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Exponential Decay of Timoshenko System with Fractional Delays and Source Terms

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Abstract: The objective of this paper is to analyse the asymptotic behavior of a Timoshenko beam system with fractional delays and nonlinear external sources. Under suitable conditions on the damping, delay and initial data, we obtain exponential decay rate of the solution.

Keywords: *Timoshenko system; energy decay; nonlinear systems; fractional delay.*

Mathematics Subject Classification (2020): 35B40, 47D03, 74D05, 93D23, 93D15.

1 Introduction

In this work, we study the following Timoshenko system with fractional delays:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x+\psi\right)_x + a_1\partial_t^{\alpha,\beta}\varphi\left(t-s\right) + \alpha_1\varphi_t = \left|\varphi\right|^{p-2}\varphi, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x+\psi\right) + a_2\partial_t^{\alpha,\beta}\psi\left(t-s\right) + \alpha_2\psi_t = \left|\psi\right|^{q-2}\psi, \\ \varphi(x=0,t) = \psi(x=0,t) = \varphi(x=L,t) = \psi(x=L,t) = 0, \\ \varphi(x,t=0) = \varphi_0(x),\ \psi(x,t=0) = \psi_0(x), \\ \varphi_t(x,t=0) = \varphi_1(x),\ \psi_t\left(x,0\right) = \psi_1(x), \\ \varphi_t\left(x,t-s\right) = f_0\left(x,t-s\right), t\in(0,s), \\ \psi_t\left(x,t-s\right) = g_0\left(x,t-s\right), t\in(0,s), \end{cases} \tag{1}$$

where $x\in\Omega=(0,L), L>0, t\in\mathbb{R}_+^\star, \rho_1,\rho_2,a_1,a_2,\alpha_1,\alpha_2,b$ and k are positive real constants. The constant $s>0$ is the time delay and the exponents p and q satisfy $p>2$

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and $q > 2$. The functions $\varphi_0, \varphi_1, \psi_0, \psi_1, f_0, g_0$ are the initial data belonging to suitable spaces. The well known notation $\partial_t^{\alpha,\beta}$ stands for the generalized Caputo's fractional derivative, see [17, 18], it is defined as

$$\partial_t^{\alpha,\beta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\ e^{-\beta(t-s)}\ u_s(s)\ ds, \quad 0 < \alpha < 1,\ \beta > 0.$$

The problem (1) is considered without internal and external forces, this type of systems has been introduced in [19]. It describes the transverse vibration of a thick beam of length L, where φ is the transverse displacement of the beam, ψ is the rotation angle of the filament of the beam, and ρ_1, ρ_2, k and b account for some physical properties of the beam, see [11]. In our case, the Timoshenko beam is subject to internal forces given by fractional delay terms and frictional damping, and to external forces represented by source terms. Physically, the occurrence of fractional delay terms in many systems can lead to undesirable dynamics such as degraded performance, reduced robustness, or even instability. Generally, these harmful effects are controlled by various dissipation terms; for more results, see [1, 2].

In the last decades, the study of the well-posedness and stability/instability of evolution equations with time delay has received considerable attention of researchers. Many authors have shown that the time delay can be a source of instability that is asymptotically stable in the absence of time delay, see in this direction [3, 15]. More results in this context can be found in [4, 5, 8, 10, 20].

For the Timoshenko system with time delay, we mention the work [7], in which the following problem is considered:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) + a_1\varphi(x, t-\tau_1) + \alpha_1\varphi_t(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x + \psi)(x,t) + a_2\psi(x, t-\tau_2) + \alpha_2\psi_t(x,t) = 0. \end{cases} \tag{2}$$

The authors obtained the exponential decay rate when the weights of time delays are smaller than the corresponding damping. By adopting the spectral analysis approach, A. Adnane et al. [1] showed the same result by considering the time delay of fractional type rather than the time delay in the system (2) without sources.

In the absence of delay, the problem of existence and energy decay for a single wave equation with damping and/or source terms has been extensively studied by several authors. They showed the damping term assures global existence in the absence of source term, whereas without the damping term, the source term causes finite time blow-up of the solution. Hence, it is valuable to study the asymptotic behavior of a single wave equation with terms having opposite effects, see [6, 12, 13]. For more results about systems with various other damping and source terms, we refer the reader to [9, 14, 16].

The purpose of this paper is to analyse the influence of the damping terms, delay terms and source terms on the solutions to (1). Under suitable assumptions, we establish local existence, global existence and asymptotic behavior of solutions to (1). As far as we know, this type of problems has never been considered before in the literature.

This paper is structured as follows. In Section 2, we state some assumptions, the augmented problem (8), and lemmas for this analysis. Section 3 is devoted to the proof of the local and global existence results by using the semi-group approach. In Section 4, we state and prove the exponential decay rate result by using the multiplier method and appropriate Lyapunov functional.

2 Preliminaries and Tools

Here, we shall reformulate the initial problem (1) into the augmented system (8). To this end, we need the following results.

Lemma 2.1 *(see [2], p. 286) Let ϖ be a function defined for $\alpha \in (0,1)$ as*

$$\varpi(\nu) = |\nu|^{\frac{2\alpha-1}{2}}, \quad \nu \in \mathbb{R}.$$

Then the relationship between the "input" U and "output" O of the system

$$\begin{cases} \phi_t(x,\nu,t) + \left(\nu^2+\beta\right)\phi(x,\nu,t) - U(x,t)\varpi(\nu) = 0, \\ \phi(x,\nu,t=0) = 0, \\ O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\phi(x,\nu,t)\varpi(\nu)\,d\nu, \end{cases} \tag{3}$$

where $\nu \in \mathbb{R},\ t>0,\ \beta>0,$ is given by

$$O = I^{1-\alpha,\beta}U,$$

here,

$$I^{\alpha,\beta}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\ e^{-\beta(t-\tau)}w(\tau)\ d\tau.$$

Lemma 2.2 *([2], p. 286) If*

$$\lambda \in D_\beta = \mathbb{C}\setminus\,]-\infty,-\beta[\,,$$

then

$$\int_{-\infty}^{+\infty}\frac{\varpi^2(\nu)}{\lambda+\beta+\nu^2}\ d\nu = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\beta)^{\alpha-1}.$$

The constants a_i, α_i are supposed to satisfy

$$a_i\beta^{\alpha-1} < \alpha_i \quad \text{for } i=1,2. \tag{4}$$

As in ([1], p. 1063), we can introduce new variables

$$z_1(x,\rho,t) = \varphi_t(x,t-s\rho),\ \rho\in(0,1),\ t>0, \tag{5}$$

$$z_2(x,\rho,t) = \psi_t(x,t-s\rho),\ \rho\in(0,1),\ t>0. \tag{6}$$

Then

$$z_{it}(x,\rho,t) = \frac{-1}{s}z_{i\rho}(x,\rho,t),\ \rho\in(0,1),\ t>0, \tag{7}$$

with $i=1,2$. For $\nu\in\mathbb{R},\ \rho\in(0,1)$, we denote $z_{it} = \frac{\partial}{\partial t}(z_i)$ and $z_{i\rho} = \frac{\partial}{\partial\rho}(z_i)$, then by (7) and Lemma 2.1, the initial system (1) is equivalent to

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x+\psi)_x + b_1\phi_1 * \varpi + \alpha_1\varphi_t = |\varphi|^{p-2}\varphi, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi) + b_2\phi_2 * \varpi + \alpha_2\psi_t = |\psi|^{p-2}\psi, \\ \phi_{1t}(x,\nu,t) + \left(\nu^2+\beta\right)\phi_1(x,\nu,t) - z_1(x,1,t)\varpi(\nu) = 0, \\ sz_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, \\ \phi_{2t}(x,\nu,t) + \left(\nu^2+\beta\right)\phi_2(x,\nu,t) - z_2(x,1,t)\varpi(\nu) = 0, \\ sz_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, \\ \varphi(x=L,t) = \varphi(x=0,t) = \psi(x=L,t) = \psi(x=0,t) = 0, \\ z_1(x,\rho=0,t) = \varphi_t(x,t), z_2(x,\rho=0,t) = \psi_t(x,t), \\ \varphi(x,t=0) = \varphi_0,\ \varphi_t(x,t=0) = \varphi_1, \\ \psi(x,t=0) = \psi_0,\ \psi_t(x,t=0) = \psi_1, \\ z_1(x,\rho,0) = f_0(x,-s\rho), z_2(x,\rho,0) = g_0(x,-s\rho), \\ \phi_1(x,\nu,t=0) = \phi_2(x,\nu,t=0) = 0, \end{cases} \tag{8}$$

where $\nu \in \mathbb{R}$ and

$$\phi_i * \varpi = \int_{-\infty}^{+\infty} \phi_i(x,\nu,t)\,\varpi(\nu)\ d\nu,$$

and

$$b_i = (\pi)^{-1} \sin(\alpha\pi)\, a_i, \quad i = 1,2.$$

To prove the dissipativity of the energy $\mathcal{E}$, we need the following lemma.

Lemma 2.3 *(See [2], p. 286) For $z \in L^2(\Omega)$ and $\nu\phi \in L^2(\Omega \times (-\infty,+\infty))$, we have*

$$\begin{aligned}\left|\int_\Omega z(x,\rho,t)\int_{-\infty}^{+\infty} \varpi(\nu)\,\phi(x,\nu,t)\ d\nu\ dx\right| &\le A_0 \int_\Omega |z(x,\rho,t)|^2\ dx \\ &+ \frac{1}{4}\int_\Omega \int_{-\infty}^{+\infty} \left(\nu^2+\beta\right)|\phi(x,\nu,t)|^2\ d\nu\ \ dx,\end{aligned}$$

where

$$A_0 = \int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{\nu^2+\beta}\ d\nu.$$

The energy associated to (8) is defined by

$$\begin{aligned}&\mathcal{E}(t) = \tfrac{1}{2}\left[\rho_1\|\varphi_t\|^2 + k\|\varphi_x+\psi\|^2 + \rho_2\|\psi_t\|^2 + b\|\psi_x\|^2\right] \\ &+ \sum_{i=1}^{2} \tfrac{b_i}{2}\int_0^L \int_{-\infty}^{+\infty} |\phi_i(x,\nu,t)|^2\ d\nu\ dx + \sum_{i=1}^{2} v_i s \int_0^L \int_0^1 |z_i(x,\rho,t)|^2\ d\rho\ dx \\ &- \tfrac{1}{p}\|\varphi\|^p - \tfrac{1}{q}\|\psi\|^q,\end{aligned} \tag{9}$$

where v_i satisfies

$$A_0 b_i < v_i < \alpha_i - b_i A_0, \quad i = 1,2. \tag{10}$$

Lemma 2.4 *Let (4) hold. Then the energy (9) satisfies*

$$\begin{aligned}&\tfrac{d\mathcal{E}(t)}{dt} \le -C\sum_{i=1}^{2}\int_\Omega \left(|z_i(x,1,t)|^2 + |z_i(x,0,t)|^2\right)\ dx \\ &- \sum_{i=1}^{2} \tfrac{b_i}{2}\int_0^L \int_{-\infty}^{+\infty} \left(\nu^2+\beta\right)|\phi_i(x,\nu,t)|^2\ d\nu\ dx \le 0\end{aligned} \tag{11}$$

for $C > 0$ and $b_i = (\pi)^{-1}\sin(\alpha\pi)\, a_i, i = 1,2.$

Proof. By multiplying $(8)_1$ by φ_t and integrating over $(0,L)$, integrating by parts and using the boundary conditions, we find

$$\begin{aligned}&\tfrac{d}{dt}\left[\tfrac{\rho_1}{2}\|\varphi_t\|^2 - \tfrac{1}{p}\|\varphi\|^p\right] + k\int_0^L (\varphi_x+\psi)\,\varphi_{xt}\ dx + \alpha_1\|\varphi_t\|^2 \\ &+ b_1\int_0^L \left(\int_{-\infty}^{+\infty} \phi_1(x,\nu,t)\,\varpi(s)\ d\nu\right)\varphi_t\ dx = 0.\end{aligned} \tag{12}$$

Multiplying $(8)_2$ by ψ_t and integrating over $(0,L)$, we have

$$\begin{aligned}&\tfrac{d}{dt}\left[\tfrac{\rho_2}{2}\|\psi_t\|^2 + \tfrac{b}{2}\|\psi_x\|^2 - \tfrac{1}{q}\|\psi\|^q\right] + \alpha_2\|\psi_t\|^2 + k\int_0^L (\varphi_x+\psi)\,\psi_t\ dx \\ &+ b_2\int_0^L \left(\int_{-\infty}^{+\infty} \phi_2(x,\nu,t)\,\varpi(\nu)\ d\nu\right)\psi_t\ dx = 0.\end{aligned} \tag{13}$$

Multiplying $(8)_j$ by $b_i\phi_i$ with $(i,j)=(1,3)$, respectively $(i,j)=(2,5)$, and integrating over $(0,L)\times\mathbb{R}$, we obtain

$$\begin{array}{l} b_i \int_0^L \int_{-\infty}^{+\infty} \left(\frac{d}{2dt} \left|\phi_i\left(x,\nu,t\right)\right|^2 + \left(\nu^2+\beta\right)\left|\phi_i\left(x,\nu,t\right)\right|^2 \right) \, d\nu \, dx \\ -b_i \int_0^L z_i\left(x,1,t\right) \int_{-\infty}^{+\infty} \varpi\left(\nu\right)\phi_i\left(x,\nu,t\right) \, d\nu \, dx = 0. \end{array} \tag{14}$$

Multiplying $(8)_j$ by $2v_iz_i$ with $(i,j)=(1,4)$, respectively $(i,j)=(2,6)$, and integrating over $(0,L)\times(0,1)$, we have

$$\begin{array}{l} \frac{d}{dt}\left\{ s v_i \int_0^L \int_0^1 \left|z_i\left(x,\rho,t\right)\right|^2 \, d\rho \, dx \right\} \\ +v_i \int_0^L \left[\left|z_i\left(x,1,t\right)\right|^2 - \left|z_i\left(x,0,t\right)\right|^2 \right] \, dx = 0. \end{array} \tag{15}$$

Summing (12), (13), (14) and (15) and due to the fact that $\varphi_t\left(x,t\right)=z_1\left(x,0,t\right)$, $\psi_t\left(x,t\right)=z_2\left(x,0,t\right)$, we have

$$\begin{array}{l} \frac{d\mathcal{E}(t)}{dt} = -\sum\limits_{i=1}^{2}\left(\alpha_i - v_i\right)\int_0^L \left|z_i\left(x,0,t\right)\right|^2 \, dt \\ -\sum\limits_{i=1}^{2} b_i \int_0^L z_i\left(x,0,t\right) \int_{-\infty}^{+\infty} \phi_i\left(x,\nu,t\right)\varpi\left(\nu\right) \, d\nu \, dx \\ -\sum\limits_{i=1}^{2} b_i \int_0^L \int_{-\infty}^{+\infty} \left(\nu^2+\beta\right)\left|\phi_i\left(\nu\right)\right|^2 \, d\nu \, dx \\ +\sum\limits_{i=1}^{2} b_i \int_0^L z_i\left(x,1,t\right) \int_{-\infty}^{+\infty} \phi_i\left(x,\nu,t\right)\varpi\left(\nu\right) \, d\nu \, dx \\ -\sum\limits_{i=1}^{2} v_i \int_0^L \left|z_i\left(x,1,t\right)\right|^2 \, dx. \end{array}$$

Thanks to Lemma 2.2 and putting $C=\min_{i=1,2}\left(v_i - A_0 b_i, \alpha_i - v_i - b_i A_0\right) > 0,\ i=1,2,$ the estimate (11) is established.

3 Unique Local and Global Weak Solution

Set $u=\varphi_t$ and $v=\psi_t$ and denote $U=\left(\varphi,u,\psi,v,\phi_1,\phi_2,z_1,z_2\right)^T$, then (8) takes the abstract form

$$\left\{ \begin{array}{l} U_t(t) = AU(t) + \mathbb{F}\left(U(t)\right), \\ U_0 = \left(\varphi_0,\varphi_1,\psi_0,\psi_1,0,0,f_0\left(-\rho s\right),g_0\left(-\rho s\right)\right)^T, \quad \text{for } \rho\in(0,1), \end{array} \right. \tag{16}$$

where the operator A is defined by

$$AU = \left(u, \frac{k}{\rho_1}\left(\varphi_x+\psi\right)_x - \frac{b_1}{\rho_1}\phi_1\star\varpi - \frac{\alpha_1}{\rho_1}u, v, \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\left(\varphi_x+\psi\right) - \frac{b_2}{\rho_2}\phi_2\star\varpi - \frac{\alpha_2}{\rho_2}v, \right.$$

$$\left. -\left(\nu^2+\beta\right)\phi_1 + z_1\left(x,1\right)\varpi\left(\nu\right), -\left(\nu^2+\beta\right)\phi_2 + z_2\left(x,1\right)\varpi\left(\nu\right), -\frac{1}{s}z_{1\rho}\left(x,\rho\right), -\frac{1}{s}z_{2\rho}\left(x,\rho\right) \right)^T,$$

where

$$\phi_i\star\varpi = \int_{-\infty}^{+\infty} \phi_i\left(x,\nu\right)\varpi\left(\nu\right) \, d\nu, \quad i=1,2,$$

for $i=1,2$, the domain is given by

$$D(A)=\left\{\begin{array}{l} U\in\mathcal{H}\ :\ (\varphi,\psi)\in\left(H^2(\Omega)\right)^2,\ (u,v)\in\left(H_0^1(\Omega)\right)^2,\\ z_i\in L^2\left(\Omega\times H^1(0,1)\right)\quad\text{for } i=1,2,\quad u=z_1(.,0),v=z_2(.,0),\\ \nu\phi_i\in L^2(\Omega\times(-\infty,+\infty))\quad\text{for } i=1,2,\\ \left(\nu^2+\beta\right)\phi_i-z_i(x,1)\varpi(\nu)\in L^2(\Omega\times(-\infty,+\infty)), \end{array}\right\}$$

where $\mathcal{H}$ is given as

$$\mathcal{H}=\left(H_0^1(\Omega)\times L^2(\Omega)\right)^2\times\left(L^2(\Omega\times(-\infty,+\infty))\right)^2\times\left(L^2(\Omega\times(0,1))\right)^2$$

and equipped with the inner product

$$\begin{array}{l}\left\langle U,\bar{U}\right\rangle_{\mathcal{H}}=k\int_\Omega(\varphi_x+\psi)\left(\bar{\varphi}_x+\bar{\psi}\right)\,dx+b\int_\Omega\psi_x\bar{\psi}_x\,dx+\rho_1\int_\Omega u\bar{u}+\rho_2\int_\Omega v\bar{v}\,dx\\ +\sum\limits_{i=1}^{2}b_i\int_\Omega\int_{-\infty}^{+\infty}\phi_i(x,\nu)\bar{\phi}_i(x,\nu)\,d\nu\,dx+2\sum\limits_{i=1}^{2}v_is\int_\Omega\int_0^1 z_i(x,\rho)\bar{z}_i(x,\rho)\,d\rho\,dx,\end{array}$$

for all $\bar{U}=\left(\bar{\varphi},\bar{u},\bar{\psi},\bar{v},\bar{\phi}_1,\bar{\phi}_2,\bar{z}_1,\bar{z}_2\right)$.

Theorem 3.1 *(Unique local weak solution) Assume that $p>2$ and $q>2$. Let (10) hold. Then, for any $U_0\in\mathcal{H}$, the system (16) has a unique local weak solution*

$$U\in \mathrm{C}([0,T],\mathcal{H}).$$

Moreover, if $U_0\in D(A)$, then

$$U\in\mathrm{C}([0,T],D(A))\cap\mathrm{C}^1([0,T],\mathcal{H}).$$

Proof. It will be proved that A is a maximal dissipative operator. We have

$$\begin{array}{l}\frac{d\mathcal{E}(t)}{dt}=\frac{1}{2}\frac{d}{dt}\|U\|^2=\langle AU,U\rangle\le -C\sum\limits_{i=1}^{2}\int_\Omega|z_i(x,1,t)|^2\,dx-C\sum\limits_{i=1}^{2}\int_\Omega|z_i(x,0,t)|^2\,dx\\ -\sum\limits_{i=1}^{2}\frac{b_i}{2}\int_0^L\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)|\phi_i(x,\nu,t)|^2\,d\nu\,dx\le 0,\end{array}$$

therefore A is dissipative.

Now, it will be shown that $I-A$ is surjective. Indeed, let $F=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)^T\in\mathcal{H}$, and look for $U\in D(A)$ such that $(I-A)U=F$. This is equivalent to

$$\left\{\begin{array}{l}\varphi-u=f_1(x),\\ \left(1+\frac{\alpha_1}{\rho_1}\right)u-\frac{k}{\rho_1}(\varphi_x+\psi)_x+\frac{b_1}{\rho_1}\phi_1\star\varpi=f_2(x),\\ \psi-v=f_3(x),\\ \left(1+\frac{\alpha_2}{\rho_2}\right)v-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}(\varphi_x+\psi)+\frac{b_2}{\rho_2}\phi_2\star\varpi=f_4(x),\\ \left(1+\nu^2+\beta\right)\phi_1-z_1(x,1)\varpi(\nu)=f_5(x,\nu),\\ \left(1+\nu^2+\beta\right)\phi_2-z_2(x,1)\varpi(\nu)=f_6(x,\nu),\\ z_1+\ \frac{1}{s}z_{1\rho}=f_7(x,\rho)\quad\rho\in(0,1),\\ z_2+\frac{1}{s}z_{2\rho}=f_8(x,\rho)\quad\rho\in(0,1).\end{array}\right. \tag{17}$$

Suppose $(\varphi,\psi)\in\left(H_0^1(\Omega)\right)^2$, then by $(17)_1$ and $(17)_3$, we obtain

$$u=\varphi-f_1\in H_0^1(\Omega), \tag{18}$$

$$v = \psi - f_3 \in H_0^1(\Omega), \tag{19}$$

and from $(17)_{7,8}$, we get

$$z_1(x,\rho) = e^{-s\rho} z_1(x,0) + s e^{-s\rho} \int_0^\rho e^{\tau s} f_7(x,\tau)\ d\tau, \tag{20}$$

$$z_2(x,\rho) = e^{-s\rho} z_2(x,0) + s e^{-s\rho} \int_0^\rho e^{\tau s} f_8(x,\tau)\ d\tau. \tag{21}$$

Using (18) and (19), we have

$$u(x) = z_1(x,0) = \varphi - f_1(x), \tag{22}$$

$$v(x) = z_2(x,0) = \psi - f_3(x). \tag{23}$$

Substituting (22) and (23) respectively in (20) and (21), we get, for all $x \in (\Omega)$, $\rho \in (0,1)$,

$$\begin{aligned} z_1(x,\rho) &= e^{-s\rho}\left[\varphi - f_1(x)\right] + s e^{-s\rho} \textstyle\int_0^\rho e^{-s\tau} f_7(x,\tau)\ d\tau\ \in L^2(\Omega \times (0,1)), \\ z_2(x,\rho) &= e^{-s\rho}\left[\psi - f_3(x)\right] + s e^{-s\rho} \textstyle\int_0^\rho e^{s\tau} f_8(x,\tau)\ d\tau \in L^2(\Omega \times (0,1)). \end{aligned} \tag{24}$$

Returning back to $(17)_{7,8}$, we find that

$$z_{1\rho} = s f_7(x,\rho) - s z_1 \in L^2(\Omega \times (0,1)), z_{2\rho} = s f_8(x,\rho) - s z_2 \in L^2(\Omega \times (0,1)).$$

Using $(17)_5$ and $(17)_6$, we obtain

$$\phi_1 = \frac{f_5 + z_1(x,1)\varpi(\nu)}{1+\nu^2+\beta} \in L^2(\Omega \times (-\infty,+\infty)), \tag{25}$$

$$\phi_2 = \frac{f_6 + z_2(x,1)\varpi(\nu)}{1+\nu^2+\beta} \in L^2(\Omega, \times (-\infty,+\infty)). \tag{26}$$

Therefore

$$\begin{aligned} \nu\phi_1 &= \tfrac{\nu}{1+\nu^2+\beta}\left[f_5 + z_1(x,1)\varpi(\nu)\right] \in L^2((0,L) \times (-\infty,+\infty)), \\ \nu\phi_2 &= \tfrac{\nu}{1+\nu^2+\beta}\left[f_6 + z_2(x,1)\varpi(\nu)\right] \in L^2(\Omega \times (-\infty,+\infty)). \end{aligned}$$

Inserting $(17)_1$ and (25) in $(17)_2$, respectively $(17)_3$, and (26) in $(17)_4$, we have

$$\begin{cases} \left(1+\frac{\alpha_1}{\rho_1}\right)\varphi - \frac{k}{\rho_1}(\varphi_x+\psi)_x = f_2 - \frac{b_1}{\rho_1}\left[\frac{f_5+z_1(x,1)\varpi(\nu)}{1+\nu^2+\beta}\right] \star \varpi + \left(1+\frac{\alpha_1}{\rho_1}\right) f_1, \\ \left(1+\frac{\alpha_2}{\rho_2}\right)\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) = f_4 - \frac{b_2}{\rho_2}\left[\frac{f_6+z_2(x,1)\varpi(\nu)}{1+\nu^2+\beta}\right] \star \varpi \\ + \left(1+\frac{\alpha_2}{\rho_2}\right) f_3. \end{cases} \tag{27}$$

By replacing (20) and (21) for $\rho = 1$ in (27), we get

$$\begin{cases} \left(1+\frac{b_{11}e^{-s}}{\rho_1}+\frac{\alpha_1}{\rho_1}\right)\varphi - \frac{k}{\rho_1}(\varphi_x+\psi)_x = f_2 + \left(1+\frac{\alpha_1}{\rho_1}\right) f_1 \\ -\frac{b_1}{\rho_1}\int_{-\infty}^{+\infty} \frac{\varpi(\nu) f_5(\nu)}{1+\nu^2+\beta}\ d\nu, + \frac{b_1}{\rho_1} f_{1,7} \int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{1+\nu^2+\beta} d\nu, \\ \left(1+\frac{b_{22}}{\rho_2}e^{-s}+\frac{\alpha_2}{\rho_2}\right)\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) = f_4 + \left(1+\frac{\alpha_2}{\rho_2}\right) f_3 \\ -\frac{b_2}{\rho_2}\int_{-\infty}^{+\infty} \frac{\varpi(\nu) f_6(\nu)}{1+\nu^2+\beta}\ d\nu + \frac{b_2}{\rho_2} f_{3,8} \int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{1+\nu^2+\beta} d\nu, \end{cases} \tag{28}$$

where, for i=1, 2,

$$b_{ii} = b_i \int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{1+\nu^2+\beta} d\nu, \quad f_{1,7} = f_1 - se^{-s} \int_0^1 e^{\tau s} f_7(x,\tau)\ d\tau,$$

and

$$f_{3,8} = f_3 - se^{-s} \int_0^1 e^{\tau s} f_8(x,\tau)\ d\tau.$$

Let $(\bar{\varphi}, \bar{\psi}) \in (H_0^1((0,L)))^2$. Multiply $(28)_1$ by $\rho_1 \bar{\varphi}$ and $(28)_2$ by $\rho_2 \bar{\psi}$. Integrating by parts, and summing the obtained result, we get

$$M(\varphi, \psi; \bar{\varphi}, \bar{\psi}) = L(\bar{\varphi}, \bar{\psi}), \tag{29}$$

here, the bilinear form

$$M : (H_0^1((0,L)) \times H_0^1((0,L)))^2 \to \mathbb{R}$$

is defined by

$$\begin{array}{l} M(\varphi, \psi; \bar{\varphi}, \bar{\psi}) = (\rho_1 + b_{11} e^{-s} + \alpha_1) \int_0^L \varphi \bar{\varphi}\ dx + (\rho_2 + b_{22} e^{-s} + \alpha_2) \int_0^L \psi \bar{\psi}\ dx \\ +k \int_0^L (\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi})\ dx + b \int_0^L \psi \bar{\psi}_x\ dx, \end{array}$$

and the linear form

$$L : (H_0^1((0,L)))^2 \to \mathbb{R}$$

by

$$\begin{array}{l} L(\bar{\varphi}, \bar{\psi}) = \rho_1 \int_0^L f_2 \bar{\varphi}\ dx + (\rho_1 + \alpha_1 \rho_1) \int_0^L f_1 \bar{\varphi} dx - b_1 \int_0^L \left\{ \int_{-\infty}^{+\infty} \frac{\varpi(\nu) f_5(x,\nu)}{1+\nu^2+\beta}\ d\nu \right\} \bar{\varphi} dx \\ -b_2 \int_0^L \left\{ \int_{-\infty}^{+\infty} \frac{\varpi(\nu) f_6(x,\nu)}{1+\nu^2+\beta}\ d\nu \right\} \bar{\psi}\ dx + \rho_2 \int_0^L f_4 \bar{\psi}\ dx + (\rho_2 + \alpha_2) \int_0^L f_3 \bar{\psi}\ dx \\ +b_1 \left(\int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{1+\nu^2+\beta} d\nu \right) \int_0^L \left\{ f_1 - se^{-s} \int_0^1 e^{\tau s} f_7(x,\tau)\ d\tau \right\} \bar{\varphi} dx \\ +b_2 \left(\int_{-\infty}^{+\infty} \frac{\varpi^2(\nu)}{1+\nu^2+\beta} d\nu \right) \int_0^L \left\{ f_3 - se^{-s} \int_0^1 e^{\tau s} f_8(x,\tau)\ d\tau \right\} \bar{\psi} dx. \end{array}$$

It is not hard to see the bilinear operator M is coercive and continuous and L is continuous. Then, applying the Lax-Milgram Theorem to find $\forall (\bar{\varphi}, \bar{\psi}) \in (H_0^1((0,L)))^2$, we see that the system (29) has a unique weak solution $(\varphi, \psi) \in (H_0^1((0,L)))^2$. Owing to the classical elliptic regularity, we find by (29) that

$$(\varphi, \psi) \in (H^2((0,L)))^2.$$

It remains only to prove

$$\nu^2 + \beta \phi_i - z_i(x,1) \varpi(\nu) \in L^2((0,L) \times (-\infty, +\infty)), i = 1, 2.$$

Indeed, we have from $(17)_5$ and $(17)_6$,

$$\begin{array}{l} (\nu^2 + \beta) \phi_1 - z_1(x,1) \varpi(\nu) = f_5 - \phi_1 \in L^2((0,L) \times (-\infty, +\infty)), \\ (\nu^2 + \beta) \phi_2 - z_2(x,1) \varpi(\nu) = f_6 - \phi_2 \in L^2((0,L) \times (-\infty, +\infty)). \end{array}$$

Therefore, $U \in D(A)$. Thus, the operator $I - A$ is surjective. Now, we prove that

$$F : \mathcal{H} \to \mathcal{H}$$

is locally Lipschitz. For $U, \bar{U} \in \mathcal{H}$, we have

$$\|\mathbb{F}(U) - \mathbb{F}(\bar{U})\|_{\mathcal{H}}^2 \quad \leq C\left[\|\varphi - \bar{\varphi}\|_{H_0^1(\Omega)}^2 + \|\psi - \bar{\psi}\|_{H_0^1(\Omega)}^2\right]. \tag{30}$$

Thus, $\mathbb{F}$ is locally Lipschitz. This completes the proof.

We show the global existence result. First, we introduce the following useful functionals:

$$\begin{aligned} I_1(t) = b_1 \int_0^L \int_{-\infty}^{+\infty} |\phi_1(x,\nu,t)|^2 \ d\nu \ dx + k\|\varphi_x + \psi\|^2 \\ +\tfrac{b}{2}\|\psi_x\|^2 - \|\varphi\|_p^p + sv_1 \int_\Omega \int_0^1 |z_1(x,\rho,t)|^2 \ d\rho \ , \end{aligned} \tag{31}$$

$$\begin{aligned} I_2(t) = b_2 \int_\Omega \int_{-\infty}^{+\infty} |\phi_2(x,\nu,t)|^2 \ d\nu \ dx + \tfrac{b}{2}\|\psi_x\|^2 \\ -\|\psi\|_q^q + sv_2 \int_\Omega \int_0^1 |z_2(x,\rho,t)|^2 \ d\rho \ dx, \end{aligned} \tag{32}$$

$$\begin{aligned} J_1(t) = \tfrac{b_1}{2} \int_\Omega \int_{-\infty}^{+\infty} |\phi_1(x,\nu,t)\, d\nu \ dx|^2 + \tfrac{k}{2}\|\varphi_x + \psi\|^2 + \tfrac{b}{4}\|\psi_x\|^2 \\ -\tfrac{1}{p}\|\varphi\|_p^p + sv_1 \int_\Omega \int_0^1 |z_1(x,\rho,t)|^2 \ d\rho \ dx, \end{aligned} \tag{33}$$

and

$$\begin{aligned} J_2(t) = \tfrac{b_2}{2} \int_\Omega \int_{-\infty}^{+\infty} |\phi_2(x,\nu,t)|^2 \ d\nu \ dx + \tfrac{b}{4}\|\psi_x\|^2 \\ - \tfrac{1}{q}\|\psi\|_q^q + sv_2 \int_\Omega \int_0^1 |z_2(x,\rho,t)|^2 \ d\rho \ dx. \end{aligned} \tag{34}$$

We easily see that

$$\mathcal{E}(t) = \frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\psi_t\|^2 + J_1(t) + J_2(t). \tag{35}$$

Lemma 3.1 *Suppose that conditions (4), $p > 2$ and $q > 2$ hold. Then, for $U_0 \in \mathcal{H}$ satisfying*

$$\begin{cases} \widetilde{B} = \max\left(C_{\star\star}^p \left(\frac{2p}{p-2}\mathcal{E}(0)\right)^{\frac{p-2}{2}},\ C_{\star\star}^q \left(\frac{2q}{q-2}\mathcal{E}(0)\right)^{\frac{q-2}{2}}\right) < 1, \\ I_i(0) > 0 \quad \textit{for } i = 1, 2, \end{cases} \tag{36}$$

we have for all $t > 0$,

$$I_i(t) > 0, \quad \textit{for } i = 1, 2.$$

Proof. As $I_i(0) > 0$ for $i = 1, 2$, by continuity of φ and ψ, there exists $T^\star < T$ such that

$$I_i(t) \geq 0 \quad \text{for all } t \in [0, t^\star], \quad i = 1, 2, \tag{37}$$

and with a straight forward calculation, we can find

$$\begin{aligned} \tfrac{2p}{p-2} J_1(t) = k\|\varphi_x + \psi\|^2 + b_1 \int_\Omega \int_{-\infty}^{+\infty} |\phi_1(x,\nu,t)|^2 \ d\nu dx + \tfrac{b}{2}\|\psi_x\|^2 \\ +\tfrac{2(p-1)}{p-2} sv_1 \int_\Omega \int_0^1 |z_1(x,\rho,t)|^2 \ d\rho \ dx + \tfrac{2}{p-2} I_1(t) \geq k\|\varphi_x + \psi\|^2 + \tfrac{b}{2}\|\psi_x\|^2, \end{aligned} \tag{38}$$

$$\begin{aligned} \tfrac{2q}{q-2} J_2(t) = b_2 \int_\Omega \int_{-\infty}^{+\infty} |\phi_2(x,\nu,t)|^2 \ d\nu \ dx + \tfrac{b}{2}\|\psi_x\|^2 \\ + \tfrac{2(q-1)}{q-2} sv_2 \int_\Omega \int_0^1 |z_2(x,\nu,t)|^2 \ d\nu \ dx + \tfrac{2}{q-2} I_2(t) \geq \tfrac{b}{2}\|\psi_x\|^2. \end{aligned} \tag{39}$$

Exploiting (35), (38), (39) and Lemma 2.4, we find

$$\frac{b}{2}\|\psi_x\|^2 + k\|\varphi_x + \psi\|^2 \leq \frac{2p}{p-2}\mathcal{E}(t) \leq \frac{2p}{p-2}\mathcal{E}(0) \quad \text{for all } t \in [0, t^\star], \tag{40}$$

and

$$\frac{b}{2}\|\psi_x\|^2 \leq \frac{2q}{q-2}\mathcal{E}(t) \leq \frac{2q}{q-2}\mathcal{E}(0) \quad \text{for all } t \in [0, t^\star]. \tag{41}$$

Applying Sobolev-Poincaré's inequality and taking into consideration (36), (40) and (41), we get

$$\begin{aligned} \|\varphi\|_p^p &\leq C_\star^p \|\varphi_x\|^p \leq C_{1\star}^p \left[\sqrt{k}\|\varphi_x+\psi\| + \sqrt{\tfrac{b}{2}}\|\psi_x\|\right]^p \\ &\leq C_{\star\star}^p \left(\tfrac{2p}{p-2}\mathcal{E}(0)\right)^{\frac{q-2}{2}} \left[k\|\varphi_x+\psi\|^2 + \tfrac{b}{2}\|\psi_x\|^2\right] \leq k\|\varphi_x+\psi\|^2 + \tfrac{b}{2}\|\psi_x\|^2, \end{aligned} \tag{42}$$

and

$$\|\psi\|_q^q \leq C_\star^q \|\psi_x\|_2^q = C_{\star\star}^q \left[\tfrac{2q}{q-2}\mathcal{E}(0)\right]^{\frac{p-2}{2}} \tfrac{b}{2}\|\psi_x\|_2^2 \leq \tfrac{b}{2}\|\psi_x\|_2^2. \tag{43}$$

This implies that

$$I_i(t) > 0 \quad \text{for } i = 1, 2 \;\forall\; t \in [0, t^\star].$$

By repeating this procedure and using the fact that

$$\lim_{t\to T^\star} \max\left(C_{\star\star}^p \left(\frac{2p}{p-2}\mathcal{E}(0)\right)^{\frac{p-2}{2}}, \; C_{\star\star}^q \left(\frac{2q}{q-2}\mathcal{E}(0)\right)^{\frac{q-2}{2}} \right) < 1,$$

we can take $T^\star = T$.

Theorem 3.2 *(Global existence) Assume that condition (10), $p > 2$ and $q > 2$ are satisfied. Then, for $U_0 \in D(A)$ satisfying (36), the solution of system* (8) *is global in time.*

Proof. It suffices to show that $\|\varphi_x + \psi\|^2 + \|\psi_x\|^2 + \|\psi_t\|^2 + \|\varphi_t\|^2$ is bounded independently of t.
Indeed, by (35), (38), (39), we get

$$\begin{aligned} &\mathcal{E}(0) \geq \mathcal{E}(t) = \tfrac{1}{2}\left[\|\varphi_t\|^2 + \|\psi_t\|^2\right] + J_1(t) + J_2(t) \\ &\geq \min\left(\tfrac{1}{2}, \tfrac{(p-2)}{2p}k, \tfrac{q-2}{2q}\tfrac{b}{2}\right)\left[\|\varphi_t\|^2 + \|\psi_t\|^2 + \|\varphi_x+\psi\|^2 + \|\psi_x\|^2\right], \end{aligned}$$

which implies that

$$\|\varphi_t\|^2 + \|\psi_t\|^2 + \|\varphi_x+\psi\|^2 + \|\psi\|^2 \leq CE(0),$$

where C is a constant depending only on p, q, k and b.

4 Decay Rate Result

Our next step is devoted to the proof of the decay result to the problem (8). For this purpose, we prepare some Lemmas and present some appropriate functionals. Firstly, we define

$$k_1(t) = \sum_{i=1}^{2} \int_\Omega \rho_i \varphi_t^i \varphi^i \, dx + \sum_{i=1}^{2} \frac{bi}{2} \int_\Omega \int_{-\infty}^{+\infty} \left(\nu^2 + \beta\right) \left|M_i(x,\nu,t)\right|^2 \, d\nu \, dx, \tag{44}$$

and

$$k_2(t) = s\sum_{i=1}^{2}\int_\Omega\int_0^1 e^{-s\rho}\left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx, \tag{45}$$

where

$$M_i\left(x,\nu,t\right) = \int_0^t \phi_i\left(x,\nu,z\right)\ dz - \frac{s\varpi\left(\nu\right)}{\nu^2+\beta}\int_0^1 f_0^i\left(x,-\rho s\right)\ d\rho + \frac{\varphi_0^i\varpi\left(\nu\right)}{\nu^2+\beta} \tag{46}$$

with

$$\left(f_0^i\left(x,-\rho s\right),\varphi_0^i\left(x\right),\varphi^i\left(x,t\right)\right) = \begin{cases}\left(f_0\left(x,-\rho s\right),\varphi_0\left(x\right),\varphi\left(x,t\right)\right) & i=1,\\ \left(g_0\left(x,-\rho s\right),\psi_0\left(x\right),\psi\left(x,t\right)\right) & i=2.\end{cases}$$

Lemma 4.1 *[1] Let $(\varphi,\phi_1,z_1,\psi,\phi_2,z_2)$ be a regular solution of problem (8), then we have*

$$\left(\nu^2+\beta\right)M_i\left(x,\nu,t\right) = -s\varpi\left(\nu\right)\int_0^1 z_i\left(x,\rho,t\right)\ d\rho + \varphi^i\left(x,t\right)\varpi\left(\nu\right) - \phi_i\left(x,\nu,t\right),$$

and

$$\begin{aligned}&\int_\Omega\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)\phi_i\left(x,\nu,t\right)M_i\left(x,\nu,t\right)\ d\nu\ dx = \int_\Omega\varphi^i\left(x,t\right)\int_{-\infty}^{+\infty}\phi_i\left(x,\nu,t\right)\varpi\left(\nu\right)\ d\nu\ dx\\ &-s\int_\Omega\int_0^1 z_i\left(x,\rho,t\right)\int_{-\infty}^{+\infty}\varpi\left(\nu\right)\phi_i\left(x,\nu,t\right)d\nu\ d\rho\ dx - \int_\Omega\int_{-\infty}^{+\infty}\left|\phi_i\left(x,\nu,t\right)\right|^2 d\nu\ dx, i=1,2.\end{aligned}$$

Lemma 4.2 *[1] Let $(\varphi,\phi_1,z_1,\psi,\phi_2,z_2)$ be a regular solution of the problem (8), then we have*

$$\begin{aligned}\left|\int_\Omega\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)\left|M_i\left(x,\nu,t\right)\right|^2\ d\nu\ dx\right| &\le 3s^2A_0\int_\Omega\int_0^1\left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx\\ &+3A_0C_\star^2\|\varphi_x^i\|_2^2 + \tfrac{3}{\beta}\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_i\left(x,\nu,t\right)\right|^2\ d\nu\ dx,\ i=1,2.\end{aligned}$$

Lemma 4.3 *Assume (4) with $p>2$ and $q>2$ hold. The functional k_1 defined in (44) satisfies*

$$\begin{aligned}&k_1'(t) \le -C_1\|\varphi_x+\psi\|^2 - C_2\|\psi_x\|^2 + C\|\varphi_t\|^2 + C\|\psi_t\|^2\\ &-\sum_{i=1}^{2}b_i\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_i\left(x,\nu,t\right)\right|^2\ d\nu\ dx + \|\varphi\|_p^p + \|\psi\|_q^q\\ &+s^2\sum_{i=1}^{2}v_i\int_\Omega\int_0^1\left|z_i\left(x,\rho,t\right)\right|^2\ d\rho\ dx + \sum_{i=1}^{2}\tfrac{b_i}{4}\int_\Omega\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)\left|\phi_i\left(x,\nu,t\right)\right|^2\ d\nu\ dx,\end{aligned} \tag{47}$$

where C_1, C_2, C are positive constants.

Proof. Differentiating k_1 with respect to t, using $(8)_1$ and $(8)_2$, by integration by parts and using Lemma 4.1, we obtain

$$\begin{aligned}&k_1'(t) = -k\|\varphi_x+\psi\|^2 - b\|\psi_x\|^2 + \rho_1\|\varphi_t\|^2 + \rho_2\|\psi_t\|^2\\ &-\sum_{i=1}^{2}b_i\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_i\left(x,\nu,t\right)\right|^2\ d\nu\ dx - \alpha_1\int_\Omega\varphi\varphi_t\ dx - \alpha_2\int_\Omega\psi\psi_t\ dx\\ &-s\sum_{i=1}^{2}b_i\int_\Omega\int_0^1 z_i\int_{-\infty}^{+\infty}\varpi\phi_i\left(x,\nu,t\right)\ d\nu\ d\rho\ dx + \|\varphi\|_p^p + \|\psi\|_q^q.\end{aligned} \tag{48}$$

Now, we will estimate the last three terms of the RHS as follows. Using Lemma 2.3 and due to the fact that $b_iA_0 < v_i,\ i = 1, 2$, and then integrating over $(0, 1)$ with respect to ρ, we can write

$$\begin{aligned} &-\sum_{i=1}^{2} b_i \int_\Omega s z_i \int_{-\infty}^{+\infty} \varpi\phi(x,\nu,t)\ d\nu\ dx \leq s^2 \sum_{i=1}^{2} v_i \int_\Omega \int_0^1 |z_i(x,\rho,t)|^2\ d\rho\ dx \\ &+\sum_{i=1}^{2} \frac{b_i}{4} \int_\Omega \int_{-\infty}^{+\infty} \left(\nu^2+\beta\right) |\phi_i(x,\nu,t)|^2\ d\nu\ dx. \end{aligned} \tag{49}$$

By Young and Poincaré's inequality, we have

$$\begin{aligned} -\alpha_1 \int_\Omega \varphi\varphi_t\ dx + \alpha_2 \int_\Omega \psi\psi_t\ dx \quad &\leq \tfrac{\alpha_1}{4\delta}\|\varphi_t\|^2 + \tfrac{\alpha_2}{4\delta}\|\psi_t\|^2 + C\delta\alpha_1\|\varphi_x+\psi\|^2 \\ &+ C\delta\left(\alpha_2+\alpha_1 C\right)\|\psi_x\|^2. \end{aligned} \tag{50}$$

Inserting (49) and (50) in (48), we arrive at

$$\begin{aligned} &k_1'(t) \leq -(k-C\delta\alpha_1)\|\varphi_x+\psi\|^2 - (b-C\delta(\alpha_2+\alpha_1 C))\|\psi_x\|^2 + \left(\tfrac{\alpha_1}{4\delta}+\rho_1\right)\|\varphi_t\|^2 \\ &+\left(\tfrac{\alpha_2}{4\delta}+\rho_2\right)\|\psi_t\|^2 - \sum_{i=1}^{2} b_i \int_\Omega \int_{-\infty}^{+\infty} |\phi_i(x,\nu,t)|^2\ d\nu\ dx + s^2 \sum_{i=1}^{2} v_i \int_\Omega \int_0^1 |z_i(x,\nu,t)|^2\ d\rho\ dx \\ &+\sum_{i=1}^{2} \tfrac{b_i}{4} \int_\Omega \int_{\infty}^{+\infty} \left(\nu^2+\beta\right)|\phi_i(x,\nu,t)|^2\ d\nu\ dx + \|\varphi\|_p^p + \|\psi\|_q^q, \end{aligned}$$

we choose $\delta = \min\left(\frac{b}{2C(\alpha_2+\alpha_1 C)}, \frac{k}{2C\alpha_1}\right)$, then setting $C_1 = k - C\delta\alpha_1$ and $C_2 = b - C\delta(\alpha_2 + \alpha_1 C)$, we get (47).

Lemma 4.4 *With the same hypotheses as in Lemma 4.3, the functional k_2 defined in (45) satisfies*

$$k_2'(t) \leq -se^{-s} \sum_{i=1}^{2} \int_\Omega \int_0^1 |z_i(x,\rho,t)|^2\ d\rho\ dx + \|\varphi_t\|^2 + \|\psi_t\|^2. \tag{51}$$

Proof. We take the derivative of k_2 with respect to t, and using $(8)_4$ and $(8)_6$, we get

$$k_2'(t) = \sum_{i=1}^{2} \int_\Omega |z_i(x,0,t)|^2 - \sum_{i=1}^{2} \int_\Omega e^{-s} |z_i(x,1,t)|^2\, d\rho\, dx - s\sum_{i=1}^{2} \int_\Omega \int_0^1 e^{-s\rho} |z_i(x,\rho,t)|^2\, d\rho\, dx.$$

We have $z_i(x,0,t) = \varphi_t^i(x,t)$, and since $e^{-s\rho} \geq e^{-s}$, we obtain (51).

Now, we introduce the perturbed modified energy, named Lyapunov function, as

$$\mathcal{L}(t) = N\mathcal{E}(t) + \varepsilon k_1(t) + k_2(t)$$

for $\varepsilon > 0$ and $N > 0$.

Lemma 4.5 *For ε_1 small and N large enough, we have*

$$\frac{N}{2}\mathcal{E}(t) \leq \mathcal{L}(t) \leq 2N\mathcal{E}(t), \quad \forall\ t \geq 0. \tag{52}$$

Proof. The application of Young and Poincaré's inequalities gives

$$\mathcal{L}(t) \le N\mathcal{E}(t) + \frac{\varepsilon}{2}\left[\rho_1\|\varphi_t\|^2 + \rho_1 C_\star^2\|\varphi_x+\psi\|^2\right] + \frac{\varepsilon}{2}\left[\rho_2\|\psi_t\|^2 + C_{\star\star}^2\left\{\rho_2 + C_\star^2\rho_1\right\}\|\psi_x\|^2\right]$$
$$+\sum_{i=1}^{2}\frac{b_i}{2}\varepsilon\int_\Omega\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)\left|M_i(x,\nu,t)\right|^2\,d\nu\,dx + s\sum_{i=1}^{2}\int_\Omega\int_0^1 e^{-s\rho}\left|z_i(x,\rho,t)\right|^2\,d\rho\,dx.$$

Using $\mathcal{E}(t)$, I_1, I_2, Lemma 4.2 and the fact that $b_iA_0 < v_i$ for $i=1,2$, we get

$$2N\mathcal{E}(t) - \mathcal{L}(t) \ge \frac{\rho_1}{2}\left[N-\varepsilon\right]\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\left[N-\varepsilon\right]\|\psi_t\|_2^2$$
$$+\frac{N}{p}I_1 + \frac{N}{q}I_2 + \frac{1}{2}\left[\frac{Nk(p-2)}{p} - \varepsilon C_\star^2\left[3v_1+\rho_1\right]\right]\|\varphi_x+\psi\|^2$$
$$+\frac{1}{2}\left[\frac{bN(pq-q-p)}{pq} - \varepsilon C_{\star\star}^2\left\{C_\star^2\left(3v_1+\rho_1\right)+3v_2+\rho_2\right\}\right]\|\psi_x\|^2$$
$$+s\int_\Omega\int_0^1\left(\left[\frac{Nv_1(p-1)}{p}-1-\frac{3}{2}s\varepsilon v_1\right]\left|z_1(x,\rho,t)\right|^2+\left[\frac{Nv_1(q-1)}{q}-1-\frac{3}{2}s\varepsilon v_2\right]\left|z_2(x,\rho,t)\right|^2\right)d\rho\,dx$$
$$+\frac{b_1}{2}\left[\frac{N(p-2)}{p}-\frac{3\varepsilon}{\beta}\right]\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_1(x,\nu,t)\right|^2d\nu\,dx+\frac{b_2}{2}\left[\frac{N(q-2)}{q}-\frac{3\varepsilon}{\beta}\right]\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_2(x,\nu,t)\right|^2d\nu\,dx.$$

On the other hand, we can estimate the following:

$$\mathcal{L}(t) - \frac{N}{2}\mathcal{E}(t) \ge \frac{N}{2}\mathcal{E}(t) - \frac{\varepsilon}{2}\left[\rho_1\|\varphi_t\|^2 + C_\star^2\rho_1\|\varphi_x+\psi\|^2\right]$$
$$+\frac{\varepsilon}{2}\left[\rho_2\|\psi_t\|^2 + C_{\star\star}^2\left\{C_\star^2\rho_1+\rho_2\right\}\|\psi_x\|^2\right] + s\sum_{i=1}^{2}e^{-s}\int_\Omega\int_0^1 z_i(x,\rho,t)\,d\rho\,dx$$
$$+\sum_{i=1}^{2}\frac{b_i\varepsilon}{2}\int_\Omega\int_{-\infty}^{+\infty}\left(\nu^2+\beta\right)\left|M_i(x,\nu,t)\right|^2\,d\nu\,dx.$$

Using Lemma 4.2 and the fact that $b_iA_0 < v_i,\ i=1,2$, we obtain

$$\mathcal{L}(t) - \frac{N}{2}\mathcal{E}(t) \ge \frac{\rho_1}{2}\left[\frac{N}{2}-\varepsilon\right]\|\varphi_t\|^2 + \frac{\rho_2}{2}\left[\frac{N}{2}-\varepsilon\right]\|\psi_t\|^2 + \frac{N}{2}pI_1 + \frac{N}{2}qI_2$$
$$+\frac{1}{2}\left[\frac{kN(p-2)}{2p} - C_\star^2\varepsilon\left(\rho_1+3v_1\right)\right]\|\varphi_x+\psi\|^2$$
$$+\frac{1}{2}\left[\frac{Nb(qp-p-q)}{2pq} - \varepsilon C_{\star\star}^2\left\{\rho_2+3v_2+C_\star^2\left(\rho_1+3v_1\right)\right\}\right]\|\psi_x\|^2$$
$$+\frac{b_1}{2}\left(\frac{N(p-2)}{2p}-\frac{3\varepsilon}{\beta}\right)\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_1(x,\nu,t)\right|^2\,d\nu\,dx$$
$$+\frac{b_2}{2}\left(\frac{N(q-2)}{2q}-\frac{3\varepsilon}{\beta}\right)\int_\Omega\int_{-\infty}^{+\infty}\left|\phi_2(x,\nu,t)\right|^2\,d\nu\,dx$$
$$+s\left[\frac{v_1N(p-1)}{2p}+e^{-s}-\frac{3}{2}\varepsilon v_1 s\right]\int_\Omega\int_0^1\left|z_1(x,\rho,t)\right|^2\,d\rho\,dx$$
$$+s\left[\frac{v_2N(q-1)}{2q}+e^{-s}-\frac{3}{2}sv_2\varepsilon\right]\int_\Omega\int_0^1\left|z_2(x,\rho,t)\right|^2\,d\rho\,dx.$$

Finally, if we pick ε small and N large enough, we deduce that

$$\mathcal{L}(t) - \frac{N}{2}\mathcal{E}(t) \ge 0 \quad\text{and}\quad 2N\mathcal{E}(t) - \mathcal{L}(t) \ge 0.$$

Hence, we conclude that

$$\mathcal{E}(t) \sim \mathcal{L}(t) \quad \forall t > 0.$$

Theorem 4.1 *(Exponential decay rate) Let $p > 2$ and $q > 2$. Assume that (4) holds for $i = 1, 2$, and $U_0 \in \mathcal{H}$ satisfying (36), then the unique solution of* (8) *satisfies*

$$\mathcal{E}(t) \le ke^{-mt} \quad \forall t \ge 0,$$

for some positive constants k and m independent of t.

Proof. We remember that

$$\mathcal{L}(t) = N\mathcal{E}(t) + \varepsilon k_1(t) + k_2(t).$$

By means of Lemma 4.3 and Lemma 4.4, we get for all $t \geq 0$,

$$\begin{aligned}
&\mathcal{L}'(t) \leq -(NC - \varepsilon C - 1)\|\varphi_t\|^2 - (NC - \varepsilon C - 1)\|\psi_t\|^2 \\
&- \sum_{i=1}^{2} \frac{b_i}{2}\left[N - \frac{\varepsilon}{2}\right] \int_0^L \int_{-\infty}^{\infty} \left(\nu^2 + \beta\right) |\phi_i(x,\nu,t)|^2 \, d\nu \, dx - C_1\varepsilon\|\varphi_x + \psi\|^2 - C_2\varepsilon\|\psi_x\|^2 \\
&- \sum_{i=1}^{2} \varepsilon b_i \int_\Omega \int_{-\infty}^{\infty} |\phi_i(x,\nu,t)|^2 \, d\nu \, dx - \sum_{i=1}^{2} s\left(e^{-s} - v_i s\varepsilon\right) \int_\Omega \int_0^1 |z_i(x,\rho,t)|^2 \, d\rho \, dx \\
&+\varepsilon\left[\|\varphi\|_p^p + \|\psi\|_q^q\right].
\end{aligned}$$

We now choose ε small enough such that $e^{-s} - v_i s\varepsilon > 0, i = 1, 2$. Pick N large enough such that $N > max\left(\frac{C\varepsilon+1}{C}, \frac{\varepsilon}{2}\right)$. Thus, $\exists m_1 > 0$ so that

$$\mathcal{L}'(t) \leq -m_1\mathcal{E}(t) \quad \forall t \geq 0.$$

By Lemma 4.5, it follows that $\mathcal{E}(t)$ and $\mathcal{L}(t)$ are equivalent $\forall t > 0$. Then, $\exists m > 0$ such that

$$\mathcal{L}'(t) \leq -m\mathcal{L}(t) \quad \forall t \geq 0. \tag{53}$$

Hence, the solution of (53) is given by

$$\mathcal{L}'(t) \leq \mathcal{L}(0)\, e^{-mt} \quad \forall t \geq 0,$$

so, we have

$$\mathcal{E}(t) \leq ke^{-mt} \quad \forall t \geq 0,$$

with $k > 0$. This completes the proof.

Example

Consider the problem (1) with $\Omega = (0, 2\pi), \rho_1 = \rho_2 = 1$, $p = q = 3 > 2, b = 1, K = \frac{1}{2}, \varphi_0(x) = \psi_0(x) = \frac{1}{\sqrt{24\pi C}} \sin x$, $\varphi_1(x) = \psi_1(x) = -\frac{1}{\sqrt{24\pi C}} \sin x$, where C is the maximal value between two constants denoted by the same notation C_{**} and they are given by (42) and (43). The initial delays $f_0(x, t - s) = g_0(x, t - s) = 0$ for $t \in (0, s)$. We set $v_i = 2b_iA_0$ and $\alpha_i = 4b_iA_0$ for i=1,2. Then we have

1. The initial condition $U_0 = \frac{1}{\sqrt{24\pi C}}(sinx, -sinx, sinx, -sinx, 0, 0, 0, 0) \in D(A)$.

2. By Lemma 2.2, we have $A_0 = \frac{\pi\beta^{\alpha-1}}{sin(\pi\alpha)}$, from the definition of b_i, it follows that $\alpha_i = 4a_i\beta^{\alpha-1}$. Then the condition (4) is satisfied.

3. It easy to notice that the relation (10) holds.

4. From the expression of the energy (9), we get $\mathcal{E}(0) = \frac{1}{12C^3}$. Thus, $\widetilde{B} = \frac{1}{\sqrt{2}} < 1$. By a simple and direct calculation, we find $I_1(0) = 3I_2(0) = \frac{1}{48C^3} > 0$. Then we deduce that the conditions (36) are verified.

So, by Theorem 3.1 and Theorem 3.2, the problem (1) has a unique local and global solution. Furthermore, by Theorem 4.1, we get the decay result.

Conclusion

In this paper, we prove the well-posedness result of problem (1) using the semi-group theories. Then, we prove that the solution decay exponentially by means of the multiplier approach. Finally, we provide an example in which our results can be applied. The main contribution of this work is the extension of the previous results from [2]. It will be interesting to extend our results to the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - k\left(\varphi_x + \psi\right)_x + a_1 \partial_t^{\alpha,\beta} \varphi\left(0t - s\right) = |\varphi|^{p-2}\varphi, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + a_2 \partial_t^{\alpha,\beta} \psi\left(t - s\right) = |\psi|^{q-2}\psi, \\ \varphi(x, t = 0) = \varphi_0(x),\ \psi(x, t = 0) = \psi_0(x), \\ \varphi_t(x, t = 0) = \varphi_1(x),\ \psi_t\left(x, 0\right) = \psi_1(x), \\ \varphi_t\left(x, t - s\right) = f_0\left(x, t - s\right), t \in (0, s), \\ \psi_t\left(x, t - s\right) = g_0\left(x, t - s\right), t \in (0, s) \end{cases}$$

under the following boundary conditions:

$$\begin{cases} (\varphi_x + \psi)(L, t) + \alpha_1 \varphi_t(L, t) = 0, \\ \psi_x(L, t) + \alpha_2 \phi_t(L, t) = 0, \end{cases}$$

which will be an open problem.

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