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Blow up of Nonlinear Hyperbolic Equation with Variable Damping and Source Terms

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Abstract: In this work, we consider a nonlinear hyperbolic equation with variable damping and source terms. Our aim is to prove that the solution with negative initial energy blows up in finite time.

Keywords: *hyperbolic equation; damping term; source term; variable exponents; blow up.*

Mathematics Subject Classification (2020): 35B40, 37D30, 37K58, 46E35.

1 Introduction

In this work, we consider the following problem

$$\begin{cases} u_{tt} - div\,(A\nabla u) + u_t\,|u_t|^{m(.)-2} = u\,|u|^{p(.)-2} & \text{in } \Omega \times (0,T)\,, \\ u = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in } \Omega, \end{cases} \qquad (P)$$

where $T > 0$, Ω is a bounded domain of $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) with a smooth boundary $\partial\Omega$. $A = A(x,t)$ is an $n \times n$ symmetric matrix with real coefficients. The exponents $m(.)$ and $p(.)$ are given measurable functions on Ω.

When $A =$ Identity, the bibliography of works concerning problems of existence and nonexistence of global solution is truly long. **In the case of constant damping and source terms,** Ball [3] in 1977, considered the wave equation with source term and proved the blow up of solution when the energy of the initial data is negative. Haraux and Zuazua [8] in 1988, proved that the damping term of polynomial or arbitrary growth

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assured the global estimates of the wave equation for arbitrary initial data. The interaction between the damping and the source term was considered by Levine [9] in 1974, in the linear damping case $m = 2$. He showed that the solutions with negative initial energy blow up in finite time. Georgiev and Todorova [6] in 1994, extended Levine's result to the nonlinear damping case $m > 2$. They showed that solutions with any initial data are global if the damping term dominates the source term, then blow up in finite time if the source term dominates the damping term and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [10] in 2001, proved that any negative initial energy solution blow up in finite time. **In the case of variable damping and source term**, these problems have been considered by many authors using the Lebesgue spaces with variable exponent [5]. For instance, Antontsev [2] in 2011, considered the wave equation with $p(x,t)$-Laplacian and variable source term. In his work, he proved existence and blow up results under some assumptions on the initial energy data. In a recent study, Messaoudi and Talahmeh [11] in 2017, considered the quasilinear wave equation with variable exponent nonlinearities and proved that the solution with negative or positive initial energy blows up in finite time. In the same year, Messaoudi et al. [12] considered the nonlinear wave equation with variable source and damping terms and proved the blow up of solution with negative energy of initial data. In 2018, Ghegal et al. [7] considered the same system. They used the stable set method to prove the global existence result. Then, by some integral inequality, they showed the stability of this solution.

When $A(x,t) = a(x,t)$, where a is a given function, Sun et al. [13] in 2016, showed a result of blow up of solution when the energy of initial data is positive.

When $A = A(x,t)$, Boukhatem and Benyatou [4], in 2012, considered the hyperbolic equation with constant damping and source terms. They obtained a result of blow up of solution when the initial energy is positive.

In this work, we consider the case of variable coefficients ($A = A(x,t)$), variable damping and source terms and we show that the solution of (P) with negative initial energy blows up in finite time.

This paper consists of two sections in addition to Introduction. In Section 2, we give the assumptions and preliminary results needed to obtain our result. In Section 3, we prove the main result.

2 Assumptions and Preliminary Results

In this paper, we study the blow up behavior of the system (P) under the following assumptions:

- (H1) *For the matrix A:* Assume that

    1. A is of class $C^1\left(\bar{\Omega} \times [0,+\infty[\right)$.
    2. There exists a constant $a_0 > 0$ such that for all $\xi \in \mathbb{R}^n$, we have

$$A\xi\xi \geq a_0 \mid \xi \mid^2 \quad \text{and} \quad A'\xi\xi \leq 0.$$

- (H2) *For the exponents:* The exponents $m(.)$ and $p(.)$ are measurable functions on Ω such that

1. The following log-Holder continuity condition is satisfied:

$$|q(x) - q(y)| \leq -\frac{A}{\log|x-y|} \quad \text{for all } x, y \in \Omega, \quad \text{with} \quad |x-y| < \delta,$$

$$\text{where} \quad A > 0 \quad \text{and} \quad 0 < \delta < 1.$$

2. We have

$$2 \leq m_1 \leq m(x) \leq m_2, \quad n = 1, 2.$$
$$2 \leq m_1 \leq m(x) \leq m_2 \leq \frac{2n}{n-2}, \quad n \geq 3,$$

with $m_1 := ess\inf_{x\in\Omega} m(x)$ and $m_2 := ess\sup_{x\in\Omega} m(x)$.

3. We assume that

$$2 \leq p_1 \leq p(x) \leq p_2, \quad n = 1, 2.$$
$$2 < p_1 \leq p(x) \leq p_2 \leq 2\frac{n-1}{n-2}, \quad n \geq 3,$$

with $p_1 := ess\inf_{x\in\Omega} p(x)$ and $p_2 := ess\sup_{x\in\Omega} p(x)$.

4. We assume that

$$m_2 < p_1 \leq p_2 \leq \frac{2n}{n-2}.$$

- (H3) *For the initial energy data:* we assume that

$$E(0) < 0,$$

where

$$E(0) := \frac{1}{2} \parallel u_1 \parallel_2^2 + \frac{1}{2}\int_\Omega A(x,0)\nabla u_0 \nabla u_0 dx - \int_\Omega \frac{1}{p(x)} \mid u_0 \mid^{p(x)} dx.$$

Now, we introduce some preliminary results needed to prove our main result. The existence and uniqueness result for problem (P) is given in the following Theorem.

Theorem 2.1 *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then the problem* (P) *has a unique local solution*

$$\begin{aligned} u &\in L^\infty((0,T); H_0^1(\Omega)), \\ u_t &\in L^\infty((0,T); L^2(\Omega)) \cap L^{m(.)}(\Omega \times (0,T)), \\ u_{tt} &\in L^2((0,T); H_0^1(\Omega)) \end{aligned} \quad ,$$

for some $T > 0$.

We define the energy functional for the local solution u of problem (P) by

$$E(t) = \frac{1}{2} \parallel u_t \parallel_2^2 + \frac{1}{2}\int_\Omega A\nabla u \nabla u dx - \int_\Omega \frac{1}{p(x)} |u|^{p(x)} dx, \qquad \forall t \in [0,T].$$

The following Lemma shows that E is a non-increasing function of t.

Lemma 2.1 *We have*

$$E'(t) = \frac{1}{2}\int_{\Omega} A'\nabla u\nabla u dx - \int_{\Omega} \mid u_t \mid^{m(x)} dx \leq 0, \qquad \forall\, t \in [0,T].$$

Proof. We multiply the first equation in (P) by u_t, integrate it over Ω, we obtain

$$\int_{\Omega} u_t u_{tt} dx - \int_{\Omega} u_t div\,(A\nabla u)\, dx - \int_{\Omega} u_t u\, |u|^{p(x)-2}\, dx = -\int_{\Omega} |u_t|^{m(x)}\, dx. \tag{1}$$

First, we have

$$\int_{\Omega} u_t u_{tt} dx = \frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 \quad \text{and} \quad \int_{\Omega} u_t u\, |u|^{p(x)-2}\, dx = \frac{d}{dt}\int_{\Omega}\frac{1}{p(x)} \mid u \mid^{p(x)} dx. \tag{2}$$

On the other hand, by the generalized Green formula, we find

$$-\int_{\Omega} u_t div\,(A\nabla u)\, dx = \int_{\Omega} A\nabla u\nabla u_t dx. \tag{3}$$

But

$$\begin{aligned} \frac{d}{dt}\int_{\Omega} A\nabla u\nabla u dx &= \int_{\Omega}\frac{d(A\nabla u)}{dt}\nabla u dx + \int_{\Omega} A\nabla u\nabla u_t dx \\ &= \int_{\Omega} A'\nabla u\nabla u dx + \int_{\Omega} A\nabla u_t\nabla u dx + \int_{\Omega} A\nabla u\nabla u_t dx. \end{aligned}$$

Since A is symmetric

$$\int_{\Omega} A\nabla u\nabla u_t dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega} A\nabla u\nabla u dx - \frac{1}{2}\int_{\Omega} A'\nabla u\nabla u dx$$

(3) becomes

$$-\int_{\Omega} u_t div\,(A\nabla u)\, dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega} A\nabla u\nabla u dx - \frac{1}{2}\int_{\Omega} A'\nabla u\nabla u dx. \tag{4}$$

We replace (2) and (4) in (1) to obtain

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2\, dx + \frac{1}{2}\int_{\Omega} A\nabla u\nabla u dx - \int_{\Omega}\frac{1}{p(x)} \mid u \mid^{p(x)} dx\right\} \\ =\;& \frac{1}{2}\int_{\Omega} A'\nabla u\nabla u dx - \int_{\Omega} \mid u_t \mid^{m(x)} dx. \end{aligned}$$

This implies the desired result. We set

$$H(t) = -E(t), \qquad \forall t \in [0,T].$$

Lemma 2.2 *We have*

$$0 < H(0) \leq H(t) \leq \frac{1}{p_1}\int_{\Omega} \mid u \mid^{p(x)} dx, \qquad \forall t \in [0,T]. \tag{5}$$

Proof.

- Since $E(0) < 0$, we find $H(0) = -E(0) > 0$.

- From the definition of H and the monotonicity of E, we have
$$H(0) \leq H(t), \qquad \forall t \in [0, T].$$

- We have
$$H(t) = -\frac{1}{2} \|u_t\|_2^2 - \frac{1}{2} \int_\Omega A\nabla u \nabla u dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx.$$
$(H1-2)$ implies that
$$H(t) \leq \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx.$$
By $(H2-3)$, we arrive at
$$H(t) \leq \frac{1}{p_1} \int_\Omega |u|^{p(x)} dx.$$

Let C be a generic positive constant and it may change from line to line.
The following two Lemmas are also needed in our work.

Lemma 2.3 *There exists a constant $C > 0$ such that*
$$\int_\Omega \mid u \mid^{p(x)} dx \geq C \|u\|_{p_1}^{p_1} \tag{6}$$
and
$$\int_\Omega \mid u \mid^{m(x)} dx \leq C \left(\left(\int_\Omega \mid u \mid^{p(x)} dx \right)^{\frac{m_2}{p_1}} + \left(\int_\Omega \mid u \mid^{p(x)} dx \right)^{\frac{m_1}{p_1}} \right). \tag{7}$$

Proof. **Proof of** (6)**:** We have
$$\int_\Omega \mid u \mid^{p(x)} dx = \int_{\Omega_+} \mid u \mid^{p(x)} dx + \int_{\Omega_-} \mid u \mid^{p(x)} dx, \tag{8}$$
where
$$\Omega_+ = \{x \in \Omega / \mid u(x,t) \mid \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega / \mid u(x,t) \mid < 1\}.$$
We have
$$\int_{\Omega_+} \mid u \mid^{p(x)} dx \geq \int_{\Omega_+} \mid u \mid^{p_1} dx \tag{9}$$
and
$$\int_{\Omega_-} \mid u \mid^{p(x)} dx \geq \int_{\Omega_-} \mid u \mid^{p_2} dx.$$

Since $p_1 \le p_2$,

$$\int_{\Omega_-} \mid u \mid^{p(x)} dx \ge C\left(\int_{\Omega_-} \mid u \mid^{p_1} dx\right)^{\frac{p_2}{p_1}}. \tag{10}$$

We replace (9) and (10) in (8) to obtain

$$\int_{\Omega} \mid u \mid^{p(x)} dx \ge \int_{\Omega_+} \mid u \mid^{p_1} dx + C\left(\int_{\Omega_-} \mid u \mid^{p_1} dx\right)^{\frac{p_2}{p_1}}.$$

This implies that

$$\int_{\Omega} \mid u \mid^{p(x)} dx \ge \int_{\Omega_+} \mid u \mid^{p_1} dx \quad \text{and} \quad C\left(\int_{\Omega} \mid u \mid^{p(x)} dx\right)^{\frac{p_1}{p_2}} \ge \int_{\Omega_-} \mid u \mid^{p_1} dx.$$

By addition, we find

$$\int_{\Omega} \mid u \mid^{p(x)} dx + C\left(\int_{\Omega} \mid u \mid^{p(x)} dx\right)^{\frac{p_1}{p_2}} \ge \|u\|_{p_1}^{p_1}.$$

So

$$\left[1 + C\left(\int_{\Omega} \mid u \mid^{p(x)} dx\right)^{\frac{p_1}{p_2}-1}\right]\int_{\Omega} \mid u \mid^{p(x)} dx \ge \|u\|_{p_1}^{p_1}.$$

But, by (5) and $(H2-3)$, we find

$$(p_1 H(0))^{\frac{p_1}{p_2}-1} \ge \left(\int_{\Omega} \mid u \mid^{p(x)} dx\right)^{\frac{p_1}{p_2}-1}.$$

Then

$$\left[1 + C\,(p_1 H(0))^{\frac{p_1}{p_2}-1}\right]\int_{\Omega} \mid u \mid^{p(x)} dx \ge \|u\|_{p_1}^{p_1}.$$

Consequently, we obtain (6).

Proof of (7)**:** We have

$$\begin{aligned}\int_{\Omega} \mid u \mid^{m(x)} dx &= \int_{\Omega_+} \mid u \mid^{m(x)} dx + \int_{\Omega_-} \mid u \mid^{m(x)} dx \\ &\le \int_{\Omega_+} \mid u \mid^{m_2} dx + \int_{\Omega_-} \mid u \mid^{m_1} dx.\end{aligned}$$

Since $m_1 \le m_2 < p_1$,

$$\begin{aligned}\int_{\Omega} \mid u \mid^{m(x)} dx &\le C\left[\left(\int_{\Omega_+} \mid u \mid^{p_1} dx\right)^{\frac{m_2}{p_1}} + \left(\int_{\Omega_-} \mid u \mid^{p_1} dx\right)^{\frac{m_1}{p_1}}\right] \\ &\le C\left(\|u\|_{p_1}^{m_2} + \|u\|_{p_1}^{m_1}\right).\end{aligned}$$

By (6), we find the desired result.

Lemma 2.4 *For all*

$$0 < \alpha \le \min\left\{\frac{p_1-2}{2p_1}, \frac{p_1-m_2}{p_1(m_2-1)}\right\} \quad and \quad k>1,$$

we have

$$\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \le C\left(\int_\Omega A\nabla u\nabla u dx + \int_\Omega \mid u \mid^{p(x)} dx\right) \tag{11}$$

and

$$\begin{aligned}\int_\Omega \mid u \mid\mid u_t \mid^{m(x)-1} dx \quad &\le \quad C\frac{k^{1-m_1}}{m_1}\left(\int_\Omega A\nabla u\nabla u dx + \int_\Omega \mid u \mid^{p(x)} dx\right) \\ &+ \quad \frac{(m_2-1)k}{m_2} H^{-\alpha}(t)H'(t).\end{aligned} \tag{12}$$

Proof. **Proof of** (11): We have

$$\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx = \int_\Omega \left[\frac{H(t)}{H(0)}\right]^{\alpha(m(x)-1)} [H(0)]^{\alpha(m(x)-1)} \mid u \mid^{m(x)} dx.$$

Since $\dfrac{H(t)}{H(0)} \ge 1$, by $(H2-2)$, we find

$$\begin{aligned}&\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \le \int_\Omega \left[\frac{H(t)}{H(0)}\right]^{\alpha(m_2-1)} [H(0)]^{\alpha(m(x)-1)} \mid u \mid^{m(x)} dx \\ \le \quad & [H(t)]^{\alpha(m_2-1)} \int_\Omega [H(0)]^{\alpha(m(x)-m_2)} \mid u \mid^{m(x)} dx.\end{aligned} \tag{13}$$

But

$$[H(0)]^{\alpha(m(x)-m_2)} \le C \quad \text{for all} \quad x\in\Omega.$$

Indeed,

$$\text{if} \quad H(0)\le 1, \text{ then} \quad [H(0)]^{\alpha(m(x)-m_2)} \le [H(0)]^{\alpha(m_1-m_2)},$$

$$\text{if} \quad H(0)> 1, \text{ then} \quad [H(0)]^{\alpha(m(x)-m_2)} \le [H(0)]^{\alpha(m_2-m_2)} = 1.$$

Then (13) becomes

$$\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \le C\,[H(t)]^{\alpha(m_2-1)} \int_\Omega \mid u \mid^{m(x)} dx.$$

By (5) and (7), we find

$$\begin{aligned}&\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \\ \le \quad & C\left(\left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_2}{p_1}+\alpha(m_2-1)} + \left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_1}{p_1}+\alpha(m_2-1)}\right).\end{aligned}$$

We apply Lemma 4.1 from [12] to

$$2 \leq s = m_1 + \alpha p_1(m_2 - 1) \leq p_1,$$

and

$$2 \leq s = m_2 + \alpha p_1(m_2 - 1) \leq p_1,$$

and by $(H1-1)$, we obtain (11).
Proof of (12): By the Young inequality

$$XY \leq \frac{\delta^{\mu}}{\mu} X^{\mu} + \frac{\delta^{-\theta}}{\theta} Y^{\theta} \quad \text{for all} \quad X,\, Y \geq 0, \;\; \delta > 0 \quad \text{and} \quad \frac{1}{\mu} + \frac{1}{\theta} = 1 \tag{14}$$

with

$$X = |u|\,, \quad Y = \mid u_t \mid^{m(x)-1}, \quad \mu = m(x) \quad \text{and} \quad \theta = \frac{m(x)}{m(x)-1},$$

we find

$$\begin{aligned} \int_{\Omega} \mid u \mid\mid u_t \mid^{m(x)-1} dx &\leq \int_{\Omega} \frac{\delta^{m(x)}}{m(x)} \mid u \mid^{m(x)} dx + \int_{\Omega} \frac{m(x)-1}{m(x)} \delta^{-\frac{m(x)}{m(x)-1}} \mid u_t \mid^{m(x)} dx \\ &\leq \frac{1}{m_1} \int_{\Omega} \delta^{m(x)} \mid u \mid^{m(x)} dx \\ &+ \frac{m_2 - 1}{m_1} \int_{\Omega} \delta^{-\frac{m(x)}{m(x)-1}} \mid u_t \mid^{m(x)} dx. \end{aligned}$$

Let $k > 0$. If we take

$$\delta = \left(kH^{-\alpha}(t)\right)^{-\frac{m(x)-1}{m(x)}} > 0,$$

then we find

$$\begin{aligned} \int_{\Omega} \mid u \mid\mid u_t \mid^{m(x)-1} dx &\leq \frac{1}{m_1} \int_{\Omega} k^{1-m(x)} H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \\ &+ \frac{(m_2-1)k}{m_1} H^{-\alpha}(t) \int_{\Omega} \mid u_t \mid^{m(x)} dx. \end{aligned} \tag{15}$$

But, from the definition of H, Lemma 2.1 and $(H1-2)$, we have

$$\int_{\Omega} |u_t|^{m(x)} dx = \frac{1}{2} \int_{\Omega} A' \nabla u \nabla u dx + H'(t) \leq H'(t).$$

Then, for $k > 1$, (15) becomes

$$\int_{\Omega} \mid u \mid\mid u_t \mid^{m(x)-1} dx \leq \frac{k^{1-m_1}}{m_1} \int_{\Omega} H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx + \frac{(m_2-1)k}{m_2} H^{-\alpha}(t) H'(t).$$

By (11), we obtain the result.

3 Main Result

In this section, we state and prove our main result.

Theorem 3.1 *The solution of problem* (P) *blows up in finite time.*

Proof. We proceed in 4 steps.
Step 1. For $\epsilon > 0$, we consider the following functional:

$$L(t) = H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t dx, \qquad \forall t \in [0, T].$$

If we derive the function L with respect to t, we obtain

$$L'(t) = (1-\alpha) H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|_2^2 + \epsilon \int_\Omega u u_{tt} dx, \quad \forall t \in [0, T]. \tag{16}$$

But

$$\begin{aligned} \int_\Omega u u_{tt} dx &= \int_\Omega u div(A\nabla u) dx - \int_\Omega u u_t \mid u_t \mid^{m(x)-2} dx \\ &+ \int_\Omega \mid u \mid^{p(x)} dx. \end{aligned}$$

By the generalized Green formula, we obtain

$$\int_\Omega u u_{tt} dx = -\int_\Omega A\nabla u \nabla u dx - \int_\Omega u u_t \mid u_t \mid^{m(x)-2} dx + \int_\Omega \mid u \mid^{p(x)} dx. \tag{17}$$

Replacing (17) in (16), we find

$$\begin{aligned} L'(t) &\geq (1-\alpha) H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|_2^2 - \epsilon \int_\Omega A\nabla u \nabla u dx \\ &- \epsilon \int_\Omega \mid u \mid\mid u_t \mid^{m(x)-1} dx + \epsilon \int_\Omega \mid u \mid^{p(x)} dx. \end{aligned}$$

By (12), we obtain

$$\begin{aligned} L'(t) &\geq \left[1 - \alpha - \epsilon \frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|_2^2 \\ &- \epsilon \left(1 + C\frac{k^{1-m_1}}{m_1}\right) \int_\Omega A\nabla u \nabla u dx \\ &+ \epsilon \left(1 - C\frac{k^{1-m_1}}{m_1}\right) \int_\Omega \mid u \mid^{p(x)} dx. \end{aligned} \tag{18}$$

Add and subtract $\epsilon(1-\eta)p_1 H(t)$ for $0 < \eta < 1$ in the right-hand side of (18) and use the definition of H to obtain

$$\begin{aligned} L'(t) &\geq \left[1 - \alpha - \epsilon \frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t) H'(t) + \epsilon(1-\eta)p_1 H(t) + \epsilon \|u_t\|_2^2 \\ &- \epsilon \left(1 + C\frac{k^{1-m_1}}{m_1}\right) \int_\Omega A\nabla u \nabla u dx + \epsilon \left(1 - C\frac{k^{1-m_1}}{m_1}\right) \int_\Omega \mid u \mid^{p(x)} dx \\ &- \epsilon(1-\eta)p_1 \left(-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\int_\Omega A\nabla u \nabla u dx + \frac{1}{p_1}\int_\Omega \mid u \mid^{p(x)} dx\right). \end{aligned} \tag{19}$$

Then

$$\begin{aligned} L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right]H^{-\alpha}(t)H'(t)+\epsilon(1-\eta)p_1H(t) \\ &+ \epsilon\left(\eta-C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega |\ u\ |^{p(x)}\ dx+\epsilon\left(\frac{(1-\eta)p_1}{2}+1\right)\epsilon\left\|u_t\right\|_2^2 \\ &+ \epsilon\left(\frac{p_1-2}{2}-\frac{\eta p_1}{2}-C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega A\nabla u\nabla u dx. \end{aligned}$$

For the fixed k sufficiently large, then for η sufficiently small, we arrive at

$$\begin{aligned} L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right]H^{-\alpha}(t)H'(t)+\epsilon\gamma\left[H(t)+\int_\Omega |\ u\ |^{p(x)}\ dx+\epsilon\left\|u_t\right\|_2^2\right] \\ &+ \epsilon\beta\int_\Omega A\nabla u\nabla u dx, \end{aligned} \tag{20}$$

where

$$\gamma=\min\left\{(1-\eta)p_1,\eta-C\frac{k^{1-m_1}}{m_1},\frac{(1-\eta)p_1}{2}+1\right\}>0,$$

and

$$\beta=\frac{p_1-2}{2}-\frac{\eta p_1}{2}-C\frac{k^{1-m_1}}{m_1}=\frac{p_1-2}{2}-\eta\left(1+\frac{p_1}{2}\right)+\eta-C\frac{k^{1-m_1}}{m_1}>0.$$

If ϵ is chosen sufficiently small such that

$$1-\alpha-\epsilon\frac{m_2-1}{m_2}k\geq 0,$$

then, by (6), inequality (20) takes the form

$$L'(t)\geq\epsilon C\left[H(t)+\|\ u\ \|_{p_1}^{p_1}+\|\ u_t\ \|_2^2\right]. \tag{21}$$

Step 2. Since

$$L(0)=H^{1-\alpha}(0)+\epsilon\int_\Omega u_0(x)u_1(x)dx>0,$$

from the increase of L (see (21)), we find

$$L(t)\geq 0,\quad \forall t\in[0,T].$$

Step 3. By the definition of L, we find

$$L^{\frac{1}{1-\alpha}}(t)\quad\leq\quad\left[H^{1-\alpha}(t)+\epsilon\int_\Omega |\ u\ ||\ u_t\ |\ dx\right]^{\frac{1}{1-\alpha}}.$$

By the following inequality:

$$(a+b)^m\leq 2^m(a^m+b^m)\quad\text{for all } a,b\geq 0 \text{ and } m>0,$$

with

$$a = H^{1-\alpha}(t),\ b = \epsilon \int_\Omega |\, u \,||\, u_t \,|\, dx \text{ and } m = \frac{1}{1-\alpha},$$

we obtain

$$L^{\frac{1}{1-\alpha}}(t) \quad \leq \quad 2^{\frac{1}{1-\alpha}} \left[H(t) + \left(\epsilon \int_\Omega |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \right].$$

But, by the Cauchy–Schwarz inequality, we have

$$\left(\int_\Omega |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \quad \leq \quad \|\, u \,\|_2^{\frac{1}{1-\alpha}} \|\, u_t \,\|_2^{\frac{1}{1-\alpha}} .$$

From the embedding $L^{p_1}(\Omega) \hookrightarrow L^2(\Omega)$, we find

$$\left(\int_\Omega |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \leq C \,\|\, u \,\|_{p_1}^{\frac{1}{1-\alpha}} \|\, u_t \,\|_2^{\frac{1}{1-\alpha}} .$$

Apply Young's inequality (14) with

$$X = \|\, u \,\|_{p_1}^{\frac{1}{1-\alpha}}, \quad Y = \|\, u_t \,\|_2^{\frac{1}{1-\alpha}}, \quad \mu = \frac{2(1-\alpha)}{1-2\alpha} \quad \textit{and} \quad \theta = 2(1-\alpha),$$

we have

$$\left(\int_\Omega |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \leq C \left(\|\, u \,\|_{p_1}^{\frac{2}{1-2\alpha}} + \|\, u_t \,\|_2^2 \right).$$

We apply Corollary 4.4 from [12] with $2 \leq s = \frac{2}{1-2\alpha} \leq p_1$ to find

$$L^{\frac{1}{1-\alpha}}(t) \leq C \left[H(t) + \|\, u \,\|_{p_1}^{p_1} + \|\, u_t \,\|_2^2 \right], \quad \forall t \in [0,T]. \tag{22}$$

Step 4. We proceed by contradiction. By the continuation principal, we obtain that $T = +\infty$. By combining (21) and (22), we arrive at

$$L'(t) \quad \geq C L^{\frac{1}{1-\alpha}}(t), \quad \text{for all } t \geq 0.$$

A simple integration over $(0,t)$ gives

$$L(t) \geq \frac{1}{\left[L^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\alpha C t}{(1-\alpha)} \right]^{\frac{1-\alpha}{\alpha}}}, \quad \text{for all } t \geq 0.$$

This leads to a contradiction.

4 Conclusion

In this work, we study the blow up of solutions of the nonlinear hyperbolic equation with variable damping and source terms. We present the assumptions and preliminary results required to obtain our main result. We also provide the energy identity associated with the solution. Finally, we state and prove the blow up result for the solution.

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