Nonlinear Dynamics and Systems Theory, **25** (4) (2025) 399–409

The Limit-Point/Limit-Circle Problem for Fractional Differential Equations

J. R. Graef *

Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA

Received: October, 5, 2024; Revised: July 15, 2025

Abstract: In this paper, the author examines the famous limit-point/limit-circle problem introduced by Hermann Weyl more than one-hundred years ago (1910) and popularized in Volume 2 of the well known treatise by Dunford and Schwartz. They visit this problem in the case where it involves fractional derivatives; this has not been studied before.

Keywords: *fractional equation; limit-point problem; limit-circle problem; square integrability.*

Mathematics Subject Classification (2020): 34B20, 34C10.

Dedication: This paper is dedicated to the memory of T. A. (Ted) Burton on the occasion of the ninetieth anniversary of his birth.

1 Introduction

A problem with more than a one-hundred year history going back to the seminal work of Hermann Weyl in [27] is the limit-point/limit-circle problem. It began with his work on eigenvalue problems for the second order linear differential equation

$$(a(t)y')' + r(t)y = \lambda y, \quad t \in [0, \infty), \quad \lambda \in \mathbb{C}, \tag{C}$$

which he classified as being of the *limit-circle* type if every solution is square integrable (belongs to L^2), and to be of *limit-point* type if at least one solution does not belong to L^2. This problem has important connections to the solution of certain boundary value problems as can be seen in the works of Titchmarsh [25, 26].

* Corresponding author: mailto:John-Graef@utc.edu

Weyl showed that if $\Im(\lambda) \neq 0$, then (C) always has a solution $y \in L^2(\mathbb{R}_+)$ (the terminology "limit-point or limit-circle" arises somewhat naturally from the proof of this fact); and if equation (C) is of the limit-circle type for some $\lambda_0 \in \mathbb{C}$, then (C) is limit-circle for all $\lambda \in \mathbb{C}$. In particular, if equation (C) is limit-circle for $\lambda = 0$, then it is limit–circle for all values of λ, and if (C) is not limit–circle for $\lambda = 0$, then it is not limit–circle for any value of λ.

The problem then reduces to whether equation (C) with $\Im(\lambda) \neq 0$ has one (limit-point case) or two linearly independent solutions (limit-circle case) in L^2. This is known as the *Weyl Alternative*. The limit-point/limit-circle problem then becomes that of determining conditions under which each of these two cases holds.

For additional discussion on the background and history of the limit-point/limit-circle problem, we refer the reader to the classic work of Dunford and Schwartz [9], the work of Coddington and Levinson [6], and the monographs [2] and [3].

Probably the best known limit-circle result for the equation

$$(a(t)y'(t))' + r(t)y(t) = 0, \quad t \geq t_0, \tag{L}$$

is that of Dunford and Schwartz [9, Sect. XIII.6.20, p. 1410].

Theorem 1.1 *Assume that*

$$\int_0^\infty \left| \left[\frac{(a(u)r(u))'}{a^{\frac{1}{2}}(u)r^{\frac{3}{2}}(u)} \right]' + \frac{\{[a(u)r(u)]'\}^2}{4a^{\frac{3}{2}}(u)r^{\frac{5}{2}}(u)} \right| du < \infty. \tag{1}$$

If

$$\int_0^\infty [1/(a(u)r(u))^{\frac{1}{2}}]du < \infty, \tag{2}$$

then equation (L) *is of the limit–circle type, i.e., every solution $y(t)$ of* (L) *satisfies*

$$\int_{t_0}^\infty y^2(u)du < \infty.$$

Their corresponding limit–point result is the following.

Theorem 1.2 *Assume that* (1) *holds. If*

$$\int_0^\infty [1/(a(u)r(u))^{\frac{1}{2}}]du = \infty, \tag{3}$$

then equation (L) *is of the limit–point type, i.e., there is a solution $y(t)$ of* (L) *such that*

$$\int_{t_0}^\infty y^2(u)du = \infty.$$

Interest in extending these results to nonlinear equations began in the mid-twentieth century with the papers of Atkinson [1], Burlak [4], Detki [7], Elias [10], Hallam [18], Suyemoto and Waltman [24], and Wong [28], and continued with the work of Graef and Spikes [13–15, 23].

Here we wish to ask whether results in the spirit of Theorems 1.1 and 1.2 can be found for equations with fractional derivatives. In particular, we will study the nonlinear fractional differential equation

$$(N^\alpha(a(t)(N^\alpha y)(t)))(t) + r(t)y^{2k-1}(t) = 0, \tag{NF}$$

where $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\alpha \in (0, 1]$, a, $r : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous, a', $r' \in AC_{loc}(\mathbb{R}_+)$, a'', $r'' \in L^2_{loc}(\mathbb{R}_+)$, $a(t) > 0$, $r(t) > 0$, and k is a positive integer. Here, N^α is the nonconformable fractional derivative developed by Nápoles Valdes *et al.* [17, 19–21], which is defined as follows.

Definition 1.1 ([17, Definition 2.1], [21, Definition 1]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional derivative of f of order $\alpha \in (0, 1)$ is defined by

$$(N^\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon e^{t^{-\alpha}}) - f(t)}{\epsilon}$$

for all $t > 0$.

Corresponding to the nonconformable fractional derivative, we have the nonconformable fractional integral.

Definition 1.2 ([21, Definition 2]) Let $f : [0, \infty) \to \mathbb{R}$. The nonconformable fractional integral of f of order $\alpha \in (0, 1)$ is defined by

$$({}_N J^\alpha_{t_0} f)(t) = \int_{t_0}^{t} \frac{f(s)}{e^{s^{-\alpha}}} ds.$$

In light of Definitions 1.1 and 1.2, we see that the following lemma is needed.

Lemma 1.1 ([21, Theorem 3]) *If f is N^α–differentiable on (t_0, ∞) with $\alpha \in (0, 1]$, then for $t > t_0$:*

(a) If f is differentiable, then $({}_N J^\alpha_{t_0}(N^\alpha f))(t) = f(t) - f(t_0)$;

(b) $(N^\alpha({}_N J^\alpha_{t_0} f))(t) = f(t)$.

As a convenience to the reader, we next list some properties of the nonconformable fractional derivative.

Lemma 1.2 ([17, Theorem 2.3]) *Let f and g be N^α differentiable at a point $t > 0$, with $\alpha \in (0, 1]$. Then:*

(1) $N^\alpha(c) = 0$ for any constant $c \in \mathbb{R}$;

(2) $N^\alpha(fg)(t) = f(t)(N^\alpha g)(t) + g(t)(N^\alpha f)(t)$;

(3) $N^\alpha\left(\frac{f}{g}\right) = \dfrac{g(t)(N^\alpha f)(t) - f(t)(N^\alpha g)(t)}{g^2(t)}$;

(4) If f is differentiable (in the ordinary sense), then $(N^\alpha f)(t) = e^{t^{-\alpha}} f'(t)$.

One very important advantage of using the nonconformable fractional derivative is the existence of a chain rule, which we state here.

Lemma 1.3 ([17, Theorem 3.1]) *Let $\alpha \in (0, 1]$, g be N^α differentiable at $t > 0$, and f be differentiable at $g(t)$. Then*

$$N^\alpha(f \circ g)(t) = f'(g(t))(N^\alpha g)(t).$$

The following Gronwall type inequality for nonconformable fractional derivatives was obtained in [16].

Lemma 1.4 ([16, Lemma 2.7]) *Let* $c \in \mathbb{R}_+$ *and* $a,\ u : \mathbb{R} \to \mathbb{R}_+$. *If*

$$u(t) \le c + ({}_N J^{\alpha}_{t_0} au)(t), \tag{4}$$

then

$$u(t) \le c \exp\{({}_N J^{\alpha}_{t_0} a)(t)\}. \tag{5}$$

At this point, it seems that some discussion of notation is needed. If f is a function of u, then $(N^{\alpha} f)(u)$ denotes the nonconformable fractional derivative of f with respect to u. However, if f is a function of u, and u in turn is a function of z, then we denote the derivative of f with respect to z by $(N^{\alpha} f(u))(z)$, or $(N^{\alpha} f)(z)$ if no ambiguity exists. With respect to integration, in the notation $({}_N J^{\alpha}_{t_0} f)(t)$, t_0 denotes the initial point for the integration and t is the terminal point, which may be ∞.

2 Nonlinear Limit-Point and Limit-Circle Results

We first have to define what we mean by nonlinear limit-point and limit-circle solutions of equation (NF).

Definition 2.1 A solution $y(t)$ of equation (NF) is said to be of the nonlinear limit-circle type if

$$({}_N J^{\alpha}_{t_0} y^{2k})(\infty) < \infty,$$

and to be of the nonlinear limit-point type if

$$({}_N J^{\alpha}_{t_0} y^{2k})(\infty) = \infty.$$

To simplify the notation in what follows, we let

$$\gamma = 1/2(k+1) \quad \text{and} \quad \omega = (2k+1)/2(k+1).$$

We begin our analysis of equation (NF) by transforming it as follows. Let

$$s = \left({}_N J^{\alpha}_{t_0} \frac{r^{\gamma}}{a^{\omega}}\right)(t), \quad y(t) = x(s(t)), \tag{T}$$

and notice that

$$\gamma + \omega = 1 \quad \text{and} \quad \omega - \gamma = 2\omega - 1 = k/(k+1).$$

Then, by Lemma 1.3,

$$(N^{\alpha} y)(t) = (N^{\alpha} x)(s) \frac{ds(t)}{dt} = (N^{\alpha} x)(s) \left[r^{\gamma}(t)/a^{\omega}(t)\right]$$

and

$$a(t)(N^{\alpha} y)(t) = (N^{\alpha} x)(s) \left[r^{\gamma}(t) a^{1-\omega}(t)\right],$$

so that

$$\begin{aligned} N^\alpha(a(t)(N^\alpha y))(t) &= (N^{2\alpha}x)(s)[r^\gamma(t)a^{1-\omega}(t)]\,[r^\gamma(t)/a^\omega(t)] + (N^\alpha x)(s)[r^\gamma(t)a^{1-\omega}(t)]' \\ &= (N^{2\alpha}x)(s)[r^{2\gamma}(t)a^{1-2\omega}(t)] + (N^\alpha x)(s)[r^\gamma(t)a^{1-\omega}(t)]' \\ &= (N^{2\alpha}x)(s)[r^{2\gamma}(t)a^{1-2\omega}(t)] + (N^\alpha x)(s)[r^\gamma(t)a^\gamma(t)]' \\ &= (N^{2\alpha}x)(s)[r^{2\gamma}(t)a^{1-2\omega}(t)] + \gamma(N^\alpha x)(s)(r(t)a(t))^{\gamma-1}(r(t)a(t))'. \end{aligned}$$

Equation (NF) then becomes

$$(N^{2\alpha}x)(s)[r^{2\gamma}(t)a^{1-2\omega}(t)] + \alpha(N^\alpha x)(s)(r(t)a(t))^{\gamma-1}(r(t)a(t))' + r(t)x^{2k-1}(s) = 0,$$

or

$$(N^{2\alpha}x)(s) + \gamma(N^\alpha x)(s)\frac{(a(t)r(t))'}{a^\gamma(t)r^{\gamma+1}(t)} + (a(t)r(t))^{\omega-\gamma}x^{2k-1}(s) = 0,$$

which we will write as

$$(N^{2\alpha}x)(s) + \gamma P(t)(N^\alpha x)(s) + R(t)x^{2k-1}(s) = 0, \qquad (E_s)$$

where

$$P(t) = \frac{[a(t)r(t)]'}{a^\gamma(t)r^{\gamma+1}(t)} \quad \text{and} \quad R(t) = (a(t)r(t))^{\omega-\gamma}.$$

Remark 2.1 If $k = 1$, the transformation (T) does not reduce to the transformation used, for example, in [9].

3 Limit-Point and Limit-Circle Results

We first have to define what we mean by nonlinear limit-point and limit-circle solutions of equation (NF).

Definition 3.1 A solution $y(t)$ of equation (NF) is said to be of the nonlinear limit-circle type if

$$({}_N J^\alpha_{t_0} y^{2k})(\infty) < \infty,$$

and to be of the nonlinear limit-point type if

$$({}_N J^\alpha_{t_0} y^{2k})(\infty) = \infty.$$

Equation (E_s) can then be written as the system

$$\begin{cases} (N^\alpha x)(s) = z(s) - \gamma P(t)x(s), \\ (N^\alpha z)(s) = \gamma(N^\alpha P)(t)x(s) - R(t)x^{2k-1}(s). \end{cases} \qquad \text{(S)}$$

The motivation for the form of this system is due to Burton and Patula [5].

We define a Liapunov (energy) function V for this system by

$$V(s) = V(x,z,s) = \frac{z^2}{2} + R(t)\frac{x^{2k}}{2k}.$$

Then, along solutions of system (S),

$$(N^\alpha V)(s) = \gamma(N^\alpha P(t))(s)xz + x^{2k}\left[\frac{(N^\alpha R(t))(s)}{2k} - \gamma R(t)P(t)\right]. \qquad (6)$$

Now by Lemma 1.3,

$$\frac{(N^{\alpha}R(t))(s)}{2k} = \frac{R'(t)(N^{\alpha}t)(s)}{2k} = \frac{R'(t)}{2k}\frac{a^{\omega}(t)}{r^{\gamma}(t)}$$
$$= \frac{\omega-\gamma}{2k}(a(t)r(t))^{\omega-\gamma-1}(a(t)r(t))'\,\frac{a^{\omega}}{r^{\gamma}} = \gamma(a(t)r(t))'\,\frac{a^{2\omega-\gamma-1}}{r^{2\gamma-\omega+1}} \quad (7)$$

and

$$\gamma R(t)P(t) = \gamma(a(t)r(t))'\frac{a^{\omega-2\gamma}(t)}{r^{2\gamma-\omega+1}(t)} = \gamma(a(t)r(t))'\,\frac{a^{2\omega-\gamma-1}(t)}{r^{2\gamma-\omega+1}(t)}. \quad (8)$$

In view of (7) and (8), we see from (6) that

$$(N^{\alpha}V)(s) = \gamma(N^{\alpha}P(t))(s)x(s)z(s) = \gamma P'(t)\frac{a^{\omega}(t)}{r^{\gamma}(t)}x(s)z(s) \quad (9)$$

since $(N^{\alpha}P(t))(s) = P'(t)\frac{a^{\omega}(t)}{r^{\gamma}(t)}$. Notice that

$$|xz| = \frac{|R^{1/2}(t)xz|}{R^{1/2}(t)} \le \left[R(t)\frac{x^2}{2} + \frac{z^2}{2}\right]/R^{1/2}(t)$$
$$\le \left[R(t)\left(\frac{x^{2k}}{2k} + C_1\right) + \frac{z^2}{2}\right]/R^{1/2}(t) \quad (10)$$
$$\le V(s)/R^{1/2}(t) + C_1R^{1/2}(t)$$

for some $C_1 \ge 0$, a constant. Therefore,

$$(N^{\alpha}V)(s) \le \gamma(N^{\alpha}P(t))(s)V(s)/R^{1/2}(t) + \gamma|(N^{\alpha}P(t))(s)|C_1R^{1/2}(t).$$

Now if $\tau(s)$ denotes the inverse function of $s(t)$,

$$\left({}_NJ^{\alpha}_{t_0}\left\{|(N^{\alpha}P(\tau))(s)|/R^{\frac{1}{2}}(\tau)\right\}\right)(s) = \left({}_NJ^{\alpha}_{t_0}\left\{|\{(ar)'/a^{\gamma}r^{\gamma+1}\}'|/(ar)^{(\omega-\gamma)/2}\right\}\right)(s)$$

and

$$\left({}_NJ^{\alpha}_{t_0}\,|(N^{\alpha}P(\tau))|\,R^{\frac{1}{2}}(\tau))\right)(s) = \left({}_NJ^{\alpha}_{t_0}|\left\{(ar)'/a^{\gamma}r^{\gamma+1}\right\}'|(ar)^{(\omega-\gamma)/2}\right)(t).$$

Integrating $(N^{\alpha}V)(s)$ gives

$$V(s) \le V(t_0) + \gamma\left({}_NJ^{\alpha}_{t_0}\,|N^{\alpha}P(\tau)|\,V/R^{\frac{1}{2}}(\tau)\right)(s) + C_1\gamma\left({}_NJ^{\alpha}_{t_0}\,|N^{\alpha}P(\tau)|\,R^{\frac{1}{2}}(\tau)\right)(s)$$
$$= V(t_0) + \gamma\left({}_NJ^{\alpha}_{t_0}\left\{|\{(ar)'/a^{\gamma}r^{\gamma+1}\}'|/(ar)^{(\omega-\gamma)/2}\right\}\right)(s)$$
$$+ C_1\gamma\left({}_NJ^{\alpha}_{t_0}|\left\{(ar)'/a^{\gamma}r^{\gamma+1}\right\}'|(ar)^{(\omega-\gamma)/2}\right)(s). \quad (11)$$

We can now formulate our limit-circle result.

Theorem 3.1 *Assume that*

$$\left({}_NJ^{\alpha}_{t_0}\left\{|\{ar)'/a^{\gamma}r^{\gamma+1}\}'|/(ar)^{(\omega-\gamma)/2}\right\}\right)(\infty) < \infty \quad (12)$$

and

$$\left({}_N J_{t_0}^{\alpha} \left\{|(ar)' a^{\alpha} r^{\alpha+1}\right\}' |(ar)^{(\beta-\alpha)/2}\right)(\infty) < \infty. \tag{13}$$

If

$$\left({}_N J_{t_0}^{\alpha} \frac{1}{(ar)^{\gamma-\omega}}\right)(\infty) < \infty,$$

then any solution y of equation (NF) *is of the nonlinear limit-circle type, that is,*

$$\left({}_N J_{t_0}^{\alpha} y^{2k}\right)(\infty) < \infty.$$

Proof. From the analysis above, we arrive at (11). We see that condition (12) ensures that the second term on the right-hand side of (11) is bounded, so by Lemma 1.4, for some constant $C_2 > 0$,

$$V(s) \le C_2 \exp\left({}_N J_{t_0}^{\alpha} |(N^{\alpha} P(\tau))| R^{\frac{1}{2}}(\tau)\right)(s).$$

Condition (13) then shows that $V(s)$ is bounded, say, $V(s) \le C_3$ for some $C_3 > 0$. Therefore,

$$(a(t)r(t))^{\omega-\gamma} y^{2k}(t) = (a(t)r(t))^{\omega-\gamma} x^{2k}(s) \le 2kC_3,$$

and so it follows that

$$\left({}_N J_{t_0}^{\alpha} y^{2k}(t)\right) \le 2kC_3 \left({}_N J_{t_0}^{\alpha}[1/(a(u))r(u))^{\omega-\gamma}]\right)(\infty) < \infty$$

by condition (14), and so all solutions of equation (NF) are of the nonlinear limit–circle type. □

Notice that if we are in the linear case (i.e., $k = 1$), then in reconstructing $V(s)$ in (10), the constant $C_1 \equiv 0$, and so condition (13) is not needed in the theorem.

Next, we wish to formulate and prove a limit-point result for equation (NF).

Theorem 3.2 *In addition to conditions* (12) *and* (13), *assume that there are constants D_1, $D_2 > 0$ such that*

$$\left|(N^{\alpha}(ar))(t)/a^{1/2}(t)r^{3/2}(t)\right| \le D_1 \tag{14}$$

and

$$|a^{\frac{1}{2}}(t)(N^{\alpha} r)(t)/r^{\frac{3}{2}}(t)| \le D_2. \tag{15}$$

In addition, assume that

$$\left({}_N J_{t_0}^{\alpha} \left\{[(N^{\alpha}(ar))(t)]^2 / ar^3\right\}\right)(\infty) < \infty \tag{16}$$

and

$$\left({}_N J_{t_0}^{\alpha} \{a[(N^{\alpha} r)(t)]^2/r^3\}\right)(\infty) < \infty. \tag{17}$$

If

$$\left({}_N J_{t_0}^{\alpha}[1/(ar)^{\omega-\gamma}]\right)(\infty) = \infty, \tag{18}$$

then equation (NF) *is of the nonlinear limit-point type, that is, there is a solution y of* (NF) *such that*

$$\left({}_N J_{t_0}^{\alpha} y^{2k}\right)(\infty) = \infty.$$

Proof. Suppose that equation (NF) is of the nonlinear limit-circle type, and let y be one such solution. Then, since $y^2 \le y^{2k} + 1$ for all $y \in \mathbb{R}$ and (17) holds,

$$\left({}_N J^{\alpha}_{t_0}\{[(N^{\alpha}(ar)(t)]^2 y^2/ar^3\}\right)(s) \le D_1^2 \left({}_N J^{\alpha}_{t_0} y^{2k}\right)(t) + \left({}_N J^{\alpha}_{t_0}\{[(N^{\alpha}(ar)(t)]^2/ar^3\}\right)(s) < \infty. \quad (19)$$

Now if we multiply equation (NF) by $y(t)/r(t)$, use the identity $y(t)(N^{\alpha}(a(t)(N^{\alpha}y)(t)))(t) = y(t)(N^{\alpha}(a(t)(N^{\alpha}y)))(t) - a(t)[(N^{\alpha}y)(t)]^2$, and integrate by parts, we then obtain

$$a(t)\left(N^{\alpha}y)(t)y/r\right)(t) - a(t_1)(N^{\alpha}y)(t_1)y(t_1)/r(t_1) + \left({}_N J^{\alpha}_{t_1}[a(t)(N^{\alpha}y)(t)y(t)(N^{\alpha}r)(t)/r^2]\right)(t) + \left({}_N J^{\alpha}_{t_1} y^{2k}\right)(t) - \left({}_N J^{\alpha}_{t_1}\{a[(N^{\alpha}y)(t)]^2/r\}\right)(t) = 0 \quad (20)$$

for any $t_1 \ge t_0$. An application of the Schwarz inequality gives

$$\left|\left({}_N J^{\alpha}_{t_1}[a(N^{\alpha}y)(t)y(N^{\alpha}r)(t)/r^2]\right)(t)\right| \le \left[\left({}_N J^{\alpha}_{t_1}\{a[(N^{\alpha}y)(t)]^2/r\}\right)(t)\right]^{\frac{1}{2}} \left[\left({}_N J^{\alpha}_{t_1}[ay^2/(N^{\alpha}r)(t)]^2/r^3\}\right)(t)\right]^{\frac{1}{2}}.$$

From (15), we have

$$a(t)y^2(t)[(N^{\alpha}r)(t)]^2/r^3(t) \le \{a(t)[(N^{\alpha}r)(t)]^2/r^3(t)\}[y^{2k}(t)+1] \le D_2^2 y^{2k}(t) + a(t)[(N^{\alpha}r)(t)]^2/r^3(t),$$

so, integrating this expression, applying (17), and using the fact that y is a nonlinear limit circle solution give

$$\left({}_N J^{\alpha}_{t_1}\{ay^2[(N^{\alpha}r)(t)]^2/r^3\}\right)(\infty) \le C_4 < \infty$$

for some $C_4 > 0$. If y is not eventually monotonic, let $\{t_j\} \to \infty$ be an increasing sequence of zeros of $(N^{\alpha}y)(t)$. Then from (20), we have

$$C_4 H^{\frac{1}{2}}(t_j) + C_5 \ge H(t_j),$$

where

$$H(t) = \left({}_N J^{\alpha}_{t_1}\{a[(N^{\alpha}y)(t)]^2/r\}\right)(t)$$

and $C_5 > 0$ is a constant. It follows that $H(t_j) \le C_6 < \infty$ for all j and some constant $C_6 > 0$, so

$$\left({}_N J^{\alpha}_{t_0}\left\{a[(N^{\alpha}y)(t)]^2/r\right\}\right)(\infty) < \infty. \quad (21)$$

If $y(t)$ is eventually monotonic, then $y(t)(N^{\alpha}y)(t) \le 0$ for $t \ge t_1$ for sufficiently large $t_1 \ge t_0$ since y is a nonlinear limit-circle type solution. Using this in (20), we can repeat the style of argument used above to again see that (21) holds.

Finally, we define $V(s)$ as we did in the proof of Theorem 3.1, namely,

$$V(s) = z^2/2 + (a(t)r(t))^{\omega-\gamma}x^{2k}/2k;$$

then

$$(N^{\alpha}V)(s) \ge -\gamma|(N^{\alpha}P)(s)|V(s)/R^{\frac{1}{2}}(t) - \gamma|(N^{\alpha}P)(t)|C_1 R^{\frac{1}{2}}(t),$$

so

$$(N^\alpha V)(s) + \gamma|(N^\alpha P)(s)|V(s)/R^{\frac{1}{2}}(t) \geq -\gamma|(N^\alpha P)(t)|C_1 R^{\frac{1}{2}}(t). \tag{22}$$

If we let G and $g : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$G(t) = \gamma|(N^\alpha P(t))(s)|/R^{\frac{1}{2}}(t)$$

and

$$g(t) = \gamma|(N^\alpha P(t))(s)|C_1 R^{\frac{1}{2}}(t),$$

(22) can be written as

$$(N^\alpha V)(s) + G(t)V(s) \geq -g(t).$$

Therefore,

$$\left(N^\alpha \left(V \exp\left({}_N J^\alpha_{t_0} G(\tau)\right)(s)\right)\right) \geq -g(t) \exp\left({}_N J^\alpha_{t_0} G(\tau)\right)(s). \tag{23}$$

Condition (12) ensures that

$$\exp\left({}_N J^\alpha_{t_0} G(\tau)\right)(\infty) \leq C_7 < \infty$$

for some constant $C_7 > 0$, and condition (13) implies that

$$C_7 \left({}_N J^\alpha_{t_0} g(\tau)\right)(\infty) \leq C_8 < \infty$$

for some $C_8 > 0$.

Let $y(t)$ be any solution of (NF) such that $V(t_0) = V(x(t_0), z(t_0), t_0) > C_8 + 1$. Integrating (23), we have

$$V(s) \exp\left({}_N J^\alpha_{t_0} G(\tau)\right)(s) \geq V(t_0) - C_8 > 1,$$

and so

$$V(s) \geq 1/C_8$$

for $s \geq 0$. Dividing both members of this last inequality by $(a(t)r(t))^{\omega-\gamma}$ and rewriting the left-hand side in terms of t, we have

$$\begin{aligned} &a(t)[(N^\alpha y)(t)]^2/2r + \gamma(a(t)r(t))' y(t) y'(t)/r^2(t) \\ &\qquad + \gamma^2[(a(t)r(t))']^2 y^2(t)/2a(t)r^3(t) + y^{2k}(t)/2k \geq 1/C_8(a(t)r(t))^{\omega-\gamma}. \end{aligned} \tag{24}$$

If $y(t)$ is a nonlinear limit–circle solution of (NF), then (19) and (21) hold. By the Schwarz inequality,

$$\begin{aligned} &\left|\left({}_N J^\alpha_{t_0}\left\{(N^\alpha(ar))(t) y (N^\alpha y)(t)/r^2\right\}\right)(\infty)\right| \\ &\qquad \leq \left[\left({}_N J^\alpha_{t_0}\left\{[N^\alpha(ar)(t)]^2 y^2/ar^3)\right\}(\infty)\right)\right]^{\frac{1}{2}} \\ &\qquad\qquad \left[\left({}_N J^\alpha_{t_0}\left\{a[(N^\alpha y)(t)]^2/r)\right\}\right)(\infty)\right]^{\frac{1}{2}} < \infty \end{aligned}$$

by (19) and (21). Since $y(t)$ is a nonlinear limit-circle type solution, an integration of (24) contradicts (18). □

Remark 3.1 From the proof of Theorem 3.2, we can see that if conditions (14) and (16) hold, then (19) is a necessary condition for the existence of a nonlinear limit-circle solution of equation (NF). The same thing can be said about (21) if (15) and (17) hold.

Based on Theorems 3.1 and 3.2, we have the following necessary and sufficient condition for equation (NF) to be of the nonlinear limit-circle type.

Theorem 3.3 *Let conditions* (12)–(17) *hold. Then equation* (NF) *is of the nonlinear limit-circle type if and only if*

$$\left({}_N J^{\alpha}_{t_0}[1/(ar)^{\omega-\gamma}]\right)(\infty) = \left({}_N J^{\alpha}_{t_0}[1/(ar)^{k/(k+1)}]\right)(\infty) < \infty. \tag{25}$$

We conclude this paper with a brief discussion of some possible directions for further research. One somewhat obvious possibility is to explore sublinear equations, that is, equations of the form

$$(N^{\alpha}(a(t)(N^{\alpha}y)(t)))(t) + r(t)y^{\delta}(t) = 0,$$

where $0 < \delta < 1$. Of course, equations with more general nonlinear terms such as $f(y)$ instead of y^{2k-1} in (NF), is another possible direction for further research. Adding a forcing term to equation (NF) should not cause major difficulties. Exploring similar results to those in this paper for equations with a delay argument or for equations with a neutral term, would also be of interest.

Another interesting possible direction would be to look at the relationship between limit-point and limit-circle solutions of (NF) and other asymptotic properties of solutions such as boundedness, oscillation, convergence to zero, stability, etc.

Equations of higher order are another possible direction of interest. This would require the notion of deficiency indicies; in this regard, the works of Devinatz [8], Dunford and Schwartz [9], Everitt [11], Fedorjuk [12], and Naimark [22] would be useful. As a final suggestion, equation (NF) with $r(t) < 0$ is another possibility, but in that case, the continuability of solutions becomes an issue.

References

[1] F. V. Atkinson. Nonlinear extensions of limit–point criteria. *Math. Z.* **130** (1973) 297–312.

[2] M. Bartušek, Z. Došlá and J. R. Graef. *The Nonlinear Limit–Point/Limit–Circle Problem.* Birkhäuser, Boston, 2004.

[3] M. Bartušek and J. R. Graef. *The Strong Nonlinear Limit-Point/Limit-Circle Problem.* Trends in Abstract and Applied Analysis, Vol. **6**, World Scientific, Singapore, 2018.

[4] J. Burlak. On the non–existence of L_2-solutions of nonlinear differential equations. *Proc. Edinburgh Math. Soc.* **14** (1965) 257–268.

[5] T. A. Burton and W. T. Patula. Limit circle results for second order equations. *Monatsh. Math.* **81** (1976) 185–194.

[6] E. A. Coddington and N. Levinson. *Theory of Ordinary Differential Equations.* McGraw–Hill, New York, 1955.

[7] J. Detki. The solvability of a certain second order nonlinear ordinary differential equation in $L^p(0,\infty)$. *Math. Balk.* **4** (1974) 115–119. (In Russian).

[8] A. Devinatz. The deficiency index problem for ordinary self adjoint differential operators. *Bull. Amer. Math. Soc.* **79** (1973), 1109–1127.

[9] N. Dunford and J. T. Schwartz. *Linear Operators; Part II: Spectral Theory.* Wiley, New York, 1963.

[10] J. Elias. On the solutions of n-th order differential equation in $L^2(0,\infty)$. In: *Qualitative Theory of Differential Equations Szeged (Hungary), 1979*, ed. M. Farkas. Colloquia Mathematica Societatis János Bolyai, Vol. **30**, North-Holland, Amsterdam, 181–191 (1981).

[11] W. N. Everitt. On the deficiency index problem for ordinary differential operators 1910–1976. *Differential Equations, Proceedings Uppsala 1977*, Almqvist & Wiksell, Stockholm, 62–81 (1977).

[12] M. V. Fedorjuk. Asymptotics of solutions of ordinary linear differential equations of n-th order. *Dokl. Akad. Nauk SSSR* **165** (1965) 777–779.

[13] J. R. Graef. Limit circle criteria and related properties for nonlinear equations. *J. Differential Equations* **35** (1980) 319–338.

[14] J. R. Graef. Limit circle type results for sublinear equations. *Pacific J. Math.* **104** (1983) 85–94.

[15] J. R. Graef and P. W. Spikes. On the nonlinear limit–point/limit–circle problem. *Nonlinear Anal.* **7** (1983) 851–871.

[16] J. R. Graef. Asymptotic properties of general nonlinear differential equations containing nonconformable fractional derivatives. *J. Math. Appl.* **47** (2024), 23–37.

[17] P. M. Guzmán, G. Langton, L. M. L. M. Bittencurt, J. Medina, and J. E. Nápoles Valdes. A new definition of a fractional derivative of local type. *J. Math. Anal.* **9** (2018) 88–98.

[18] T. G. Hallam. On the nonexistence of L^p solutions of certain nonlinear differential equations. *Glasgow Math. J.* **8** (1967) 133–138.

[19] F. Martínez and J. E. Nápoles Valdes. Towards a non-conformable fractional calculus of n-variables. *J. Math.* **43** (2020) 87–98.

[20] J. E. Nápoles Valdes, P. M. Guzman, and L. M. Lugo. On the stability of solutions of nonconformable differential equations. *Stud. Univ. Babeş-Bolyai Math.*, to appear.

[21] J. E. Nápoles Valdes and C. Tunç. On the boundedness and oscillation of non-conformable Liénard equation. *J. Fract. Calc. Appl.* **11** (2020) 92–101.

[22] M. A. Naimark. *Linear Differential Operators, Part II.* George Harrap & CO., LTD, London, 1968.

[23] P. W. Spikes. Criteria of limit circle type for nonlinear differential equations. *SIAM J. Math. Anal.* **10** (1979) 456–462.

[24] L. Suyemoto and P. Waltman. Extension of a theorem of A. Winter. *Proc. Amer. Math. Soc.* **14** (1963) 970–971.

[25] E. C. Titchmarsh. *Eigenfunction Expansions Associated with Second-Order Differential Equations*, Part I. Oxford Univ. Press, Oxford (1962).

[26] E. C. Titchmarsh. On the uniqueness of the Green's function associated with a second-order differential equation. *Canad. J. Math.* **1** (1949) 191–198.

[27] H. Weyl. Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen. *Math. Ann.* **68** (1910) 220–269.

[28] J. S. W. Wong. Remark on a theorem of A. Wintner. *Enseignement Math.* (2) **13** (1967) 103–106.