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Analysis and Existence of Optimal Control in Industrial Economic Growth with Investment Using the Ramsey-Cass-Koopmans Model

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Abstract: Economic growth is associated with an increase in the production of goods and services. Consumption and investment influence the increased production of goods and services. Consumption parameters can be assessed based on utility, while investments can be affected by the level of capital stock. This paper applies a modification and analysis of the Ramsey-Cass-Koopmans model to the economic growth of two industries, focusing on investment strategies to maximize consumption utility. The analysis of the Ramsey-Cass-Koopmans model showed that the model is valid as it has a positive and unique solution. This study performs optimal control by maximizing the consumption utility of each industry, where the control is given in the form of per capita consumption. In this paper, consumption control can be interpreted as a form of savings. In addition, the existence of optimal control is proven, indicating that the problem can be solved.

Keywords: *Ramsey-Cass-Koopmans model; utility; optimal control; investment.*

Mathematics Subject Classification (2020): 49J15, 49K15, 49L20, 93-10.

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1 Introduction

A country's development is determined by economic growth. This is demonstrated by increasing a country's ability to provide goods and services to its population [1]. Consumption and investment drive the growth in the production of goods and services. Consumption and investment factors cannot be separated. Therefore, consumption and investment are interconnected and influence each other. Consumption allows firms to generate the income needed to increase investment in the capital stock, while investment can increase production. Utility is the value or benefit obtained from consumption activities or use of goods and services. Furthermore, utility is an essential factor that influences consumption. The importance of consumption utility lies in its ability to increase productivity and efficiency in the production of goods and services [2]. One approach to maximizing utility is modelling economic growth problems using the Ramsey model, which can then be analyzed using optimal control theory.

Optimal control theory focuses on determining controls that influence processes while adhering to specific constraints [3–5]. Optimal control theory also serves as an alternative for solving economic growth problems, including those related to the Ramsey model. The Ramsey model was first introduced in 1928 by Frank P. Ramsey [6]. The Ramsey model is a neoclassical economic growth model that maximizes the utility of capital-bound consumption under dynamic constraints. David Cass and Tjalling Koopmans further developed this model in separate works, and it is now known as the Ramsey-Cass-Koopmans model [7, 8]. In their developments, several previous studies have used this model [9–13]. Olivia Bundau and Adina Juratoni [14] discussed the Ramsey growth model in infinite and continuous time with the aim of maximizing global utility using the Pontryagin Maximum Principle. Then Kajanovičová et al. [15] discussed optimal control of the Ramsey-Cass-Koopmans economic growth model with non-constant population growth using the Maximum Principle, which aims to maximize consumption utility with control in the form of per capita consumption. Further research by Frerick et al. [16] discussed the multi-object Ramsey-Cass-Koopmans model for Ramsey-type equilibrium problems with heterogeneous agents.

This paper discusses a modification and analysis of the nonlinear dynamics of the Ramsey-Cass-Koopmans model of the economic growth of two industries that are interrelated by investment. The optimal control is aimed to maximize the utility of the amount of consumed production. The analyses conducted in this study are the positivity analysis, the uniqueness analysis, and the existence of optimal control [4]. Positivity and uniqueness aim to validate the model, while the existence of optimal control verifies whether a control that maximizes utility exists. The control variable for maximizing utility is defined as per capita consumption. When consumption is controlled, consumption expenditure is reduced, and the remaining production output can be reinvested or saved as savings.

2 Mathematical Model of Economic Growth with Investment

The mathematical model of economic growth used in this paper is a modification of the multi-object Ramsey-Cass-Koopmans model [16]. In this paper, it is assumed that the second industry has a high demand for goods. Thus, the first industry provides some of its capital by investing in the second industry. The investment return is assumed to be a profit of 5% from the investment, and this problem can be illustrated as follows.

Individual A owns and manages two industries in different regions, namely, the first industry in region X and the second industry in region Y. In the first industry, consumer demand for goods in region X is not too high; thus, the first industry can invest in the second industry. While consumer demand for goods in region Y in the second industry exceeds the demand for goods in the first industry in region X, the first industry helps by providing capital or investing in the second industry. With this additional capital, it is expected to maximize *output* or production results. For example, if individual A owns and manages two industries and has problems in one industry, then individual A can solve the problems in the first industry. This process aims to maximize consumption utility and increase capital stock growth in both industries.

The relationship between the two industries can be formulated by the following mathematical model of economic growth:

$$\begin{aligned} \tfrac{dK_1(t)}{dt} &= F_1(\mathcal{A}_1, K_1(t), L_1(t)) - \delta_1 K_1(t) + \delta_2 K_2(t) - C_1(t), \\ \tfrac{dK_2(t)}{dt} &= F_2(\mathcal{A}_2, K_2(t), L_2(t)) + \delta_1 K_1(t) - \delta_2 K_2(t) - C_2(t) \end{aligned} \tag{1}$$

with

$K_1(t)$	:	First industry capital stock at time t
$K_2(t)$	:	Second industry capital stock at time t
$L_1(t)$	:	Total labor of the first industry at time t
$L_2(t)$	:	Total labor of the second industry at time t
$C_1(t)$	:	Amount of production output consumed by the first industry at time t
$C_2(t)$	:	Amount of production output consumed by the second industry at time t
δ_1	:	Investment rate
δ_2	:	Investment return rate
$F_1(\mathcal{A}_1, K_1(t), L_1(t))$	:	First industry output
$F_2(\mathcal{A}_2, K_2(t), L_2(t))$	:	Second industry output
$\mathcal{A}$	:	Technological advancement factor.

$K(t)$ and $C(t)$ are continuous functions, and the production function F used in the model is the Cobb-Douglas production function

$$F(\mathcal{A}, K(t), L(t)) = \mathcal{A}K(t)^{\alpha} L(t)^{1-\alpha} \tag{2}$$

with $\mathcal{A} > 0$ being a constant. The production output (F) describes the relationship between the technological advancement factor ($\mathcal{A}$), the capital stock (K), and the amount of labor (L) with $\dot{L}(t) = nL(t)$, where $L(t)$ experiences exponential growth with a constant growth rate of the amount of labor (n).

In economic analysis, to enable more accurate and fair comparisons between groups with different populations, it is necessary to convert the Equation (1) into per capita model:

- Capital stock per capita (k):

$$k(t) = \frac{K(t)}{L(t)},$$
$$K(t) = k(t)L(t).$$

Then

$$\dot{K}(t) = \dot{k}(t)L(t) + k(t)\dot{L}(t),$$
$$\dot{K}(t) = \dot{k}(t)L(t) + k(t)nL(t).$$

- Amount of production *output* consumed per capita (c):

$$c(t) = \frac{C(t)}{L(t)}.$$

- Production output per capita (f):

$$f(t) = \frac{F(t)}{L(t)},$$

where F is

$$\begin{aligned} F(t) &= \mathcal{A}K(t)^{\alpha}L(t)^{1-\alpha} \\ &= \mathcal{A}k(t)^{\alpha}L(t)^{\alpha}L(t)^{1-\alpha} \\ &= \mathcal{A}k(t)^{\alpha}L(t). \end{aligned}$$

Therefore, the output of per capita production is

$$f(t) = \frac{F(t)}{L(t)} = \frac{\mathcal{A}k(t)^{\alpha}L(t)}{L(t)} = \mathcal{A}k(t)^{\alpha}.$$

Furthermore, assume labor $L(t) = L_1(t) = L_2(t)$, then Equation (1) becomes:

1. The capital stock of the first industry (k_1) can be given as follows:

$$(\dot{k}_1(t) + k_1(t)n_1)L(t) = (f_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - c_1(t))L(t).$$

   Then, simplify both segments by multiplying by $\frac{1}{L(t)}$:

$$\dot{k}_1(t) = f_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - k_1(t)n_1 - c_1(t).$$

   Substitute $f_1(t) = \mathcal{A}_1 k_1(t)^{\alpha_1}$ so that

$$\dot{k}_1(t) = \mathcal{A}_1 k_1(t)^{\alpha_1} - \delta_1 k_1(t) + \delta_2 k_2(t) - k_1(t)n_1 - c_1(t).$$

2. The capital stock of the second industry (k_2) can be given as follows:

$$(\dot{k}_2(t) + k_2(t)n_2)L(t) = (f_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - c_2(t))L(t).$$

   Then, simplify both segments by multiplying by $\frac{1}{L(t)}$:

$$\dot{k}_2(t) = f_2(t) + \delta_1 k_1 - \delta_2 k_2(t) - k_2(t)n_2 - c_2(t).$$

   Substitute $f_2(t) = \mathcal{A}_2 k_2(t)^{\alpha_2}$ so that

$$\dot{k}_2(t) = \mathcal{A}_2 k_2(t)^{\alpha_2} + \delta_1 k_1(t) - \delta_2 k_2 - k_2(t)n_2 - c_2(t).$$

Thus, Equation (1) can be expressed as

$$\begin{array}{rcl} \dot{k}_1(t) &=& \mathcal{A}_1 k_1(t)^{\alpha_1} - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t) - c_1(t), \\ \dot{k}_2(t) &=& \mathcal{A}_2 k_2(t)^{\alpha_2} + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t) - c_2(t) \end{array} \quad (3)$$

for $k_i(0) = k_{i0},\ i = 1, 2.$

3 Positive and Unique Solution

It can be seen from Equation (3) that the model is considered valid if it has a positive solution at any time t. It means that if the model has initial conditions $k_1(t_0) > 0$ and $k_2(t_0) > 0$, then $k_1(t) > 0$ and $k_2(t) > 0$ for every $t > t_0$. First, it will be shown that Equation (3) is valid. Suppose X is the set of all $x(t) = (k_1, k_2)$ for each time t as the solution of a controlled model with the initial conditions $x(t_0) = (k_1(t_0), k_2(t_0))$ and the set

$$\Omega_{(k_1(t_0),k_2(t_0))} := \{k_1(t), k_2(t)|t_0 \le t \le t_f, 0 < k_1(t), k_2(t)\}. \tag{4}$$

If the initial conditions in the Equation (3) satisfy $k_1(t_0 = 0) > 0$ and $k_2(t_0 = 0) > 0$, then it can be said that the Equation (3) is valid if the set $\Omega_{(k_1(t_0),k_2(t_0))}$ is a positive invariant set. The definition of a positive invariant set can be given as follows.

Definition 3.1 *(Positively invariant set). Let $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$ be a dynamic system with the initial conditions $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$. Suppose Ω is a subset of $\mathbb{R}^n$. Then Ω is said to be a positive invariant set if $\boldsymbol{x}_0 \in \Omega$ implies $\boldsymbol{x}(t, x_0) \in \Omega$ for every $t \ge t_0$.*

The positive invariant set of (4) can be proven by the following theorem.

Theorem 3.1 *Let*

$$\Omega_{(k_1(t_0),k_2(t_0))} := \{k_1(t), k_2(t)|t_0 \le t \le t_f, 0 < k_1(t), k_2(t)\}$$

be a subset of all solutions of the Equation (3) with the initial conditions $k_1(t_0 = 0), k_2(t_0 = 0)$. If $k_1(t_0), k_2(t_0) > 0$, then $\Omega_{(k_1(t_0),k_1(t_0))}$ is a positive invariant set.

Proof. Define the functions $\dot{k}_1(t)$ and $\dot{k}_2(t)$ as follows:

$$\begin{aligned} \dot{k}_1(t) &= f_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t) - c_1(t) \\ &= f_1(t) - c_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t), \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{k}_2(t) &= f_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t) - c_2(t) \\ &= f_2(t) - c_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t). \end{aligned} \tag{6}$$

Based on Krasovskii et al. [13], the consumption function is defined as

$$\begin{aligned} C(t) &= (1 - s(t))F(t), \\ c(t)L(t) &= (1 - s(t))f(t)L(t). \end{aligned}$$

Both segments are multiplied by $\frac{1}{L(t)}$:

$$\begin{aligned} c(t) &= (1 - s(t))f(t), \\ c(t) &= f(t) - s(t)f(t), \\ s(t)f(t) &= f(t) - c(t), \end{aligned}$$

where $s(t)$ represents the investment of current savings invested in capital growth at time t with $s(t) < 1$. Thus, the Equations (5) – (6) can be written as

$$\begin{aligned} \dot{k}_1(t) &= f_1(t) - c_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t) \\ &= s_1(t) f_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t) \\ &= s_1(t) \mathcal{A}_1 k_1(t)^{\alpha_1} - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t), \end{aligned}$$

$$\begin{aligned}\dot{k}_2(t) &= f_2(t) - c_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t)\\ &= s_2(t) f_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t)\\ &= s_2(t)\mathcal{A}_2 k_2(t)^{\alpha_2} + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t).\end{aligned}$$

Assume that there exists $t \in (0, t_f]$ such that $k_1(t) \le 0$ or $k_2(t) \le 0$. First, suppose $k_1(t) \le 0$ and $k_{1*} = \{t \in (0, t_f] \mid k_1(t) \le 0\}$, then let $t^* = \inf k_{1*}$. It can be seen that $t^* \ne 0$, so there exists $k_1(t) > 0, \forall t \in [0, t^*)$ and

$$\begin{aligned}\dot{k}_2(t) &= s_2(t)\mathcal{A}_2 k_2(t)^{\alpha_2} + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t),\\ \dot{k}_2(t) &> -\delta_2 k_2(t) - n_2 k_2(t), \quad \forall t \in [0, t_f),\\ \dot{k}_2(t) + \delta_2 k_2(t) + n_2 k_2(t) &> 0.\end{aligned}$$

Assume that there exists $t \in (0, t^*)$ such that $k_2(t) \le 0$. Then suppose $k_{2*} = \{t \in (0, t^*) \mid k_2(t) \le 0\}$ and $t^*_{k_2} = \inf k_{2*}$. It can be seen that $t^*_{k_2} \ne 0$. Then there exists $k_2(t) > 0, \forall t \in [0, t^*_{k_2})$ and

$$\begin{aligned}\dot{k}_2(t) + \delta_2 k_2(t) + n_2 k_2(t) &> 0, \quad \forall t \in [0, t^*_{k_2}),\\ \dot{k}_2(t) + (\delta_2 + n_2) k_2(t) &> 0,\\ e^{(\delta_2+n_2)t}\dot{k}_2(t) + e^{(\delta_2+n_2)t}(\delta_2 + n_2) k_2(t) &> 0,\\ \frac{d}{dt}\left(e^{(\delta_2+n_2)t} k_2(t)\right) &> 0,\\ \int_0^{t^*_{k_2}} \frac{d}{dt}\left(e^{(\delta_2+n_2)t} k_2(t)\right) dt &> 0,\\ e^{(\delta_2+n_2)t^*_{k_2}} k_2(t) - k_2(0) &> 0,\\ k_2(t^*_{k_2}) &> k_2(0) e^{-(\delta_2+n_2)t^*_{k_2}}.\end{aligned}$$

We obtain that $k_2(t^*_{k_2}) > k_2(0)e^{-(\delta_2+n_2)t^*_{k_2}} > 0$. However, this contradicts the statement $k_2(t^*_{k_2}) \le 0$. Therefore, it can be concluded that $k_2(t) > 0$ for any $t \in [0, t^*)$. It means that

$$\begin{aligned}\dot{k}_1(t) &= s_1(t)\mathcal{A}_1 k_1(t)^{\alpha_1} - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t),\\ \dot{k}_1(t) &> -\delta_1 k_1(t) - n_1 k_1(t), \qquad \forall t \in [0, t^*),\\ \dot{k}_1(t) + \delta_1 k_1(t) + n_1 k_1(t) &> 0,\\ \dot{k}_1(t) + (\delta_1 + n_1) k_1(t) &> 0,\\ e^{(\delta_1+n_1)t}\dot{k}_1(t) + e^{(\delta_1+n_1)t}(\delta_1 + n_1) k_1(t) &> 0,\\ \frac{d}{dt}\left(e^{(\delta_1+n_1)t^*_{k_1}} k_1(t)\right) &> 0,\\ \int_0^{t^*} \frac{d}{dt}\left(e^{(\delta_1+n_1)t} k_1(t)\right) dt &> 0,\\ e^{(\delta_1+n_1)t^*} k_1(t) - k_1(0) &> 0,\\ k_1(t^*) &> k_1(0) e^{-(\delta_1+n_1)t^*}\end{aligned}$$

so that $k_1(t^*) > k_1(0)e^{-(\delta_1+n_1)t^*} > 0$ holds. However, this contradicts the statement $k_1(t^*) \le 0$. Therefore, it can be concluded that $k_1(t) > 0$ for any $t \in [0, t_f]$. Furthermore, in the same way, for $k_2(t)$, it is obtained that $k_2(t) > 0$ for every $t \ge t_0$.

It has been proved that the set $\Omega_{(k_1(t_0),k_2(t_0))}$ defined in Equation (4) is a positive invariant set. It means that if the initial condition in Equation (3) with control is positive, then the solution of the model is positive for any time t. However, it is not guaranteed that this model has a unique solution for a given initial condition.

Now, to guarantee that the solution of Equation (3) exists and is unique, we can use the concept of the Lipschitz condition in Equation (3) [17]. For that, we prove that Equation (3) satisfies the Lipschitz condition for $\alpha_1 = \alpha_2 = 1$ as given in the following theorem.

Theorem 3.2 *The mathematical model* (3) *that satisfies a given initial condition* $k_1(t_0), k_2(t_0) > 0$*, has a unique solution.*

Proof. Let $X = (k_1, k_2)$ and

$$\varphi(X) = \begin{bmatrix} \frac{dk_1}{dt} \\ \\ \frac{dk_2}{dt} \end{bmatrix}.$$

The Ramsey-Cass-Koopmans model with investment can be written as

$$\varphi(X) = \begin{bmatrix} \mathcal{A}_1 k_1 - \delta_1 k_1 + \delta_2 k_2 - n_1 k_1 - c_1 \\ \\ \mathcal{A}_2 k_2 + \delta_1 k_1 - \delta_2 k_2 - n_2 k_2 - c_2 \end{bmatrix}.$$

Note that for $X_a = (k_{1a}, k_{2a})$ and $X_b = (k_{1b}, k_{2b})$,

$$\varphi(X_a) - \varphi(X_b) = \begin{bmatrix} (\mathcal{A}_1 - \delta_1 - n_1)(k_{1a} - k_{1b}) + \delta_2(k_{2a} - k_{2b}) \\ \\ (\mathcal{A}_2 - \delta_2 - n_2)(k_{2a} - k_{2b}) + \delta_1(k_{1a} - k_{1b}) \end{bmatrix}.$$

Furthermore, by using the Euclidean norm of $\mathbb{R}^2$ and based on the triangle inequality, we obtain

$$\| \varphi(X_a) - \varphi(X_b) \| \leq \left\| \begin{bmatrix} (\mathcal{A}_1 - \delta_1 - n_1)(k_{1a} - k_{1b}) \\ \\ (\mathcal{A}_2 - \delta_2 - n_2)(k_{2a} - k_{2b}) \end{bmatrix} \right\| + \left\| \begin{bmatrix} \delta_2(k_{2a} - k_{2b}) \\ \\ \delta_1(k_{1a} - k_{1b}) \end{bmatrix} \right\|.$$

Since $\mathcal{A}_1, \mathcal{A}_2, \delta_1, \delta_2, n_1, n_2$ are constant, then there exists $M > 0$ such that

$$|\mathcal{A}_1 - \delta_1 - n_1|, |\mathcal{A}_2 - \delta_2 - n_2|, |\delta_1|, |\delta_2| \leq M.$$

Then

$$\begin{aligned} \| \varphi(X_a) - \varphi(X_b) \| &\leq M \left\| \begin{bmatrix} k_{1a} - k_{1b} \\ \\ k_{2a} - k_{2b} \end{bmatrix} \right\| + M \left\| \begin{bmatrix} k_{2a} - k_{2b} \\ \\ k_{1a} - k_{1b} \end{bmatrix} \right\| \\ &= M \left\| \begin{bmatrix} k_{1a} - k_{1b} \\ \\ k_{2a} - k_{2b} \end{bmatrix} \right\| + M \left\| \begin{bmatrix} k_{2a} - k_{2b} \\ \\ k_{1a} - k_{1b} \end{bmatrix} \right\| \\ &= M \| X_a - X_b \| + M \| X_a - X_b \| \\ &= 2M \| X_a - X_b \| \end{aligned}$$

so that $\varphi(X)$ is a Lipschitz function. It means that we obtain

$$X(t) = X(t_0) + \int_{t_0}^{t} \varphi(X)\ dt.$$

It is proved that X has a unique solution for the initial condition $k_i(t_0) > 0,\ i = 1, 2$.

4 The Existence of Optimal Control

In this paper, the objective function aims to maximize consumption utility through per capita consumption control. This approach is generally used when utility is directly based on consumption level. Consumption level is considered as the main factor affecting the consumption utility. The utility used is the logarithmic utility function, that is,

$$u(c(t)) = \ln c(t).$$

Then the objective function based on the Ramsey-Cass-Koopman model can be defined as follows:

$$J = \max_{c_1(t)\in U_1} \int_0^\infty \ln c_1(t) e^{-\rho t} dt + \max_{c_2(t)\in U_2} \int_0^\infty \ln c_2(t) e^{-\rho t} dt \tag{7}$$

for the discount factor ρ adjusts future consumption utility values according to individual time preferences, with constraints based on Equation (3). From Equation (7), we obtain that this statement is equivalent to

$$J = \max_{(c_1(t),c_2(t))\in U} \int_0^\infty (\ln c_1(t) + \ln c_2(t)) e^{-\rho t} dt \tag{8}$$

for $U = (U_1, U_2)$. Furthermore, by considering the equation

$$c(t) = (1 - s(t)) f(t) = (1 - s(t)) \mathcal{A} k(t),$$

we can represent the objective function (7) as

$$J = \max \int_0^\infty (\ln(1 - s_1(t)) \mathcal{A} k_1(t) + \ln(1 - s_2(t)) \mathcal{A} k_2(t)) e^{-\rho t} dt, \tag{9}$$

where $s_1(t)$ and $s_2(t)$ denote investment in the form of a portion of current output saved and invested in capital growth at time t with $s(t) < 1$ in the first and second industries, respectively.

We have shown in Section 3 that the Equation (3) has a unique and positive solution. Using the result from Fleming and Rishel [18]) we prove the existence of the optimal control by checking the following points.

1. The set S defined as

   $$S = \{(s_1(t), s_2(t)) \mid 0 \le s_1(t), s_2(t) \le g, \forall t \in [0, t_f]\}$$

   is a nonempty set. This can be seen from Theorems 3.1 and 3.2, where every control $s \in S$ has a unique and positive solution.

2. The set S is a closed convex set.

   Let $s_1(t),\ s_2(t) \in S$. It can be easily seen that $0 \leq s_1(t), s_2(t) \leq g$ for every $t \in [0, t_f]$; so, with every $\lambda \in [0,1]$, we obtain

$$0 \leq \lambda s_1(t) + (1-\lambda)\, s_2(t) \leq g, t \in [0, t_f].$$

   Therefore

$$\lambda s_1(t) + (1-\lambda)\, s_2(t) \in S.$$

   This shows that S is a convex set. Now, we show that S is a closed set. It is enough to show that for every convergent sequence $(s_n(t))_{n\in\mathbb{N}} = (s_{1n}(t), s_{2n}(t)) \subseteq S$, $\lim_{n\to\infty} s_n(t) \in S$. It means that for $s_n(t) \to (s_1(t), s_2(t))$ with $s_1(t) = \lim_{n\to\infty} s_{1n}(t)$ and $s_2 = \lim_{n\to\infty} s_{2n}(t)$, we obtain $(s_1(t), s_2(t)) \in S$. Now we define

$$\| u - v \| := \sup\{|u\,(t) - v\,(t)\,| \mid t \in [0, t_f]\}\ .$$

   Then we know that $s_n(t)$ is a convergent sequence such that for every $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ that satisfies

$$\| s_{1n}(t) - s_1(t) \| < \varepsilon$$

   and

$$\| s_{2n}(t) - s_2(t) \| < \varepsilon$$

   for every $n \geq K(\varepsilon)$. Therefore, we obtain

$$\begin{array}{ll} & \| s_{in}(t) - s_i(t) \| < \varepsilon \\ (\Rightarrow) & |s_{in}(t) - s_i(t)| < \sup\{|s_{in}(t) - s_i(t)| \mid t \in [0, t_f]\} < \varepsilon \\ (\Rightarrow) & -\varepsilon < s_i(t) - s_{in}(t) < \varepsilon \\ (\Rightarrow) & -\varepsilon \leq s_{in}(t) - \ \varepsilon < s_i(t) < \varepsilon + s_{in}(t) \leq \varepsilon + g \\ (\Rightarrow) & -\varepsilon < s_i(t) < g + \varepsilon \end{array}$$

   for $i = 1, 2$. Since it holds for every $\varepsilon > 0$, we obtain $0 \leq s_1(t), s_2(t) \leq g$ and $(s_1(t), s_2(t)) \in S$. Consequently, we show that S is a closed set. Furthermore, it can be proven that S is a closed convex set.

3. Note that the dynamic Equation (3) can be expressed as

$$\begin{array}{rcl} \dot{k}_1(t) & = & s_1(t)\mathcal{A}_1 k_1(t) - \delta_1 k_1(t) + \delta_2 k_2(t) - n_1 k_1(t), \\ \dot{k}_2(t) & = & s_2(t)\mathcal{A}_2 k_2(t) + \delta_1 k_1(t) - \delta_2 k_2(t) - n_2 k_2(t). \end{array} \tag{10}$$

   It can be seen that the right-hand side of equation (10) is a linear function in the state and control variables. Then we know that $k_i(t)$ is continuous in the interval $[0, t_f]$ and $s_i(t)$ is a bounded function with $0 \leq s_i(t) \leq g$ for $i = 1, 2$. So we prove that the Equation (10) is bounded.

4. Let

$$U(s_1(t), s_2(t)) = e^{-\rho t} \ln((1 - s_1(t))\mathcal{A}k_1(t)) + \ln((1 - s_2(t))\mathcal{A}k_2(t))$$

   and we know that $s_1(t), s_2(t), k_1(t)$ and $k_2(t)$ are bounded functions on $[0, t_f]$. It means that $U(s_1(t), s_2(t))$ is a bounded function. Suppose that $U_1(s_1(t) = \ln((1 - s_1(t))\mathcal{A}k_1(t))$ and $U_2(s_2(t) = \ln((1 - s_2(t))\mathcal{A}k_2(t))$. Then we obtain

$$\frac{\partial^2 U_1}{\partial s_1^2} = -\frac{1}{(1 - s_1(t))^2}\mathcal{A}k_1(t) < 0$$

and
$$\frac{\partial^2 U_2}{d\partial s_2^2} = -\frac{1}{(1-s_2(t))^2}\mathcal{A}k_2(t) < 0.$$

It means that U_1 and U_2 are concave functions. Thus, we prove that U is a concave function.

5 Conclusion

The dynamic model of industrial economic growth with investment, as developed in this paper, extends the Ramsey-Cass-Koopmans model by incorporating a strategy focused on maximizing consumption utility. Specifically, the model is adapted to two industries, with investment flowing from the first industry to the second to achieve optimal consumption utility across both sectors. Control variables are introduced in the form of per capita consumption in the first industry (c_1) and the second industry (c_2) to maximize utility in both industries. In this context, control through per capita consumption can be interpreted as control through savings, represented by s_1 and s_2, which denote savings in the first and second industries, respectively. Analytically, the model is valid and has a unique solution, as demonstrated through the concept of positive invariant sets and the Lipschitz continuity of the model. The positivity of the resulting solution ensures that the capital stocks in both industries, K_1 and K_2, remain non-negative. This paper also analyzes the existence of optimal control, establishing that any introduced control leads to a positive solution and confirming the existence of optimal control for maximizing consumption utility.

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