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Transformation and Generalised H_∞ Optimization of Descriptor Systems

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Abstract: The generalized type H_∞ control problem is investigated for a class of linear descriptor systems with nonzero initial state. A generalized performance measure is used, which characterizes the weighted damping level of external and initial disturbances. A non-degenerate transformation of the system is proposed, which allows to apply known evaluation methods and achieve desired performance measures for ordinary lower-order systems. A numerical example of the descriptor control system is given to show the effectiveness of the obtained results.

Keywords: *descriptor system; exogenous disturbances; weighted performance measure; H_∞ control; LMI.*

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1 Introduction

In modern control theory, great attention is paid to descriptor (differential-algebraic) systems, which are used in modeling the motion of objects in mechanics, robotics, energy, electrical engineering, economics, etc. (see, e.g., [1–5]). Equations of motion, inputs and outputs of controlled objects may contain uncertain elements (parameters, external disturbances, measurement inaccuracies, etc.) that necessitate solving the problems of robust stabilization and minimize the impact of bounded disturbances on the quality of transient processes (H_∞ *optimization*).

A typical performance measure in the H_∞ optimization problem for systems with zero initial state is a damping level of external (exogenous) disturbances, which corresponds

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to the maximum value of the ratio for L_2-norms of controlled output and disturbances. For a class of the linear descriptor systems

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \tag{1}$$

this characteristic coincides with the H_∞-norm of the matrix transfer function

$$\|\mathcal{H}\|_\infty = \sup_{\omega\in\mathbb{R}} \sqrt{\lambda_{\max}(\mathcal{H}^\top(-i\omega)\mathcal{H}(i\omega))}, \quad \mathcal{H}(\lambda) = C(\lambda E - A)^{-1}B + D,$$

where $x \in \mathbb{R}^n$ is the state, $z \in \mathbb{R}^k$ is the controlled output and $w \in \mathbb{R}^s$ represents the exogenous input (external disturbances), E, A, B, C and D are the constant matrices with compatible dimensions, $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix.

In practice, it is advisable to apply generalized performance measures of the form [6,7]

$$J_0 = \sup_{0<\|w\|_P<\infty} \frac{\|z\|_Q}{\|w\|_P}, \quad J = \sup_{\{w,x_0\}\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}. \tag{2}$$

Here, $\|z\|_Q$ and $\|w\|_P$ are the weighted L_2-norms of z and w, respectively,

$$\|z\|_Q = \sqrt{\int_0^\infty z^\top Qz\,dt}, \quad \|w\|_P = \sqrt{\int_0^\infty w^\top Pw\,dt},$$

$\mathcal{W}$ is a set of admissible pairs $\{w, x_0\}$ of the system such that $0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$, $P = P^\top > 0$, $Q = Q^\top > 0$ and $X_0 = E^\top HE$ are the weight matrices, $H = H^\top > 0$ and the initial vector $x_0 = x(0_-)$ (see also [8,9]). It is obvious that $J_0 \le J$. If $P = I_s$ and $Q = I_k$, then $J_0 = \|\mathcal{H}\|_\infty$. The value of J characterizes the weighted damping level of external disturbances, as well as initial disturbances caused by the nonzero initial vector.

Well-known H_∞ control design methods are based on the statements of the Bounded Real Lemma type [10–12], which represent necessary and sufficient conditions for achieving the upper estimates of the performance measures used. These statements are formulated in terms of quadratic matrix equations and linear matrix inequalities (LMIs). For a class of linear descriptor systems, similar statements were established in [13–16]. For the available H_∞ optimization methods for such systems, see, e.g., [3,5,7,13,15,17].

This paper proposes new methods for solving the generalized H_∞ control problem for linear descriptor systems with performance measures of the form (2) based on a nonsingular transformation of such systems into ordinary ones and the application of well-known methods for synthesis of static and dynamic controllers. As a result, in a number of cases, the corresponding control synthesis algorithms are based on LMIs solving without additional rank constraints. In particular, the order of the desired dynamic controller in such synthesis algorithms does not exceed the rank of the coefficient matrix at the state derivative in the original system. Also, a distinctive feature of the obtained results compared to known results is the application of weighted performance measures, which provide new opportunities for achieving the desired characteristics of descriptor control systems. By using weight coefficients in these performance criteria, we can establish priorities between the components of controlled output and the unknown disturbances in the control system.

Note that quite effective computer tools have been created for solving LMIs, for example, the LMI Toolbox of MATLAB software [18]. The LMIRank and YALMIP tools with MATLAB [19, 20] as well as the Solve Block in Mathcad Prime software [21] can be used to solve LMIs with rank constraints.

Notations: I_n is the identity $n\times n$ matrix; $0_{n\times m}$ is the zero $n\times m$ matrix; $X = X^\top > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix; $\sigma(A)$ is the spectrum of A; $A^{-1}(A^+)$ is the inverse (pseudo-inverse) of A; $\operatorname{Ker} A$ is the kernel of A; W_A is the right null matrix of $A \in \mathbb{R}^{m\times n}$, that is, $AW_A = 0$, $W_A \in \mathbb{R}^{n\times(n-r)}$, $\operatorname{rank} W_A = n - r$, where $r = \operatorname{rank} A < n$ ($W_A = 0$ if $r = n$); $\|w\|_P$ is the weighted L_2-norm of a vector function $w(t)$; $\mathbb{C}^-$ is the open half-plane $\mathrm{Re}\lambda < 0$.

2 Definitions and Auxiliary Statements

Consider the descriptor system (1) with $\operatorname{rank} E = r < n$ and the performance measures (2). The system is said to be *admissible* if the pair of matrices $\{E, A\}$ is *regular*, *stable* and *impulse-free* [1], i.e., $\det F(\lambda) \not\equiv 0$ $(\lambda \in \mathbb{C})$, $\sigma(F) \subset \mathbb{C}^-$ and $\deg\{\det F(\lambda)\} = r$, respectively. Here, $\sigma(F)$ is the finite spectrum of the matrix pencil $F(\lambda) = A - \lambda E$. The system (1) is called *internally stable* if it is stable without disturbances ($w \equiv 0$).

The pair of matrices $\{E, A\}$ is regular if and only if there exist nonsingular matrices L and R that transform it to the canonical Weierstrass form [22]. System (1) is impulse-free if and only if [2]

$$\operatorname{rank}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + r. \tag{3}$$

Let $E = E_1E_2^\top$ be the skeletal decomposition of E, where $E_1, E_2 \in \mathbb{R}^{n\times r}$ are matrices of full rank r. Denote the corresponding orthogonal complements by $E_1^\perp, E_2^\perp \in \mathbb{R}^{n\times(n-r)}$ such that $E_i^\top E_i^\perp = 0$ and $\det\begin{bmatrix} E_i & E_i^\perp \end{bmatrix} \neq 0$, $i = 1, 2$.

Define a nonsingular transformation of system (1) by

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\; LAR = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\; x = R\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix},\; \xi_1 \in \mathbb{R}^r,\; \xi_2 \in \mathbb{R}^{n-r}, \tag{4}$$

where

$$L = \begin{bmatrix} E_1^+ \\ E_1^{\perp +} \end{bmatrix}, \quad E_1^+ = (E_1^\top E_1)^{-1}E_1^\top, \quad E_1^{\perp +} = (E_1^{\perp\top}E_1^\perp)^{-1}E_1^{\perp\top},$$

$$R = \begin{bmatrix} E_2^{+\top} & E_2^{\perp +\top} \end{bmatrix}, \quad E_2^+ = (E_2^\top E_2)^{-1}E_2^\top, \quad E_2^{\perp +} = (E_2^{\perp\top}E_2^\perp)^{-1}E_2^{\perp\top},$$

$$A_1 = E_1^+AE_2^{+\top}, \quad A_2 = E_1^+AE_2^{\perp+\top}, \quad A_3 = E_1^{\perp+}AE_2^{+\top}, \quad A_4 = E_1^{\perp+}AE_2^{\perp+\top}.$$

Note that

$$L^{-1} = \begin{bmatrix} E_1 & E_1^\perp \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} E_2^\top \\ E_2^{\perp\top} \end{bmatrix}, \quad \xi_1 = E_2^\top x, \quad \xi_2 = E_2^{\perp\top}x.$$

It is easy to establish that (3) is equivalent to the inequality $\det A_4 \neq 0$, i.e.,

$$\det(E_1^{\perp\top}AE_2^\perp) \neq 0. \tag{5}$$

Eliminating the variable $\xi_2 = -A_4^{-1}\big(A_3\xi_1 + B_2w\big)$ under the condition (5), based on the transformation (4), we obtain the ordinary system

$$\dot{\xi}_1 = \bar{A}\xi_1 + \bar{B}w, \quad z = \bar{C}\xi_1 + \bar{D}w, \quad \xi_1(0) = \xi_{10}, \tag{6}$$

where

$$\bar{A} = A_1 - A_2A_4^{-1}A_3,\ \bar{B} = B_1 - A_2A_4^{-1}B_2,\ \bar{C} = C_1 - C_2A_4^{-1}A_3,\ \bar{D} = D - C_2A_4^{-1}B_2,$$

$$LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

The spectrum of matrix $\bar{A}$ coincides with $\sigma(F)$ and the performance measures J_0 and J of impulse-free system (1) do not depend on ξ_2 and are determined by system (6) since

$$\begin{bmatrix} I_r & -A_2A_4^{-1} \\ 0 & I_{n-r} \end{bmatrix} LF(\lambda)R \begin{bmatrix} I_r & 0 \\ -A_4^{-1}A_3 & I_{n-r} \end{bmatrix} = \begin{bmatrix} \bar{A} - \lambda I_r & 0 \\ 0 & A_4 \end{bmatrix},$$

$$x_0^\top X_0 x_0 = \begin{bmatrix} \xi_{10}^\top & \xi_{20}^\top \end{bmatrix} R^\top E^\top L^\top L^{-1\top} H L^{-1} L E R \begin{bmatrix} \xi_{10} \\ \xi_{20} \end{bmatrix} = \xi_{10}^\top \bar{H} \xi_{10},$$

where $\bar{H} = E_1^\top H E_1$. Therefore, applying Lemma 4.1 from [23] to system (6), we have the following statement.

Lemma 2.1 *System* (1) *is admissible with* $J_0 < \gamma$ *if and only if* (5) *holds and there exists a matrix* $X = X^\top > 0$ *such that*

$$\bar{\Phi}(X) = \begin{bmatrix} \bar{A}^\top X + X\bar{A} + \bar{C}^\top Q\bar{C} & X\bar{B} + \bar{C}^\top Q\bar{D} \\ \bar{B}^\top X + \bar{D}^\top Q\bar{C} & \bar{D}^\top Q\bar{D} - \gamma^2 P \end{bmatrix} < 0. \tag{7}$$

The system is admissible with $J < \gamma$ *if and only if* (5) *holds and the LMIs* (7) *and*

$$0 < X < \gamma^2 \bar{H} \tag{8}$$

are feasible.

Lemma 2.1 can be used to calculate the characteristics J_0 and J of system (1) based on solving the corresponding optimization problems. At the same time, the restrictions in these problems are used exclusively in terms of LMIs:

$$J_0 = \inf\big\{\gamma :\ \bar{\Phi}(X) < 0,\ X > 0\big\}, \quad J = \inf\big\{\gamma :\ \bar{\Phi}(X) < 0,\ 0 < X < \gamma^2\bar{H}\big\}.$$

For the *worst-case* perturbation vector $w(t)$ with respect to J_0, in (2), the supremum is reached, i.e., $\|z\|_Q = J_0\|w\|_P$. If $\|z\|_Q^2 = J^2\big(\|w\|_P^2 + x_0^\top X_0 x_0\big)$, then $\{w(t), x_0\}$ is the *worst-case* pair with respect to J in system (1). The methods of finding such vectors in individual cases are proposed in [8, 24, 25]. For example, if system (1) is admissible and there exists a matrix X such that

$$A_0^\top X + X^\top A_0 + X^\top R_0 X + Q_0 = 0, \quad 0 \le E^\top X = X^\top E \le J^2 X_0,$$

where $A_0 = A + BR_1^{-1}D^\top QC$, $R_0 = BR_1^{-1}B^\top$, $Q_0 = C^\top\big(Q + QDR_1^{-1}D^\top Q\big)C$, $R_1 = J^2P - D^\top QD > 0$, then the worst-case pair $\{w(t), x_0\}$ with respect to J can be defined as $w = K_* x$ with $K_* = R_1^{-1}(B^\top X + D^\top QC)$ and $x_0 \in \mathrm{Ker}\,(E^\top X - J^2X_0)$ [25].

We present another method of finding the worst-case pair $\{w(t), x_0\}$ with respect to J for impulse-free system (1) based on the transformation (4). Under condition (5), we construct the worst-case initial vector in the form

$$x_0 = R \begin{bmatrix} \xi_{10} \\ -A_4^{-1}\big(A_3\xi_{10} + B_2 w(0)\big) \end{bmatrix}, \tag{9}$$

where $\{w(t), \xi_{10}\}$ is the worst-case pair of system (6) with respect to J.

According to the Schur complement lemma [10], the condition (7) is equivalent to the Riccati matrix inequality

$$\bar{A}_0^\top X + X\bar{A}_0 + X\bar{R}_0 X + \bar{Q}_0 < 0, \tag{10}$$

where $\bar{A}_0 = \bar{A} + \bar{B}\,\bar{R}_1^{-1}\bar{D}^\top Q\bar{C}$, $\bar{R}_0 = \bar{B}\,\bar{R}_1^{-1}\bar{B}^\top$, $\bar{Q}_0 = \bar{C}^\top\big(Q + Q\bar{D}\,\bar{R}_1^{-1}\bar{D}^\top Q\big)\bar{C}$, $\bar{R}_1 = \gamma^2 P - \bar{D}^\top Q\bar{D} > 0$. If the pair $\{\bar{A}, \bar{B}\}$ is controllable, the pair $\{\bar{A}, \bar{C}\}$ is observable, and $J_0 < \gamma$, then the corresponding Riccati matrix equation

$$\bar{A}_0^\top X + X\bar{A}_0 + X\bar{R}_0 X + \bar{Q}_0 = 0 \tag{11}$$

has the solutions X_- and X_+ such that $\sigma(\bar{A}_0 + \bar{R}_0 X_\pm) \subset \mathbb{C}^\pm$, $0 < X_- < X_+$, and every solution of inequality (10) belongs to the interval $X_- < X < X_+$ (see [26, 27]). Moreover, if $J < \gamma$ ($J \le \gamma$) and X satisfies (11), then $X < \gamma^2\bar{H}$ ($X \le \gamma^2\bar{H}$). Indeed, setting $v(\xi_1) = \xi_1^\top X\xi_1$ and

$$w = \bar{K}_*\xi_1, \quad \bar{K}_* = \bar{R}_1^{-1}\big(\bar{B}^\top X + \bar{D}^\top Q\bar{C}\big), \tag{12}$$

we get $\dot{v} + z^\top Qz - \gamma^2 w^\top Pw = 0$, where $\dot{v}$ is the derivative of the Lyapunov function v along the trajectory of system (6). Integrating the above equality from zero to infinity under the condition $J < \gamma$, we get $\|z\|_Q^2 - \gamma^2\|w\|_P^2 = \xi_{10}^\top X\xi_{10} < \gamma^2\xi_{10}^\top\bar{H}\xi_{10}$ for any $\xi_{10} \ne 0$, otherwise $J \ge \gamma$. If $J = \gamma$, then under conditions (11) and (12), the equality $\xi_{10}^\top X\xi_{10} = \gamma^2\xi_{10}^\top\bar{H}\xi_{10}$ or its equivalent $(X - \gamma^2\bar{H})\xi_{10} = 0$ is possible for some $\xi_{10} \ne 0$. At the same time, $\|z\|_Q^2 = J^2(\|w\|_P^2 + \xi_{10}^\top\bar{H}\xi_{10})$, i.e., in (2), the supremum is reached. Hence, the following statement holds.

Lemma 2.2 *Let $X > 0$ be the stabilizing solution of the Riccati equation* (11) *with $\gamma = J$. Then the structured vector of external disturbances* (12)*, where ξ_1 is a solution of the system*

$$\dot{\xi}_1 = (\bar{A} + \bar{B}\bar{K}_*)\xi_1, \quad \xi_1(0) = \xi_{10}, \tag{13}$$

and the vector (9) *with $\xi_{10} \in \operatorname{Ker}(X - J^2\bar{H})$ present the worst-case pair $\{w(t), x_0\}$ with respect to J in system* (1)*. If $X > 0$ is the stabilizing solution of* (11) *with $\gamma = J_0$ and $\xi_1 = \xi_1(t, \xi_{10})$ is a solution of* (13) *at $\xi_{10} = 0$, then* (12) *are the worst-case disturbances with respect to J_0 in system* (1)*.*

3 Main Results

Consider a class of linear descriptor control system described by

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u, \quad x(0_-) = x_0,\\ z &= C_1 x + D_{11} w + D_{12} u,\\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{14}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^s$ represents the exogenous input, $z \in \mathbb{R}^k$ is the controlled output and $y \in \mathbb{R}^l$ is the measured output. In (14), all matrix coefficients are constant, $\operatorname{rank} E = r < n$ and the pair $\{E, A\}$ is regular and impulse-free. The components of $w(t)$ can be both external disturbances acting on the system and errors of the measured output. This vector must be bounded by the weighted norm. The initial perturbations in the system are caused by the unknown initial vector x_0.

We are interested in the stabilizing control laws that guarantee the internal stability of the closed-loop system and the desired upper estimates of performance measure (2) for the system with respect to the controlled output z. Static and dynamic controllers that minimize the performance measure J are called J-*optimal*. For the identity weight matrices P and Q, the J_0-optimal control is called H_∞-*optimal*. The search for J_0- and J-optimal controllers can be performed based on achieving the corresponding estimates $J_0 < \gamma$ and $J < \gamma$ for the minimum possible value of γ.

When studying the class of systems (14), their properties such as C-, R- and I-controllability, as well as the dual properties C-, R- and I-observability, are used [3, 5]. In particular, for solvability of the generalized H_∞ optimization problems, the triple $\{E, A, B_2\}$ must be stabilizable and I-controllable. This is equivalent to the existence of a matrix K such that the pair $\{E, A + B_2K\}$ is stable and impulse-free, i.e., admissible. The I-controllability of the triple $\{E, A, B_2\}$ and I-observability of the triple $\{E, A, C_2\}$ are equivalent to the corresponding equalities [28]

$$\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B_2 \end{bmatrix} = n + r, \quad \operatorname{rank}\begin{bmatrix} E & A \\ 0 & E \\ 0 & C_2 \end{bmatrix} = n + r. \tag{15}$$

We apply the equivalent transformation (4) to system (14). Excluding the variable $\xi_2 = -A_4^{-1}\big(A_3\xi_1 + B_{12}w + B_{22}u\big)$ under condition (5), we get the ordinary system

$$\dot{\xi}_1 = \bar{A}\xi_1 + \bar{B}_1 w + \bar{B}_2 u, \quad z = \bar{C}_1\xi_1 + \bar{D}_{11}w + \bar{D}_{12}u, \quad y = \bar{C}_2\xi_1 + \bar{D}_{21}w + \bar{D}_{22}u, \tag{16}$$

where $\bar{A} = A_1 - A_2A_4^{-1}A_3$, $\bar{B}_1 = B_{11} - A_2A_4^{-1}B_{12}$, $\bar{B}_2 = B_{21} - A_2A_4^{-1}B_{22}$,
$\bar{C}_1 = C_{11} - C_{12}A_4^{-1}A_3$, $\bar{D}_{11} = D_{11} - C_{12}A_4^{-1}B_{12}$, $\bar{D}_{12} = D_{12} - C_{12}A_4^{-1}B_{22}$,
$\bar{C}_2 = C_{21} - C_{22}A_4^{-1}A_3$, $\bar{D}_{21} = D_{21} - C_{22}A_4^{-1}B_{12}$, $\bar{D}_{22} = D_{22} - C_{22}A_4^{-1}B_{22}$,

$$LB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad LB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \quad C_1R = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad C_2R = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.$$

Defining the performance measure (2) for this system, we use the expression $x_0^\top X_0 x_0 = \xi_{10}^\top \bar{H}\xi_{10}$, where $\xi_{10} = \xi_1(0)$, $\bar{H} = E_1^\top H E_1$ (see the previous section).

Thus, the J_0- and J-optimization problems for descriptor system (14) with the impulse-free pair $\{E, A\}$ are reduced to the application of well-known methods for solving similar problems for system (16).

3.1 Static controller

When using for system (16) the static output-feedback controller

$$u = Ky, \quad \det(I_m - K\bar{D}_{22}) \neq 0, \tag{17}$$

the closed-loop system has the form

$$\dot{\xi}_1 = A_*\xi_1 + B_*w, \quad z = C_*\xi_1 + D_*w, \tag{18}$$

where $A_* = \bar{A} + \bar{B}_2K_0\bar{C}_2$, $B_* = \bar{B}_1 + \bar{B}_2K_0\bar{D}_{21}$, $C_* = \bar{C}_1 + \bar{D}_{12}K_0\bar{C}_2$, $D_* = \bar{D}_{11} + \bar{D}_{12}K_0\bar{D}_{21}$ and $K_0 = (I_m - K\bar{D}_{22})^{-1}K$. The controller (17) will also be used for the original system (14).

Applying the Schur complement lemma [10], we rewrite the inequality (7) in Lemma 2.1 for system (18) as the LMI with respect to K_0:

$$\begin{bmatrix} A_*^\top X + XA_* & XB_* & C_*^\top \\ B_*^\top X & -\gamma^2 P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} = L_0^\top K_0 R_0 + R_0^\top K_0^\top L_0 + \Omega < 0, \tag{19}$$

where $R_0 = \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} & 0_{l\times k} \end{bmatrix}$, $L_0 = \begin{bmatrix} \bar{B}_2^\top X & 0_{m\times s} & \bar{D}_{12}^\top \end{bmatrix}$ and

$$\Omega = \begin{bmatrix} \bar{A}^\top X + X\bar{A} & X\bar{B}_1 & \bar{C}_1^\top \\ \bar{B}_1^\top X & -\gamma^2 P & \bar{D}_{11}^\top \\ \bar{C}_1 & \bar{D}_{11} & -Q^{-1} \end{bmatrix}.$$

Based on Lemma 2.1 and Theorem 5.1 from [7], we have the following result.

Theorem 3.1 *For system* (14)*, there is a static output-feedback controller* (17) *such that the closed-loop system is admissible and* $J < \gamma$ *if and only if* (8) *and*

$$W_{\bar{R}}^\top \begin{bmatrix} \bar{A}^\top X + X\bar{A} + \bar{C}_1^\top Q\bar{C}_1 & X\bar{B}_1 + \bar{C}_1^\top Q\bar{D}_{11} \\ \bar{B}_1^\top X + \bar{D}_{11}^\top Q\bar{C}_1 & \bar{D}_{11}^\top Q\bar{D}_{11} - \gamma^2 P \end{bmatrix} W_{\bar{R}} < 0, \tag{20}$$

$$W_{\bar{L}}^\top \begin{bmatrix} \bar{A}Y + Y\bar{A}^\top + \bar{B}_1 P^{-1}\bar{B}_1^\top & Y\bar{C}_1^\top + \bar{B}_1 P^{-1}\bar{D}_{11}^\top \\ \bar{C}_1 Y + \bar{D}_{11}P^{-1}\bar{B}_1^\top & \bar{D}_{11}P^{-1}\bar{D}_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_{\bar{L}} < 0, \tag{21}$$

$$W = \begin{bmatrix} X & \gamma I_r \\ \gamma I_r & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W = r, \tag{22}$$

where $\bar{R} = \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$ *and* $\bar{L} = \begin{bmatrix} \bar{B}_2^\top & \bar{D}_{12}^\top \end{bmatrix}$*, are feasible for some* X *and* Y*.*

The gain matrix of the controller can be found as $K = K_0(I_l + \bar{D}_{22}K_0)^{-1}$*, where* K_0 *is a solution of* (19)*.*

Note that (22) hold if and only if $X = X^\top > 0$, $Y = Y^\top > 0$ and $XY = \gamma^2 I_r$. In what follows, we present the corollaries of Lemma 2.1 and Theorem 3.1 for

$$\operatorname{rank} \bar{C}_2 = r \leq l, \quad \bar{D}_{21} = 0, \quad \bar{D}_{22} = 0, \tag{23}$$

$$\bar{D}_{11}^\top Q\bar{D}_{11} < \gamma^2 P. \tag{24}$$

Conditions (23) are satisfied if, for example,

$$\operatorname{rank}(C_2E_2) = r, \quad C_2E_2^\perp = 0, \quad D_{21} = 0, \quad D_{22} = 0.$$

Theorem 3.2 *Suppose* (23) *and* (24) *hold. The following statements are equivalent:*

1) *for system* (14)*, there is a static state-feedback controller* (17)*, for which the closed-loop system is admissible and* $J < \gamma$*;*

2) *there is a matrix* $Y > \bar{H}^{-1}$ *that satisfies* (21)*;*

3) *there exist matrices* $Y > \bar{H}^{-1}$ *and* Z *satisfying the LMI*

$$\begin{bmatrix} \gamma^2(\bar{A}Y + Y\bar{A}^\top + \bar{B}_2 Z + Z^\top \bar{B}_2^\top) & \gamma^2 \bar{B}_1 & Y\bar{C}_1^\top + Z^\top \bar{D}_{12}^\top \\ \gamma^2 \bar{B}_1^\top & -\gamma^2 P & \bar{D}_{11}^\top \\ \bar{C}_1 Y + \bar{D}_{12}Z & \bar{D}_{11} & -Q^{-1} \end{bmatrix} < 0. \tag{25}$$

When statement 2) *holds, the desired gain matrix* $K = K_0$ *of controller* (17) *in statement* 1) *can be found as a solution of* (19) *with* $X = \gamma^2 Y^{-1}$*. If statement* 3) *holds, then this matrix can be defined as a solution of the linear equation* $K\bar{C}_2 Y = Z$*.*

Proof. Given the conditions (23), we have $y = \bar{C}_2\xi_1 = \bar{C}_2 E_2^\top x$ and $l \geq r$. The equivalence of statements 1) and 2) follows from Theorem 3.1 since $W_{\bar{R}} = \begin{bmatrix} 0 & I_s \end{bmatrix}^\top$ under conditions (23). In this case, the inequality (20) takes the form (24) and does not depend on X. The desired matrix in (21) has the form $Y = \gamma^2 X^{-1}$. Therefore, instead of (8), we have the equivalent condition $Y > \bar{H}^{-1}$. Given (23), the matrix K of the controller (17) satisfying statement 1) can be an arbitrary solution K_0 of the LMI (19).

The equivalence of statements 1) and 3) follows from Lemma 2.1 for the closed-loop system (18), where $K_0 = K$. At the same time, the inequality (25) in statement 3) arises as a result of multiplying the first block row on the left-hand side and the first block column on the right-hand side of (19) by $Y = \gamma^2 X^{-1}$, taking into account (23) and the notation $Z = K\bar{C}_2 Y$. The last correlation can be solved with respect to K:

$$K = \begin{cases} Z(\bar{C}_2 Y)^{-1}, & l = r, \\ ZY^{-1}\bar{C}_2^{+} + T\bar{C}_2^{\perp\top}, & l > r, \end{cases}$$

where T is an arbitrary $m \times (l - r)$ matrix. □

Remark 3.1 Consider the case when the pair $\{E, A\}$ in system (14) is not impulse-free, but there exists a matrix $K_1 \in \mathbb{R}^{m\times l}$ such that

$$\det(I_m - K_1 D_{22}) \neq 0, \quad \det\left[E_1^{\perp\top}(A + B_2 K_{10} C_2) E_2^{\perp}\right] \neq 0, \tag{26}$$

where $K_{10} = K_{11}K_1$ and $K_{11} = (I_m - K_1 D_{22})^{-1}$. It can be established that under conditions (26), the rank relations (15) are satisfied, i.e., the system is I-controllable and I-observable.

Under the above assumptions, instead of (17), we use the controller $u = K_1 y + v$, where v is a new control in the system

$$E\dot{x} = \widetilde{A}x + \widetilde{B}_1 w + \widetilde{B}_2 v, \quad z = \widetilde{C}_1 x + \widetilde{D}_{11} w + \widetilde{D}_{12} v, \quad y = \widetilde{C}_2 x + \widetilde{D}_{21} w + \widetilde{D}_{22} v. \tag{27}$$

Here, under condition (26), the pair $\{E, \widetilde{A}\}$ is impulse-free and

$$\widetilde{A} = A + B_2 K_{10} C_2, \quad \widetilde{B}_1 = B_1 + B_2 K_{10} D_{21}, \quad \widetilde{B}_2 = B_2 K_{11},$$

$$\widetilde{C}_1 = C_1 + D_{12} K_{10} C_2, \quad \widetilde{D}_{11} = D_{11} + D_{12} K_{10} D_{21}, \quad \widetilde{D}_{12} = D_{12} K_{11},$$
$$\widetilde{C}_2 = C_2 + D_{22} K_{10} C_2, \quad \widetilde{D}_{21} = D_{21} + D_{22} K_{10} D_{21}, \quad \widetilde{D}_{22} = D_{22} K_{11}.$$

We perform an equivalent transformation of system (27) based on the relations

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad L\widetilde{A}R = \begin{bmatrix} \widetilde{A}_1 & \widetilde{A}_2 \\ \widetilde{A}_3 & \widetilde{A}_4 \end{bmatrix}, \quad L\widetilde{B}_1 = \begin{bmatrix} \widetilde{B}_{11} \\ \widetilde{B}_{12} \end{bmatrix}, \quad L\widetilde{B}_2 = \begin{bmatrix} \widetilde{B}_{21} \\ \widetilde{B}_{22} \end{bmatrix},$$

$$\widetilde{C}_1 R = \begin{bmatrix} \widetilde{C}_{11} & \widetilde{C}_{12} \end{bmatrix}, \quad \widetilde{C}_2 R = \begin{bmatrix} \widetilde{C}_{21} & \widetilde{C}_{22} \end{bmatrix},$$

$$x = R\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad \xi_1 = E_2^\top x, \quad \xi_2 = -\widetilde{A}_4^{-1}\big(\widetilde{A}_3\xi_1 + \widetilde{B}_{12} w + \widetilde{B}_{22} v\big),$$

where L and R are nonsingular matrices defined in (4). Then we can formulate analogues of Theorems 3.1 and 3.2 using the static controller $v = Ky$ for the ordinary system

$$\dot{\xi}_1 = \bar{A}\xi_1 + \bar{B}_1 w + \bar{B}_2 v, \quad z = \bar{C}_1\xi_1 + \bar{D}_{11} w + \bar{D}_{12} v, \quad y = \bar{C}_2\xi_1 + \bar{D}_{21} w + \bar{D}_{22} v, \tag{28}$$

where

$$\bar{A} = \widetilde{A}_1 - \widetilde{A}_2\widetilde{A}_4^{-1}\widetilde{A}_3, \quad \bar{B}_1 = \widetilde{B}_{11} - \widetilde{A}_2\widetilde{A}_4^{-1}\widetilde{B}_{12}, \quad \bar{B}_2 = \widetilde{B}_{21} - \widetilde{A}_2\widetilde{A}_4^{-1}\widetilde{B}_{22},$$

$$\bar{C}_1 = \widetilde{C}_{11} - \widetilde{C}_{12}\widetilde{A}_4^{-1}\widetilde{A}_3, \quad \bar{D}_{11} = \widetilde{D}_{11} - \widetilde{C}_{12}\widetilde{A}_4^{-1}\widetilde{B}_{12}, \quad \bar{D}_{12} = \widetilde{D}_{12} - \widetilde{C}_{12}\widetilde{A}_4^{-1}\widetilde{B}_{22},$$

$$\bar{C}_2 = \widetilde{C}_{21} - \widetilde{C}_{22}\widetilde{A}_4^{-1}\widetilde{A}_3, \quad \bar{D}_{21} = \widetilde{D}_{21} - \widetilde{C}_{22}\widetilde{A}_4^{-1}\widetilde{B}_{12}, \quad \bar{D}_{22} = \widetilde{D}_{22} - \widetilde{C}_{22}\widetilde{A}_4^{-1}\widetilde{B}_{22}.$$

As a result, the original system (14) with the control

$$u = \big(K_{10}C_2 + K_{11}K_0\bar{C}_2E_2^\top\big)x + \big(K_{10}D_{21} + K_{11}K_0\bar{D}_{21}\big)w$$

takes the form

$$E\dot{x} = A_0x + B_0w, \quad z = C_0x + D_0w, \tag{29}$$

where $K_0 = (I_m - K\bar{D}_{22})^{-1}K$, $\det(I_m - K\bar{D}_{22}) \neq 0$,

$$A_0 = A + B_2\big(K_{10}C_2 + K_{11}K_0\bar{C}_2E_2^\top\big), \quad B_0 = B_1 + B_2\big(K_{10}D_{21} + K_{11}K_0\bar{D}_{21}\big),$$

$$C_0 = C_1 + D_{12}\big(K_{10}C_2 + K_{11}K_0\bar{C}_2E_2^\top\big), \quad D_0 = D_{11} + D_{12}\big(K_{10}D_{21} + K_{11}K_0\bar{D}_{21}\big).$$

3.2 Dynamic controller

When using for system (16) the dynamic controller of the order p

$$\dot{\eta} = Z\eta + Vy, \quad u = U\eta + Ky, \quad \eta(0) = 0, \tag{30}$$

the closed-loop system in an extended phase space $\mathbb{R}^{r+p}$ has the form

$$\dot{\widehat{x}} = \widehat{A}_*\widehat{x} + \widehat{B}_*w, \quad z = \widehat{C}_*\widehat{x} + \widehat{D}_*w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{31}$$

where

$$\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2,\ \widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21},\ \widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2,\ \widehat{D}_* = \bar{D}_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{x} = \begin{bmatrix} \xi_1 \\ \eta \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} \bar{A} & 0_{r\times p} \\ 0_{p\times r} & 0_{p\times p} \end{bmatrix}, \quad \widehat{B}_1 = \begin{bmatrix} \bar{B}_1 \\ 0_{p\times s} \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} \bar{B}_2 & 0_{r\times p} \\ 0_{p\times m} & I_p \end{bmatrix},$$

$$\widehat{C}_1 = \begin{bmatrix} \bar{C}_1 & 0_{k\times p} \end{bmatrix}, \quad \widehat{C}_2 = \begin{bmatrix} \bar{C}_2 & 0_{l\times p} \\ 0_{p\times r} & I_p \end{bmatrix}, \quad \widehat{D}_{12} = \begin{bmatrix} \bar{D}_{12} & 0_{k\times p} \end{bmatrix}, \quad \widehat{D}_{21} = \begin{bmatrix} \bar{D}_{21} \\ 0_{p\times s} \end{bmatrix},$$

$$\widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix} = \left[\begin{array}{c|c} (I_m - K\bar{D}_{22})^{-1}K & (I_m - K\bar{D}_{22})^{-1}U \\ \hline V(I_l - \bar{D}_{22}K)^{-1} & Z + V\bar{D}_{22}(I_m - K\bar{D}_{22})^{-1}U \end{array}\right].$$

We define a performance measure $\widehat{J}$ for system (31) of the form (2) with the weight matrices P, Q and $\widehat{X}_0$, where $\widehat{X}_0$ is some block $(r+p)\times(r+p)$ matrix, whose first diagonal block is $\bar{H}$. The value of $\widehat{J}$ coincides with J since $\eta(0) = 0$.

Lemma 3.1 [23]. *Given positive definite matrices $X, Y \in R^{r\times r}$ and a number $\gamma > 0$, there are matrices $X_1 \in \mathbb{R}^{p\times r}$, $X_2 \in \mathbb{R}^{p\times p}$, $Y_1 \in \mathbb{R}^{p\times r}$ and $Y_2 \in \mathbb{R}^{p\times p}$ such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^\top \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{r+p} \tag{32}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_r \\ \gamma I_r & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq r + p. \tag{33}$$

Theorem 3.3 *For system* (14)*, there is a dynamic controller* (30) *of order* $p \le r$*, such that a closed-loop system is admissible and* $J < \gamma$ *if and only if* (8)*,* (20)*,* (21) *and* (33) *are feasible with respect to* X *and* Y*. The matrices of such controller can be defined as*

$$\left[\begin{array}{cc} K & U \\ V & Z \end{array}\right] = (I_{m+p} + \widehat{K}_0 \widehat{D}_{22})^{-1} \widehat{K}_0, \quad \widehat{D}_{22} = \left[\begin{array}{cc} \bar{D}_{22} & 0_{l\times p} \\ 0_{p\times m} & 0_{p\times p} \end{array}\right], \tag{34}$$

where $\widehat{K}_0$ *is a solution of the LMI*

$$\widehat{L}^\top \widehat{K}_0 \widehat{R} + \widehat{R}^\top \widehat{K}_0^\top \widehat{L} + \widehat{\Omega} < 0, \tag{35}$$

$$\widehat{R} = \left[\begin{array}{ccc} \widehat{C}_2 & \widehat{D}_{21} & 0_{(l+p)\times k} \end{array}\right], \quad \widehat{L} = \left[\begin{array}{ccc} \widehat{B}_2^\top \widehat{X} & 0_{(m+p)\times s} & \widehat{D}_{12}^\top \end{array}\right],$$

$$\widehat{\Omega} = \left[\begin{array}{ccc} \widehat{A}^\top \widehat{X} + \widehat{X}\widehat{A} & \widehat{X}\widehat{B}_1 & \widehat{C}_1^\top \\ \widehat{B}_1^\top \widehat{X} & -\gamma^2 P & \widehat{D}_{11}^\top \\ \widehat{C}_1 & \widehat{D}_{11} & -Q^{-1} \end{array}\right].$$

The block matrix $\widehat{X}$ *in* (35) *is formed on the basis of Lemma* 3.1 *according to* (32)*, where* X *and* Y *satisfy* (8)*,* (20)*,* (21) *and* (33)*.*

Taking into account the structure of matrices in (31), the system (16) with a dynamic controller (30) can be represented as a system in the space $\mathbb{R}^{r+p}$ with a static controller:

$$\dot{\widehat{x}} = \widehat{A}\,\widehat{x} + \widehat{B}_1\,w + \widehat{B}_2\,\widehat{u}, \quad z = \widehat{C}_1\,\widehat{x} + \widehat{D}_{11}\,w + \widehat{D}_{12}\,\widehat{u}, \quad \widehat{y} = \widehat{C}_2\,\widehat{x} + \widehat{D}_{21}\,w,$$

$$\widehat{x} = \left[\begin{array}{c} \xi_1 \\ \eta \end{array}\right], \quad \widehat{y} = \left[\begin{array}{c} y - \bar{D}_{22}u \\ \eta \end{array}\right], \quad \widehat{u} = \left[\begin{array}{c} u \\ \dot{\eta} \end{array}\right], \quad \widehat{u} = \widehat{K}_0 \widehat{y}.$$

Therefore, Theorem 3.3 can be proved as a corollary of Theorem 3.1 and Lemma 3.1.

Note that Theorems 3.1 and 3.3, without using the constraint $X < \gamma^2 \bar{H}$, give the existence criteria and methods for constructing stabilizing controllers that provide the estimate $J_0 < \gamma$ for the corresponding closed-loop systems. In the case $p = 0$, Theorem 3.3 yields a criterion for the existence of a static controller (17) with the properties specified in Theorem 3.1. The construction of dynamic controllers of the order $p = r$ satisfying Theorem 3.3 reduces to the solution of the LMI system without additional constraints. In this case, the rank constraint in (33) holds automatically.

We present the following algorithm for constructing a dynamic controller (30), which satisfies Theorem 3.3.

Algorithm 3.1

1) Calculating the transforming matrices (4) and coefficient matrices of system (16);
2) calculating $W_{\bar{R}}$ and $W_{\bar{L}}$, where $\bar{R} = \left[\begin{array}{cc} \bar{C}_2 & \bar{D}_{21} \end{array}\right]$, $\bar{L} = \left[\begin{array}{cc} \bar{B}_2^\top & \bar{D}_{12}^\top \end{array}\right]$;
3) finding matrices X and Y that satisfy (8), (20), (21) and (33);
4) constructing the decomposition $\Delta = Y - \gamma^2 X^{-1} = S^\top S \ge 0$, where $S \in \mathbb{R}^{p\times r}$, $\ker S = \ker \Delta$, and forming the block matrix

$$\widehat{X} = \left[\begin{array}{cc} X & X_1^\top \\ X_1 & X_2 \end{array}\right] > 0, \quad X_1 = \frac{1}{\gamma} SX, \quad X_2 = \frac{1}{\gamma^2} SXS^\top + I_p;$$

5) solving the LMI (35) with respect to $\widehat{K}_0$ taking into account $\det(I_m + K_0 \bar{D}_{22}) \ne 0$;
6) calculating the controller matrices according to (34).

Remark 3.2 Algorithm 3.1 can be implemented, e.g., by means of the MATLAB software. If $\Delta = 0$ in step 4) of the algorithm, i.e., $\operatorname{rank} W = r$, then solving the LMI (19), we obtain the gain matrix of static controller (17), which satisfies Theorem 3.1.

Remark 3.3 If the pair $\{E, A\}$ in system (14) is not impulse-free, but there is a matrix $K_1 \in \mathbb{R}^{m\times l}$ satisfying (26), then we set $u = K_1 y + v$, where v is a new control generated by

$$\dot{\eta} = Z\eta + Vy, \quad v = U\eta + Ky, \quad \eta(0) = 0,$$

which solves the problem for the ordinary system (28) formed on the basis of equivalent transformation of system (27) (see the previous subsection). As a result, the closed-loop descriptor system in the extended phase space has the form

$$\widehat{E}\dot{\widehat{x}} = \widehat{A}_0\widehat{x} + \widehat{B}_0 w, \quad z = \widehat{C}_0\widehat{x} + \widehat{D}_0 w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{36}$$

where

$$\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & I_p \end{bmatrix}, \quad \widehat{x} = \begin{bmatrix} x \\ \eta \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix},$$

$$\widehat{A}_0 = \begin{bmatrix} A + B_2(K_{10}C_2 + K_{11}G_1) & B_2K_{11}G_2 \\ V(\bar{C}_2E_2^\top + \bar{D}_{22}G_1) & Z + V\bar{D}_{22}G_2 \end{bmatrix}, \widehat{B}_0 = \begin{bmatrix} B_1 + B_2(K_{10}D_{21} + K_{11}G_3) \\ V(\bar{D}_{21} + \bar{D}_{22}G_3) \end{bmatrix},$$

$$\widehat{C}_0 = \begin{bmatrix} C_1 + D_{12}(K_{10}C_2 + K_{11}G_1) & D_{12}K_{11}G_2 \end{bmatrix}, \; \widehat{D}_0 = D_{11} + D_{12}(K_{10}D_{21} + K_{11}G_3),$$

$$G_1 = K_0\bar{C}_2E_2^\top, \;\; G_2 = (I_m - K\bar{D}_{22})^{-1}U, \;\; G_3 = K_0\bar{D}_{21}, \;\; K_0 = (I_m - K\bar{D}_{22})^{-1}K.$$

4 Example

Consider an electric circuit control system of the form described in (14), where [29]

$$E = \begin{bmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & -1 & 1 \\ 0 & -1/R_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{11} = D_{21} = D_{22} = 0_{2\times 1},$$

$x = \begin{bmatrix} i & v_2 & v_1 \end{bmatrix}^\top$, $z = \begin{bmatrix} v_2 & v_1 + u \end{bmatrix}^\top$, $y = \begin{bmatrix} v_2 & v_1 \end{bmatrix}^\top$, $L = 3$ is the inductance, $C = 2$ is the capacitance, $R_1 = 2$ and $R_2 = 1$ are the resistances, i is the current, v_1 and v_2 are the voltages, u is the control signal of a current source with bounded disturbance w (see Fig. 1). In this system, the pair $\{E, A\}$ is not impulse-free, the triples $\{E, A, B_2\}$ and $\{E, A, C_2\}$ are I-controllable and I-observable, respectively.

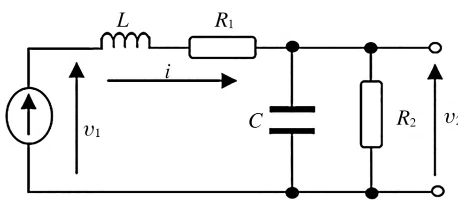

Figure 1: The electrical circuit.

We choose $K_1 = \left[\begin{array}{cc} 0 & 1 \end{array}\right]$ satisfying (26) and the weight matrices for performance measures (2): $P = 1$, $Q = I_2$, $X_0 = E^\top H E$, $H = 3I_3$. Using Theorem 3.1 for system (27) with $\gamma = 1,03624$, we find the controller

$$v = Ky, \quad K = \left[\begin{array}{cc} 0.22439 & -17.998625 \end{array}\right],$$

such that the closed-loop system is admissible and $J = 0.94402 < \gamma$. At the same time, the finite spectrum of the system coincides with $\sigma(\bar{A}) = \big\{ -0.59314 \pm 0.39471\, i \big\}$, where $\bar{A}$ is a system matrix of (28). Applying Lemma 2.2 for closed-loop system (29), the worst-case pair $\{w, x_0\}$ with respect to J is found as follows:

$$w = \bar{K}_* \xi_1, \quad \bar{K}_* = \left[\begin{array}{cc} -26.31483 & -4.74882 \end{array}\right], \tag{37}$$

$$x_0 = \left[\begin{array}{ccc} -0.32886 & 0.08162 & 1.50212 \end{array}\right]^\top. \tag{38}$$

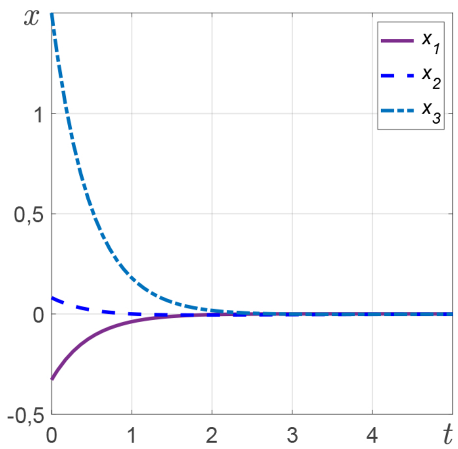

Figure 2: Behavior of a closed-loop system.

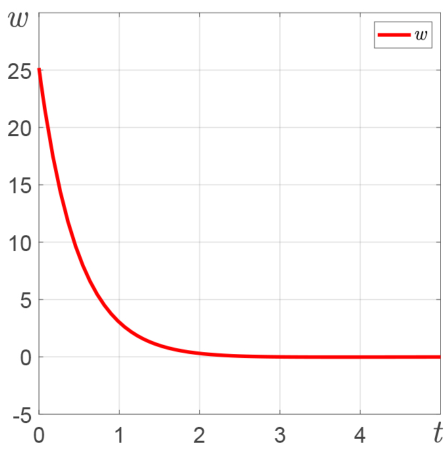

Figure 3: The worst-case perturbation with respect to J.

Fig.2 shows the behavior of the solution of the closed-loop system under the worst-case conditions (37) and (38), and Fig.3 shows the behavior of the worst-case disturbance (37).

Next, applying Algorithm 3.1, the matrices of the approximate J-optimal dynamic controller (30) of the order $p = 2$ are found for system (27) as follows:

$$\left[\begin{array}{cc} K & U \\ V & Z \end{array}\right] = \left[\begin{array}{cc|cc} 0.16824 & -2.24725 & -0.00072 & -0.15173 \\ \hline -0.00256 & 0.00014 & -0.00063 & 0.00259 \\ -0.10392 & 0.09342 & -0.01008 & -0.77037 \end{array}\right],$$

for which the closed-loop system (36) is admissible with the finite spectrum

$$\big\{ -0.72210 \pm 0.30576\, i, \ -0.77143, \ -0.00067 \big\}$$

and has the minimum value of the performance measure $J = 0.28356$.

5 Conclusion

Constructive methods for evaluating and achieving the desired damping level of external and initial disturbances in descriptor control systems have been developed. The practical implementation of these methods is based on the equivalent transformation of descriptor systems and application of well-known methods of H_∞ control theory for ordinary lower-order systems. Thus, the existence conditions and algorithms for constructing a dynamic controller of the order $p = \operatorname{rank} E$, for which the closed-loop system is admissible with weighted performance measures $J_0 < \gamma$ or $J < \gamma$, reduce to solving LMIs without additional rank constraints. In the case, when the original descriptor system is not impulse-free, it is proposed to search for an additional control that provides the specified property of this system. The equivalent transformation of the descriptor system to the ordinary one was also applied to find the worst-case external and initial disturbances with respect to the weighted performance measures. Studying the behavior of a closed-loop system under such worst-case conditions can be important in the design and testing of real controlled objects.

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