Nonlinear Dynamics and Systems Theory, **25** (4) (2025) 424–433

Bright and Dark Solitons via Homoclinic Dynamics in Helmholtz-Type DNLS Equations

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Received: October 29, 2024; Revised: July 13, 2025

Abstract: The existence of homoclinic orbits in a dynamical system has interesting consequences for its behavior. This is the case in this paper, where we present a model of the discrete nonlinear Schrödinger equation under the Helmholtz operator. We give the fundamental theorem of the existence of a homoclinic (heteroclinic) orbit for a particular class of reversible planar maps. Homoclinic structures are known to be sources of sensitivity that, under small perturbations, can bifurcate solutions. The problem of the existence of solitons has therefore been replaced by that of the existence of homoclinic solutions. We prove the existence of bright and dark solitons in a certain case of nonlinearity.

Keywords: *discrete Schrödinger equation; Helmholtz operator; homoclinic orbits; heteroclinic orbits; reversible planar maps.*

Mathematics Subject Classification (2020): 35Q55, 35Q51, 37K60, 70K44, 93-02.

1 Introduction

Over the last decade, the existence of discrete solitons in DNLS equations has become a hot topic of many studies, to mention just a few, refer to [7, 11–13, 15–17]. These include variational methods, central manifold reduction, and the Nehari manifold approach. A good number of these papers take into account DNLS equations with constant coefficients, and their conclusions have been presented in [7, 12, 15, 16, 19]. DNLS equations with periodic coefficients have recently appeared in the physics literature, and this phenomenon can be identified by numerical simulations [11, 13].

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The existence of bright solitons in different cases was then examined using Melnikov's method, assuming a small perturbation, and for the anti-integrability method [1], some localized solutions persist for weak coupling cases. In [6], the variational approach can also be used, but the allowed frequency region cannot be explicitly determined by the variational method. We are looking at the homoclinic orbit approach to the existence of soliton solutions of DNLS equations used in our paper and in [16], it is precisely a generalization of the work in [7]. Homoclinic structures are recognized as sources of sensitivity which, under small perturbations, can bifurcate solutions. The existence of homoclinic orbits in a dynamical system has interesting consequences for its behavior. The problem of the existence of solitons has therefore been replaced by the problem of the existence of homoclinic solutions. However, this approach yields the frequency Ω and the related sequence x_n simultaneously, and therefore the interval of existence of the frequency Ω. Discrete Helmholtz equations are closely related to discrete Schrodinger equations, which appear naturally in the tight-binding model of electrons in crystals [2]. Similar equations also appear in the case of studies involving time harmonic elastic waves in lattice models of crystals [3], see for example, [14], especially in the case d = 2.

We consider spatially localized standing waves for the discrete nonlinear Schrödinger equation (DNLS):

$$\dot{\psi}_n = -H\psi_n - h(\mid \psi_n \mid)\psi_n, n \in \mathbb{Z},$$

,

$$H\psi_n = \frac{1}{w_n}(\psi_{n+1} + \psi_{n-1} + d_n\psi_n),$$

where $w_n > 0, d_n \in \mathbb{R}$, and $(w_n w_{n+1})^{-1}, w_n^{-1} d_n$ are bounded sequences. It gives rise to an operator H, called Helmholtz operator [18], in the weighted Hilbert space $l^2(\mathbb{Z}; w)$ with scalar product:

$$\langle f,\ g\rangle = \textstyle\sum_{n\in\mathrm{Z}} w_n \overline{f_n} g_n\ ,\ f,\ g \in l^2(\mathbb{Z}; w)\ .$$

There is an interesting link between the Jacobi and Helmholtz operators. in [18] (Theorem 1.14, page 21).

Use the stationary wave ansatz

$$\psi_n = x_n \exp(-i\omega t),$$

where x_n is a sequence with real values and $\omega \in \mathbb{R}$.

We impose the following boundary condition at infinity: $\lim_{n\to\pm\infty} u_n = 0$, and we are looking for non-trivial solutions, i.e the solutions that are not equal to 0.

The objective of this paper is to explore the existence of homoclinic solutions for a given class of periodic difference equations.

We use the symmetry properties of reversible planar maps to improve the homoclinic orbit approach. The results of the existence of the soliton of the discrete Helmholtz-Schrodinger equation will not be obtained by the variational method or the anti-integrability method.

This paper is structured as follows. In the second Section, we outline some basics about reversible planar maps and homoclinic (heteroclinic) points. In addition, we give the fundamental theorem for the existence of a homoclinic (heteroclinic) orbit for a particular class of planar maps so that we can prove the existence results rigorously.

In Section 3, we present the conditions for the existence of bright and dark solitons for local solutions of the discrete Schrödinger equations with the Helmholtz operator.

We also examine the existence of soliton solutions for DNLS equations in certain cases of nonlinearity.

2 Homoclinic Orbits of Planar Reversible Maps

We will give a mathematical description of time-reversal symmetry in the context of dynamical systems. In the most interesting applications, $\Omega = \mathbb{R}^n$. We are interested only in the diffeomorphism of $\mathbb{R}^{2n}$. Let R be a smooth diffeomorphism satisfying the following conditions:

- $R \circ R =$ identity.
- The dimension of the fixed point set of R, $Fix(R)$, is n.

R is known as inverse involution. A diffeomorphism T is called R-reversible if $R \circ T = T^{-1} \circ R$.

Several periodic points are easy to find; they are called symmetrical periodic points and are characterized by the following proposition.

Proposition 2.1 *[5] Let $p \in Fix(R)$ and suppose that $T^k(p) \in Fix(R)$, and therefore, $T^{2k}(p) = p$, then we have*

$$T^k(p) = RT^k(p) = T^{-k}R(p) = T^{-k}(p), \quad therefore: \quad T^{2k}(p) = p.$$

So, symmetrical periodic points can be geometrically identified; we focus on the self-intersections of the set of fixed points of R under the iteration of T. We might also find homoclinic geometrically reversible diffeomorphism of R-geometrically reversible diffeomorphisms.

Proposition 2.2 *[4] Let $p \in Fix(R)$ be a symmetric fixed point of T and let $W^s(p)$ and $W^u(p)$ denote the stable and unstable manifolds of p, respectively. Then $R(W^u(p)) = W^s(p)$ and $R(W^s(p)) = W^u(p)$. In particular, if $q \in W^u(p) \cap Fix(R)$, then q is a homoclinic point.*

Let $x \in W^u(p)$ such that $\lim\limits_{n \longrightarrow \infty} T^{-n}(x) = p$, and so we have

$$p = R \lim_{n \longrightarrow \infty} (T^{-n}(x)) = \lim_{n \longrightarrow \infty} T^n(R(x)).$$

We have $R(x) \in W^s(p)$, where $RW^u(p) \subset W^s(p)$. We also have $RW^s(p) \subset W^u(p)$ such that $RW^u(p) = W^s(p)$. If $q \in W^u(p) \cap Fix(R)$. So, $q = R(q) \in W^s(p) \cap Fix(R)$ also, q is a homoclinic point [4].

Hence, to generate homoclinic points for reversible diffeomorphisms, it is sufficient to find the intersections of $W^u(p)$ with $Fix(R)$. We note that both of these propositions are valid in much more general terms. Homoclinic points which are also in $Fix(R)$ are described as symmetric homoclinic points. Homoclinic points are called regular homoclinic points if the unstable variety (and hence the stable variety) intersects $Fix(R)$ transversely at the homoclinic point.

Proposition 2.3 *[5] Let p be a symmetric fixed point and let q be a symmetric homoclinic point in $W^u(p)$. Let N be any neighborhood of p in $Fix(R)$. Then there exists an infinite number of periodic symmetric points in N.*

Proposition 2.4 *[4] Let p be a non-symmetric periodic point. Suppose $q \in W^u(p) \cap Fix(R)$. Then $q \in W^u(p) \cap W^s(R(p))$. Thus some heteroclinic points can be found geometrically as symmetric homoclinic points. Regular symmetric heteroclinic points are defined as regular homoclinic points.*

Proposition 2.5 *[4] Assume that T is an R-reversible diffeomorphism on the plane and let p be a nonsymmetric saddle point for T. Assume that a branch of $W^u(p)$ and a branch of $W^s(p)$ intersect. Suppose a branch of $W^s(p)$ intersects $Fix(R)$ transversely. Then there exist infinitely many symmetric periodic orbits entering any neighborhood of p and $R(p)$.*

A reversible class of planar maps is derived from symmetrical differential equations of the form [5, 7]

$$x_{n+1} + x_{n-1} = g(x_n). \tag{1}$$

In this paper we treat the most general case. We consider the difference expression

$$H_n x_n = \tfrac{1}{w_n}(x_{n+1} + x_{n-1} + d_n x_n),$$

where $w_n > 0, d_n \in \mathbb{R}$, and $(w_n w_{n+1})^{-1}, w_n^{-1} d_n$ are bounded sequences. It gives rise to an operator H, called the Helmholtz operator [18], in the weighted Hilbert space $l^2(\mathbb{Z}; w)$ with scalar product:

$$\langle f,\ g\rangle = \textstyle\sum_{n \in \mathrm{Z}} w_n \overline{f_n} g_n\ ,\ f,\ g \in l^2(\mathbb{Z}; w),$$

$$x_{n+1} + x_{n-1} = g(x_n, w_n, d_n, h),$$

which regularly appears in analyses of the stationary state of coupled oscillators in one-dimensional lattices [5] . The system can be expressed as a planar map, given by T, of the form

$$\begin{cases} x_{n+1} = z_n, \\ z_{n+1} = -x_n + g(z_n), \end{cases}$$

i.e.,

$$T(x, z) = (z, -x + g(z)) \quad \textit{and} \quad g_n(x_n) = d_n x_n + \omega_n h(x_n).$$

It is an easy matter to check that T is invertible and

$$\begin{cases} x_{n+1} = -z_n + g(x_n), \\ z_{n+1} = x_n \end{cases}$$

$$T^{-1}(x, z) = (-z + g(x), x).$$

Furthermore, T is a $\mathcal{C}^1$ diffeomorphism if g is $\mathcal{C}^1$. $g_n(x)$ is nonlinear and continuous at x. We have $g_{n+P}(x) = g_n(x)$ for all $n \in \mathbb{Z}$. In this work, we always suppose that g is a $\mathcal{C}^1$ function. We see that T is R_1-reversible with respect to the involution $R_1(x, z) = (z, x)$, and R_2-reversible with respect to the involution $R_2(x, z) = (-z, -x)$ since g is an odd function.

$$R_1 \circ T(x_n, z_n) = R_1(z_n, -x_n + g(z_n)) = (-x_n + g(z_n), z_n),$$

$$T^{-1} \circ R_1(x_n, z_n) = T^{-1}(z_n, x_n) = (-x_n + g(z_n), z_n).$$

Note that the fixed-point sets $Fix(R_1)$ and $Fix(R_2)$ are indicated by the lines $z = x$ and $z = -x$, denoted by S_1 and S_2, respectively. Let $d = \min_{n \in \mathrm{Z}} d_n > 1$, $f(z) = g(z) - dz$ and we fix $w = w_n > 0$.

Theorem 2.1 *Suppose that*

1. *$f(z)$ is a $\mathcal{C}^1$ and odd function, and has three real zeros, $-z_0$, 0 and z_0 ($z_0 > 0$), with $f'(0) > 0$.*

2. *$sup_{z \geq z'}((d-2)z + wf(z) < 0$ for given $z' > z_0$.*
Then the planar map T has a homoclinic orbit.

Proof. Because f as an odd function has three different real zeros, we can suppose that its real zeros are $-z_0, 0$ and z_0 with $z_0 > 0$. The planar map T has three fixed points $P(-z_0, -z_0), O(0,0)$ and $Q(z_0, z_0)$, all of which are symmetrical with the involution R_1. The origin O is hyperbolic if $f‘(0) > 0$. In addition, the unstable manifold $W^{\mathrm{u}}(O)$ and the stable manifold $W^{\mathrm{s}}(O)$ are tangent to the lines $z = \lambda_2 x$ and $z = \lambda_1 x$, respectively, where $\lambda_2 > 1$ and $0 < \lambda_1 < 1$ are eigenvalues of the Jacobian matrix of T at the origin. We first prove that the intersection of $W^{\mathrm{u}}(O)$ with the interior of the segment EQ is non-empty, where $E(0, z_0)$ lies on the z-axis. It is simple to verify that a branch of $W^{\mathrm{u}}(O)$ initially enters the interior of the triangle $\triangle OEQ$, noted by $int(\triangle OEQ)$. For any point $A(x,z) \in int(\triangle OEQ)$, When $0 < x < z < z_0$, the coordinates of the image point $T(A)$ are $(z, -x + dz + wf(z))$. Furthermore, since $f(z)$ is positive and $d \geq 0$ for $z \in (0, z_0)$, the distance between the point $T(A)$ and the line S_1 is $\frac{\sqrt{2}}{2}(wf(z) - x + (d-1)z)$, greater than the distance from A to S_1. Thus, the unstable manifold $W^{\mathrm{u}}(O)$ inside $\triangle OEQ$ never intersects the segments OE and OQ. In the next section, we show by contradiction that $W^{\mathrm{u}}(O)$ intersects the segment EQ.

Suppose that the branch of $W^{\mathrm{u}}(O)$ in the first quadrant always lies inside $\triangle OEQ$. Consider a point $B \in W^{\mathrm{u}}(O) \cap int(\triangle OEQ)$. Then all the image points $T^n(B) \in int(\triangle OEQ)$ for $n = 1, 2, \cdots$. In addition, the sequences of x-coordinates and z-coordinates of $T^n(B)$ are at the same time strictly increasing and bounded above, and therefore converge to x^* and z^*, respectively. Consequently, the sequence of points $T^n(B)$ is convergent to $N(x^*, z^*)$, which is a fixed point of T. Based on the facts that $x^* > 0$ and $z^* > 0$, it thus follows that $N = Q$. On the other part, the sequence of the distance between $T^n(B)$ and S_1 is also strictly increasing, implying that $N \neq Q$, there is a contradiction. Consequently, the unstable manifold $W^{\mathrm{u}}(O)$ pierces the segment EQ. Secondly, we show that $W^{\mathrm{u}}(O)$ in the first quadrant meets the line S_1 at some point. We note $H_0(x_0, z_0)$, the intersection point of $W^{\mathrm{u}}(O)$ with the segment EQ. Let $H_{n+1} = T(H_n), n = 0, 1, \cdots$. The coordinates of H_n are (x_n, z_n). It then follows that $z_1 = -x_0 + dz_0 + wf(z_0) = z_0 + ((d-1)z_0 + x_0) > z_0$. Since $f(z) < 0$ for $z > z_0$, we derive from assumption (ii) that $\sup_{z > z_1}((d-2)z + wf(z) < 0$.
We note

$$\sup_{z \geq z_1}((d-2)z + wf(z)) < 0, \quad so \quad \sup_{z \geq z_1}(d-2)z + wf(z) = -a, \quad (a > 0).$$

Suppose that $W^{\mathrm{u}}(O)$ in the first quadrant does not cross the line S_1. Then $W^{\mathrm{u}}(O)$ is between the z-axis and the line S_1. So, the points H_n are above the line S_1, meaning that $z_{n+1} > x_{n+1} = z_n > x_n = \cdots = z_1 > x_1 = z_0$, and $(d-2)z_n + wf(z_n) \leq -a$ for $n = 1, 2, \cdots$. Consider d_n as the distance between H_n and the line S_1. Then

$$dist_n = \frac{\sqrt{2}}{2}(z_n - x_n) = \frac{\sqrt{2}}{2}(z_n - z_{n-1}), n = 0, 1, (z_{-1} = x_0).$$

Let $z_{n+1} = -x_n + dz_n + wf(z_n)$, so $z_{n+1} - z_n = z_n - z_{n-1} + (d_n - 2)z_n + wf(z_n)$.

Therefore, $\sqrt{2}dis_{n+1} = \sqrt{2}dis_n + (d-2)z_n + wf(z_n), n = 0, 1, \ldots$. It follows that

$$\sqrt{2}dis_1 = \sqrt{2}dis_0 + (d-2)z_0,$$

$$\sqrt{2}dis_2 = \sqrt{2}dis_1 + (d_1-2)z_1 + wf(z_1),$$
$$\sqrt{2}dis_{n+1} = \sqrt{2}dis_n + (d_n-2)z_n + wf(z_n)$$

and hence

$$0 \leq \sqrt{2}dis_{n+1} = \sqrt{2}dis_0 + \sum_{i=1}^{n}[(dis_i-2)z_i + wf(z_i)] \leq \sqrt{2}dis_0 - na.$$

Let $n \to \infty$, we obtain a contradiction. As a result, the intersection of $W^{\mathrm{u}}(O)$ with the line S_1 is non empty. From Proposition 2.2, it follows that $W^{\mathrm{u}}(O)$ and $W^{\mathrm{s}}(O)$ intersect at a point q on S_1, which means that a homoclinic orbit exists. □

Let (x_0, x_0) be the point of intersection of $W^{\mathrm{u}}(O)$ and S_1. So, the homoclinic orbit $(x_n, z_n) = T^n((x_0, x_0))$ in the first quadrant has the following property: $x_n = z_{-n}$ and $x_{-n} = z_n$ for $n \geq 1$.

From the homoclinic orbit, we derive a sequence $\{x_n\}$ that satisfies (1) and $x_n \to 0$ exponentially as $n \to +\infty$ or $-\infty$.

Theorem 2.2 *Suppose that*
(i) $f(z)$ is a $\mathcal{C}^1$ and odd function, and $f(z) + 2dz$ has only three real zeros, $-z_0, 0,$ and $z_0 (z_0 > 0)$ with $f'(0) < -2d$.
(ii) $inf_{z \geq z'}(\{wf(z) + 2dz\}) > 0$, for some $z' > z_0$.
Therefore the planar map T has a homoclinic orbit.

Proof. Note first that we obtain the following symmetry if x_n satisfies the difference equation

$$wf(x_n) = x_{n-1} + x_{n+1} - dx_n, \tag{2}$$

then $\{y_n = (-1)^n x_n\}$ is a solution of the difference equation. We have $g(x_n) = x_{n-1} + x_{n+1}$. So, if n is even, we get,

$$\begin{cases} y_n = (-1)^n x_n, \\ y_{n+1} = (-1)^{n+1} x_{n+1}, \\ y_{n-1} = (-1)^{n-1} x_{n-1}. \end{cases}$$

Therefore

$$\begin{cases} y_n = x_n, \\ y_{n+1} = -x_{n+1}, \\ y_{n-1} = -x_{n-1}. \end{cases}$$

From (2), we can find

$$\begin{aligned} w\widehat{f}(y_n) &= -y_{n+1} - y_{n-1} - dy_n, \\ &= -g(y_n) - dy_n, \\ &= -wf(y_n) - d + y_n - dy_n, \\ &= -wf(y_n) - 2dy_n. \end{aligned}$$

Hence, $w\widehat{f}(z) = -wf(z) - 2dz$ and vice versa. Assumptions (i) and (ii) are satisfied for $\widehat{f}(z)$. It follows that the planar application T induced has a homoclinic orbit, implying the existence of a homoclinic orbit for the planar application T. □

From Theorem 2.2, we derive a sequence $\{x_n\}$ that satisfies (1), $sign(x_n) = -sign(x_n)$ and $x_n \to 0$ exponentially as $n \to +\infty$ or $-\infty$.

Theorem 2.3 *Suppose that $f(z)$ is a $\mathcal{C}^1$ and odd function, and admits three real zeros,$-z_0, 0$ and $z_0(z_0 > 0)$ with $f'(z_0) > 0$. Therefore, the planar application T has a heteroclinic orbit.*

Proof. The reversible map T has three fixed points, two of which, $P(-z_0, -z_0)$ and $Q(z_0, z_0)$, are hyperbolic if $f'(z_0) > 0$. Similarly to the proof of Theorem 3.1, one can verify that $W_u(Q)$ intersects the $x-$axis at $H(x, 0)$ with $0 < x < z_0$. Simple calculations show that $T(H)$ and H are symmetric with respect to S_2. Then the intersection of $W_u(Q)$ with S_2 is nonempty. Consequently, from Proposition 2.2, it follows that the intersection of $W^u(Q)$ with $W^s(P)$ is nonempty, and hence the planar map T has a heteroclinic orbit.

From Theorem 2.3, we derive a sequence $\{x_n\}$ that satisfies (1) and $x_n \to z_0$ as $n \to +\infty$ and $x_n \to -z_0$ as $n \to -\infty$.

The proof of the present theorem is the same as that of Theorem 2.2.

Theorem 2.4 *Suppose that $f(z)$ is an odd $\mathcal{C}^1$ function, and $f(z)+2dz$ has only three real zeros, $-z_0, 0$ and $z_0(z_0 > 0)$ with $f'(z_0) < -2d$, Therefore, the planar application T has a heteroclinic orbit.*

The conclusion of Theorem 2.4, implies the existence of a solution $\{x_n\}$ that satisfies (1), with the property that $\text{sign}(x_n) = -\text{sign}(x_{n+1})$ as $|x_n| \to z_0$.

3 The DNLS Equations with Helmholtz Operator and General Nonlinearities

In this section, we investigate the DNLS equations with the Helmholtz operator and general nonlinearities

$$i\frac{\partial \psi_n}{\partial t} + h(|\ \psi_n\ |)\psi_n + \frac{1}{w_n}(\psi_{n+1} + \psi_{n-1} - d_n\psi_n) = 0, \tag{3}$$

where h is a C^1 function. Great attention has been paid to localized solutions of the form $\psi_n = x_n e^{-i\Omega t}$, where x_n are time independent. Such solutions are time periodic and spatially localized. The result is a difference equation

$$-\Omega x_n + h(|\ x_n\ |)x_n + \frac{1}{w_n}(x_{n+1} + x_{n-1} - d_n x_n) = 0,$$

$$g_n(x_n) = x_{n+1} + x_{n-1},$$

$$x_{n+1} + x_{n-1} = [\omega_n(\Omega - h(|\ x_n\ |)) + d_n]x_n,$$

$$f(z) = [\omega(\Omega - h(|\ z\ |) + d]z - dz,$$

$$f(z) = \omega(\Omega - h(|\ z\ |))z.$$

Theorem 3.1 *1. Assume that h is strictly increasing in $[0,+\infty[$. Then there exists an unstaggered (staggered) bright solitons of the form $x_n e^{i\Omega t}$ with $h(0) < \Omega < h_\infty$ ($h(0) - 2d/w < \Omega < h_\infty - 2d/w$) for the system (3) with $w > 0$.*

2. Assume that h is strictly decreasing in $[0,+\infty[$. So there are bright solitons of the form $x_n e^{i\Omega t}$ with $h_\infty < \Omega < h(0)$ for the system (3) with $w < 0$.

Proof. Assume that h is strictly increasing and $w > 0$. Then it follows that $f(z)$ has only three zeros if $h(0) < \Omega < h_\infty$ and $f'(0) = (\Omega - h(0))/w < 0$ for $w > 0$. Consequently, the system (3) admits solutions of bright solitons by Theorem (2.1). Similarly, the other cases can be proved by Theorem 2.1.

Theorem 3.2 *Assume that $h'(r) > 0$ (< 0) for $r \in [0,+\infty[$. Then, there exist dark solitons of the form $x_n e^{i\Omega t}$ with $h(0) < \Omega < h_\infty$ ($h_\infty < \Omega < h(0)$) for the system (3) with $w < 0$ (> 0).*

Proof. The proof is obvious by Theorem 2.3.

We are interested in the possibility of finding non-trivial homoclinic solutions for (3). This problem comes up when we look for the discrete solitons of the periodic equation DNLS if

$$h(\mid \psi_n \mid) = \frac{\sigma \chi_n \mid \psi_n \mid^2}{1 + c_n \mid \psi_n \mid^2},$$

where $\sigma = \pm 1$, the given sequences χ_n, c_n are assumed to be T-periodic and positive. The DNLS with saturable nonlinearities can be used to describe the propagation of optical pulses in different doped fibers [9] and have been reviewed in [10]. Being spatially localized and temporally periodic solutions, the solitons decay to zero at infinity. Suppose x_n is a real valued sequence and Ω is the temporal frequency. In this case, (3) becomes

$$-\Omega x_n + \frac{\sigma \chi_n x_n^2}{1 + c_n x_n^2} x_n + \frac{1}{w_n}(x_{n+1} + x_{n-1} - d_n x_n) = 0. \tag{4}$$

The problem on the existence of solitons of (3) has therefore been replaced by the problem on the existence of homoclinic solutions of (4). Pankov [15] in 2005, considered a special case with $h(x_n) = \sigma \chi_n x_n^2$, then posed an open problem on the existence of gap solitons for asymptotically linear nonlinearities as in (4).

The existence of bright soliton solutions of type $x_n e^{-i\Omega t}$ has been studied by the variational method in [8]. The frequency Ω related to the sequence x_n, in which x_n is a minimiser for a variational method. Therefore, one must solve a variational problem first to obtain a minimizer, and then to derive the associated frequency. Thus, one cannot explicitly derive the allowed region of the frequency Ω by the variational method. This approach, however, yields the frequency Ω and the related sequence x_n simultaneously, and therefore one can obtain the interval of existence of the frequency Ω.

$h(x_n) = \sigma \chi_n x_n^2$ is strictly increasing in $[0,+\infty)$ and $h(0) = 0$, $h_\infty = \infty$. It follows that the DNLS equation is studied in one-dimensional lattice:

$$i\frac{\partial \psi_n}{\partial t} + \sigma \chi_n \psi_n^3 + \frac{1}{w_n}(\psi_{n+1} + \psi_{n-1} - d_n \psi_n) = 0. \tag{5}$$

Then, there exists a unstaggered (staggered) bright soliton of the form $x_n e^{i\Omega t}$ with $h(0) < \Omega < h_\infty$ ($h(0) - 2d/w < \Omega < h_\infty - 2d/w$) for the system (3) with $w > 0$.

The DNLS equation with saturable non-linearity is

$$i\frac{\partial\psi_n}{\partial t}+\frac{\sigma\chi_n\psi_n^2}{1+c_n\psi_n^2}x_n++\frac{1}{w_n}(\psi_{n+1}+\psi_{n-1}-d_n\psi_n)=0. \tag{6}$$

Comparing with (3), one has that $h(r)=\frac{\sigma\chi_n r^2}{1+c_n r^2}$ for r positive. Then

$$h'(r)=\frac{\sigma\chi_n 2r}{(1+c_n r^2)^2}.$$

We can see that h is strictly increasing in $[0,+\infty)$ and $h(0)=0$, $h_\infty=\infty$.

4 Conclusion

A model of a discrete nonlinear Schrodinger equation has been presented. The existence of bright soliton solutions has been studied for a discrete Schrodinger equation under the Helmholtz operator by the reversible systems approach and not by the variational method or the anti-integrability method. Chaos is often linked to homoclinic orbits in nonlinear determination dynamics. Recently, DNLS equations with periodic coefficients have been addressed in the physics literature. Future work will address the existence of homoclinic solutions for a class of periodic difference equations with saturable nonlinearity. This gives rise to a more general Jacobi operator using the method of reversible systems.

References

[1] S. Aubry. Anti-integrability in dynamical and variational problems. *Physica D: Nonlinear Phenomena* **86** (1-2) (1995) 284–296.

[2] W. A. Harrison. *Electronic Structure and the Properties of Solids: The Physics of the Chemical Bond*. Courier Corporation, 2012.

[3] L. Brillouin. Wave Propagation in Periodic Structures: Electric Filters and Crystal Lattices. In: *Dover Publications*. Mineola, New York, 1953, 80099–6.

[4] R. L. Devaney. Homoclinic bifurcations and the area-conserving Hénon mapping. *Journal of differential equations* **51** (2) (1984) 254–266.

[5] J. S.W. Lamb and J. A.G. Roberts. Time-reversal symmetry in dynamical systems: a survey. *Physica-Section D* **112** (1-2) (1998) 1–39.

[6] A. Pankov and N. Zakharchenko. On some discrete variational problems. *Acta Applicandae Mathematica* **65** (1) (2001) 295–303.

[7] W. X. Qin and X. Xiao. Homoclinic orbits and localized solutions in nonlinear Schrödingerr lattices. *Nonlinearity* **20** (10) (2007) 2305.

[8] M. I. Weinstein. Excitation thresholds for nonlinear localized modes on lattices. *Nonlinearity* **12** (3) (1999) 673.

[9] S. Gatz and J. Herrmann. Soliton propagation in materials with saturable nonlinearity. *JOSA B* **8** (11) (1991) 2296–2302.

[10] A. Pankov and V. Rothos. Periodic and decaying solutions in discrete nonlinear Schrödinger with saturable nonlinearity. *Proceedings of The Royal Society A: Mathematical, Physical and Engineering Sciences* **464** (2100) (2008) 3219–3236.

[11] J. Yang and G. Chen. Periodic discrete nonlinear schrödinger equations with perturbed and sub-linear terms. *Journal of Applied Analysis and Computation* **12** (6) (2022) 2220–2229.

[12] B. X. Zhou and C. Liu. Homoclinic solutions of discrete nonlinear Schrödinger equations with unbounded potentials. *Applied Mathematics Letters* **123** (2022) 107575.

[13] Z. Wang, Y. Hui and L. Pang. Gap solitons in periodic difference equations with sign-changing saturable nonlinearity. *AIMS Mathematics* **7** (10) (2022) 18824–18836.

[14] R. Novikov and B. L. Sharma. Inverse source problem for discrete Helmholtz equation. *arXiv preprint arXiv* **2401** (14103) (2024).

[15] A. Pankov. Gap solitons in periodic discrete nonlinear Schrödinger equations. *Nonlinearity* **19** (1) (2005) 27.

[16] A. Mehazzem, M. S. Abdelouahab and K. Haouam. Homoclinic Orbits and Localized Solutions in Discrete Nonlinear Schrodinger Equation with Long-Range Interaction. *International Journal of Nonlinear Analysis and Applications* **13**(1) (2022) 353–363.

[17] J. Kuang and Z. Guo. Homoclinic solutions of a class of periodic difference equations with asymptotically linear nonlinearities.*Nonlinear Analysis: Theory, Methods and Applications* **89** (2013) 208–218.

[18] T. Gerald. *Jacobi operators and completely integrable nonlinear lattices*. American Mathematical Soc. (72) (2000).

[19] M. U. Uddin, M. A. Nishu and M. W. Ullah. Nonlinear Damped Oscillator with Varying Coefficients and Periodic External Forces. *Nonlinear Dynamics and Systems Theory* **23** (2) (2023) 227–236.