) \text{ or } r\left(C_2\right)=n.$$

Then we get (12). Eq (20) holds for all X_1 if and only if

$$\left[\begin{array}{cc} B_1 & (A_2-A_1)F_Z \end{array}\right]=0 \text{ or } \left[\begin{array}{c} C_1F_Z \\ (A_2-A_1)F_Z \end{array}\right]=0 \text{ or } F_Z=0,$$

which then is equivalent to

$$r\left[\begin{array}{cc} A_2-A_1 & B_1 \\ C_2 & 0 \\ C_3 & 0 \end{array}\right]=r\left[\begin{array}{c} C_2 \\ C_3 \end{array}\right] \text{ or } r\left[\begin{array}{c} A_2-A_1 \\ C_1 \\ C_2 \\ C_3 \end{array}\right]=r\left[\begin{array}{c} C_2 \\ C_3 \end{array}\right] \text{ or } r\left[\begin{array}{c} C_2 \\ C_3 \end{array}\right]=n,$$

which proves (13). Hence we establish (b).

Setting $B_3=I_p$, $C_2=I_n$ in Theorem 8, we get the following result.

Corollary 9 *Consider two domains of two linear matrix functions*

$$S_1=\left\{A_1+B_1X_1C_1 \mid X_1\in\mathbb{C}^{p_1\times n_1}\right\},$$
$$S_2=\left\{A_2+B_2X_2+X_3C_3 \mid X_2\in\mathbb{C}^{p_2\times n},\ X_3\in\mathbb{C}^{l\times p_3}\right\},$$

where $A_1,\ A_2\in\mathbb{C}^{l\times n},\ B_1\in\mathbb{C}^{l\times p_1},\ B_2\in\mathbb{C}^{l\times p_2}$ *and* $C_1\in\mathbb{C}^{n_1\times n},\ C_3\in\mathbb{C}^{p_3\times n}$ *are known matrices. Then*

(a) $S_1\cap S_2\neq\varnothing$ *if and only if the following rank equalities hold:*

$$r\begin{bmatrix} A_2-A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{bmatrix}=r\begin{bmatrix} B_1 & B_2 \end{bmatrix}+r(C_3),$$

$$r\begin{bmatrix} A_2-A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{bmatrix}=r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}+r(B_2),$$

$$r\begin{bmatrix} A_2-A_1 & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix}=r\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}+r\begin{bmatrix} B_1 & B_2 \end{bmatrix}.$$

(b) $S_1\subseteq S_2$ *if and only if*

$$r\left(B_2\right)=l \textit{ or } r\left[\begin{array}{ccc} A_2-A_1 & B_1 & B_2 \\ C_3 & 0 & 0 \end{array}\right]=r\left(B_2\right)+\left(C_3\right)$$

$$\textit{or } r\left[\begin{array}{cc} A_2-A_1 & B_2 \\ C_1 & 0 \\ C_3 & 0 \end{array}\right]=r\left(C_3\right)+r\left(B_2\right) \textit{ or } r\left(C_3\right)=n.$$

From Colrollary 9, we can deduce the following result.

Corollary 10 *Let $A_4 \in \mathbb{C}^{l\times n}$, $B_4 \in \mathbb{C}^{p\times s}$, $C_4 \in \mathbb{C}^{l\times p}$, $D_4 \in \mathbb{C}^{n\times s}$, $A_5 \in \mathbb{C}^{k\times t}$, $B_5 \in \mathbb{C}^{k\times n}$, $C_5 \in \mathbb{C}^{p\times t}$be given, X_4, $X_5 \in \mathbb{C}^{n\times p}$ be unknown matrices, and assume that the system $A_4X = C_4$, $XB_4 = D_4$, and the matrix equation $B_5XC_5 = A_5$ are solvable for X_4 and X_5, respectively. Denote*

$$S_1 = \left\{X_4 \in \mathbb{C}^{n\times p} \mid \ A_4X_4 = C_4,\ X_4B_4 = D_4\right\}, \tag{21}$$
$$S_2 = \left\{X_5 \in \mathbb{C}^{n\times p} \mid B_5X_5C_5 = A_5\right\}. \tag{22}$$

Then

(a) $S_1 \cap S_2 \neq \varnothing$, that is, the system $A_4X_4 = C_4$, $X_4B_4 = D_4$ and $B_5X_5C_5 = A_5$ have a common solution if and only if

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix},$$
$$r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix},$$
$$r\begin{bmatrix} 0 & B_4 & C_5 \\ A_4 & -C_4B_4 & 0 \\ B_5 & 0 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix} + r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix}.$$

(b) $S_1 \subseteq S_2$, that is, all solutions of $A_4X_4 = C_4$, $X_4B_4 = D_4$ are solutions of $B_5X_5C_5 = A_5$ if and only if

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r(A_4) \quad or \quad r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r(B_4).$$

Proof. It follows from Lemmas 6 and 2 that, the solutions of system $A_4X_4 = C_4$, $X_4B_4 = D_4$ and equation $B_5X_5C_5 = A_5$ can be expressed, respectively, as

$$X_4 = A_4^{+}C_4 + F_{A_4}D_4B_4^{+} + F_{A_4}VE_{B_4},$$
$$X_5 = B_5^{+}A_5C_5^{+} + F_{B_5}U + WE_{C_5},$$

where V, U and W are arbitrary. So, two sets in (21) and (22) can be represented, respectively, as

$$S_1 = \left\{A_4^{+}C_4 + F_{A_4}D_4B_4^{+} + F_{A_4}VE_{B_4}\right\},$$
$$S_2 = \left\{B_5^{+}A_5C_5^{+} + F_{B_5}U + WE_{C_5}\right\}.$$

From Corollary 9, the relation $S_1 \cap S_2 \neq \varnothing$ holds if and only if the following equalities hold:

$$r\begin{bmatrix} B_5^+A_5C_5^+ - A_4^+C_4 - F_{A_4}D_4B_4^+ & F_{A_4} & F_{B_5} \\ E_{C_5} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} F_{A_4} & F_{B_5} \end{bmatrix} + r(E_{C_5}), \tag{23}$$

$$r\begin{bmatrix} B_5^+A_5C_5^+ - A_4^+C_4 - F_{A_4}D_4B_4^+ & F_{B_5} \\ E_{B_4} & 0 \\ E_{C_5} & 0 \end{bmatrix} = r\begin{bmatrix} E_{B_4} \\ E_{C_5} \end{bmatrix} + r(F_{B_5}), \tag{24}$$

$$r\begin{bmatrix} B_5^+A_5C_5^+ - A_4^+C_4 - F_{A_4}D_4B_4^+ & F_{A_4} & F_{B_5} \\ E_{B_4} & 0 & 0 \\ E_{C_5} & 0 & 0 \end{bmatrix} = r\begin{bmatrix} E_{B_4} \\ E_{C_5} \end{bmatrix} + r\begin{bmatrix} F_{A_4} & F_{B_5} \end{bmatrix}. \tag{25}$$

By Lemma 1, and simplifying by $C_4B_4 = A_4D_4$, $A_4A_4^+C_4 = C_4$, $D_4B_4^+B_4 = D_4$, $B_5B_5^+A_5 = A_5$, $A_5C_5^+C_5 = A_5$, we find that the rank equalities in (23)-(25) are equivalent, respectively, to

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix},$$
$$r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix},$$
$$r\begin{bmatrix} 0 & B_4 & C_5 \\ A_4 & -C_4B_4 & 0 \\ B_5 & 0 & A_5 \end{bmatrix} = r\begin{bmatrix} B_4 & C_5 \end{bmatrix} + r\begin{bmatrix} A_4 \\ B_5 \end{bmatrix}.$$

Thus (a) is proved.

(b) $S_1 \subseteq S_2$ holds if and only if

$$r\begin{bmatrix} B_5^+A_5C_5^+ - A_4^+C_4 - F_{A_4}D_4B_4^+ & F_{A_4} & F_{B_5} \\ E_{C_5} & 0 & 0 \end{bmatrix} = r(F_{B_5}) + r(E_{C_5}),$$
$$\text{or} \quad r\begin{bmatrix} B_5^+A_5C_5^+ - A_4^+C_4 - F_{A_4}D_4B_4^+ & F_{B_5} \\ E_{B_4} & 0 \\ E_{C_5} & 0 \end{bmatrix} = r(E_{C_5}) + r(F_{B_5}),$$

which then is equivalent to

$$r\begin{bmatrix} A_4 & C_4C_5 \\ B_5 & A_5 \end{bmatrix} = r(A_4) \quad \text{or} \quad r\begin{bmatrix} B_4 & C_5 \\ B_5D_4 & A_5 \end{bmatrix} = r(B_4).$$

Then we establish (b).

Remark 3.1 *Result (a) of Corollary 10 is the same as in [7, Theorem 2.4].*

4 Conclusion

In this study, we discussed and examined some fundamental questions associated with the connections between two domains of linear matrix functions and specific types of linear matrix equations. The general solutions can be expressed via particular explicit linear matrix functions to establish some connections between their domains through the methodical application of various established or well-known relations to ranks and ranges of matrices. Thus, they show that a variety of matrix equality and matrix set inclusion problems may be solved with the help of the matrix rank and range method.

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