Nonlinear Dynamics and Systems Theory, **25** (4) (2025) 434–444

Existence Results for a Class of Hybrid Fractional Differential Equations Involving Generalized Riemann-Liouville Fractional Derivatives

Ibtissem Merzoug [1*] and Esma Kenef [2]

[1] *Laboratory of Numerical Analysis, Optimisation and Statistics, Badji Mokhtar - Annaba University, 12, P.O.BOX, 23000 Annaba, Algeria.*
[2] *Laboratory of Physical Chemistry and Biology of Materials, Higher Normal School of Technological Education, Skikda, 21000, Algeria.*

Received: July 9, 2024; Revised: July 13, 2025

Abstract: This research paper deals with the uniqueness of solutions for a second-type hybrid fractional differential equation that involves generalized Riemann-Liouville fractional derivatives using the Banach contraction principle. We also discover at least one solution by employing certain assumptions and the Schaefer fixed point theorem. Subsequently, the Ulam–Hyers stability is discussed. Finally, we enhance our study with a relevant example.

Keywords: *hybrid fractional differential equations, generalized Riemann-Liouville fractional derivatives, existence and uniqueness of solution, Ulam-Hyers stability.*

Mathematics Subject Classification (2020): 34A12, 34K37, 93D05, 70k20.

1 Introduction

Fractional differential equations (FDEs) are a fascinating area of mathematics dealing with derivatives of non-integer order and allowing for a more nuanced description of systems with memory effects or long-range interactions. Solving FDEs can be challenging due to the non-integer order of the derivatives, requiring specialized techniques such as fractional calculus. In general, fractional differential equations provide a powerful tool for understanding complex systems with given dynamics [1, 6, 7, 9]. Indeed, though the operations of FDEs are relatively broad, they can not be applied to all systems. The researchers have shown that certain phenomena related to material heterogeneity cannot be adequately modeled using fractional derivatives. In view of this fact, a solution to this

* Corresponding author: mailto:ibtissem.merzoug@univ-annaba.dz

problem was proposed by Caputo in 1967, who introduced a fractional derivative allowing the application of initial conditions with physical meaning. In his researches, new FDEs are defined, called generalized fractional derivatives, for a more extensive collection of fractional calculus.

On the other hand, in the realm of generalized FDEs, the existence and uniqueness of solutions play a vital role in ensuring the validity and reliability of the mathematical problem. The investigation of the existence and uniqueness of solutions for differential equations involving the generalized fractional derivative has been undertaken by numerous researchers (see [3, 10, 13] and the references therein). Furthermore, the stability theory for FDEs has been a significant area of research. In particular, the Ulam-Hyers stability is attracting attention due to its importance in understanding the behavior of dynamic problems. It is essential to predict the long-term evolution and stability of solutions in different applications, making it a key focus in mathematics and science [11, 12]. Many researchers focused on developing the methods of solution of the hybrid fractional differential equations by using different kinds of fixed point theorems, for example, in [2], the researchers studied the existence of solutions for hybrid fractional integral differential equations, involving the generalized Caputo derivative. They used the hybrid fixed point theorem for some of three operators due to Dhage for proving the main results.

This paper is devoted to the study of the existence, uniqueness and stability of solutions for the following second-type hybrid fractional differential equation involving the generalized Riemann-Liouville fractional derivatives:

$$\begin{cases} D_{0^+}^{\alpha;\phi}\left(u(t)-f(t,u(t))\right)+g\left(t,u(t),D_{0^+}^{\alpha;\phi}\left(u(t)-f(t,u(t))\right)\right)=0, \quad t\in J=[0,1],\\ \lim\limits_{t\to 0}(\phi(t)-\phi(0))^{2-\alpha}(u(t)-f(t,u(t)))=0,\\ u(1)=\omega+f(1,u(1)), \qquad \omega\in\mathbb{R}. \end{cases} \tag{P}$$

where $D_{0^+}^{\alpha;\phi}$ is the ϕ -Riemann-Liouville fractional derivative with $1<\alpha<2$.
$f\in C(J\times\mathbb{R},\mathbb{R})$ and $g\in C(J\times\mathbb{R}^2,\mathbb{R})$ are non-linear functions. The function $\phi: J\to\mathbb{R}$ is a strictly increasing function such that $\phi\in C^2(J,\mathbb{R})$ and $\phi'(t)\neq 0$ for all $t\in J$.

The structure of the paper is outlined as follows. Section 2 provides a detailed overview of the foundational concepts and definitions that are pertinent to our investigation. In Section 3, we convert the differential problem into equivalent integral equations via constructing the Green function. Then we establish certain properties for it and we assume some sufficient conditions through which we prove the existence of the solution using Schaefer's fixed point theorem and the uniqueness of the solution using the Banach fixed point theorem. We also study the stability of this solution. Finally, the paper concludes with a practical example to give a clear demonstration of the concepts that are discussed.

2 Notational Preliminaries

Here, we recall some useful definitions, theorems, and lemmas, which play an important role in the results of the paper.

Definition 2.1 [2] Let $f:[a,b]\to\mathbb{R}$ be an integrable function and $\phi:[a,b]\to\mathbb{R}$ be an increasing function such that for all $t\in[a,b]$, $\phi'(t)\neq 0$. The left-sided ϕ-Riemann-Liouville fractional integral of a function f is defined as follows:

$$I_{a+}^{\alpha,\phi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha-1} f(s)\, ds.$$

Definition 2.2 [2] Let $n = [\alpha] + 1$. The left-sided ϕ-Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function f corresponding to the ϕ-Riemann-Liouville fractional integral is defined as follows:

$$D_{a^+}^{\alpha,\phi} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\phi'(t)} \frac{d}{d(t)} \right)^n \int_a^t \phi'(s)(\phi(t) - \phi(s))^{n-\alpha-1} f(s) ds.$$

Lemma 2.1 *[4, 13] Let $z : J \to \mathbb{R}$ with $1 < \alpha < 2$, then*

- $I_{0^+}^{\alpha,\phi} D_{0^+}^{\alpha,\phi} z(t) = z(t) + C_0(\phi(t) - \phi(0))^{\alpha-1} + C_1(\phi(t) - \phi(0))^{\alpha-2}$, *where* $C_0, C_1 \in \mathbb{R}$.
- $D_{0^+}^{\alpha,\phi} I_{0^+}^{\alpha,\phi} z(t) = z(t)$.

Definition 2.3 *[11].The problem* (P) *is said to be Ulam-Hyers stable (UH stable) if there exists a constant $\Theta > 0$ such that for every function $y \in C(J, \mathbb{R})$ satisfying the inequality*

$$\left| \mathrm{D}_{0^+}^{\alpha,\phi} \left(\mathrm{y(t)} - \mathrm{f(t, y(t))} \right) - \mathrm{g} \left(\mathrm{t, y(t)}, \mathrm{D}_{0^+}^{\alpha,\phi} \left(\mathrm{y(t)} - \mathrm{f(t, y(t))} \right) \right) \right| \leq \varepsilon, \quad \mathrm{t} \in \mathrm{J}, \qquad (1)$$

for each $\varepsilon > 0$, there exists an exact solution $u \in C(J, \mathbb{R})$ of the problem (P) *such that*

$$|y(t) - u(t)| \leq \Theta\varepsilon, \quad t \in J.$$

Remark 2.1 A function $y \in C(J, \mathbb{R})$ is a solution of the inequality (1) if and only if there exists a function $\psi \in C(J, \mathbb{R})$ (which depends on y) such that

1. $|\psi(t)| \leq \varepsilon, \quad t \in J.$

2. $D_{0^+}^{\alpha,\phi} \left(y(t) - f(t, y(t)) \right) = g \left(t, y(t), D_{0^+}^{\alpha,\phi}(y(t) - f(t, y(t))) \right) + \psi(t), \quad t \in J.$

Theorem 2.1 *(Banach fixed point theorem) [5] Let $\mathbb{E}$ be a non-empty closed subset of a Banach space. Then any contraction mapping A of $\mathbb{E}$ into itself has a unique fixed point, i.e.,*

$$\exists! x \in \mathbb{E} : A(x) = x.$$

Theorem 2.2 *(Schaefer fixed point theorem) [5] Let $\mathbb{E}$ be a non-empty Banach space. Let also $f : \mathbb{E} \to \mathbb{E}$ be a completely continuous mapping. If the set $\chi = \{y \in \mathbb{E} : y = \lambda f(y), 0 < \lambda < 1\}$ is bounded in $\mathbb{E}$, then f admits at least one fixed point in $\mathbb{E}$.*

3 Existence, Uniqueness and Ulam-Hyers Stability Results

The following section is devoted to stating and proving the existence, uniqueness and Ulam stability results for problem (P).

Definition 3.1 The function u from $C(J,\mathbb{R})$ is a solution to the problem (P) if it satisfies the equation

$$D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)=-g\left(t,u(t),D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)\right) \tag{2}$$

and the conditions

$$\lim_{t\to 0}\left[\phi(t)-\phi(0))^{2-\alpha}(u(t)-f(t,u(t))\right]=0, \tag{3}$$

$$u(1)=\omega+f(1,u(1)). \tag{4}$$

Lemma 3.1 *Let $h:J\longrightarrow\mathbb{R}$ be a continuous function. Then u is a solution for the second-type hybrid fractional differential equation*

$$D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)=-h(t),\quad t\in J,$$

and satisfies the conditions (3)-(4) *if and only if u is a solution of the integral equation via the Green function*

$$u(t)=\omega\gamma(t)+f(t,u(t))+\int_0^1 G(t,s)\phi'(s)h(s)ds,\quad t\in J, \tag{5}$$

where

$$G(t,s)=\frac{\gamma(t)}{\Gamma(\alpha)}\begin{cases}(\phi(1)-\phi(s))^{\alpha-1}-\dfrac{1}{\gamma(t)}(\phi(t)-\phi(s))^{\alpha-1}, & 0\le s\le t\le 1,\\ (\phi(1)-\phi(s))^{\alpha-1}, & 0\le t\le s\le 1,\end{cases} \tag{6}$$

with

- $K(t)=\phi(t)-\phi(0)$ *and* $\gamma(t)=\dfrac{(K(t))^{\alpha-1}}{(K(1))^{\alpha-1}}$ *for all* $t\in J$.

Proof. We have u as a solution of the problem (P),

$$I_{0^+}^{\alpha,\phi}\left(D_{0^+}^{\alpha,\phi}\left(u(t)-f(t,u(t))\right)\right)=-I_{0^+}^{\alpha,\phi}(h(t))+C_0(\phi(t)-\phi(0))^{\alpha-1}+C_1(\phi(t)-\phi(0))^{\alpha-2},$$

$$u(t)-f(t,u(t))=-I_{0^+}^{\alpha,\phi}(h(t))+C_0(\phi(t)-\phi(0))^{\alpha-1}+C_1(\phi(t)-\phi(0))^{\alpha-2}.$$

By using the conditions (3)-(4), we obtain $C_1=0$ and

$$C_0=\frac{1}{(\phi(1)-\phi(0))^{\alpha-1}}\left(\omega+\frac{1}{\Gamma(\alpha)}\int_0^1\phi'(s)(\phi(t)-\phi(s))^{\alpha-1}h(s)ds\right).$$

By substitution, we get

$$\begin{aligned}u(t)&=f(t,u(t))+\omega\gamma(t)-\frac{1}{\Gamma(\alpha)}\int_0^t\phi'(s)(\phi(t)-\phi(s))^{\alpha-1}h(s)ds\\&\quad+\frac{\gamma(t)}{\Gamma(\alpha)}\int_0^1\phi'(s)(\phi(1)-\phi(s))^{\alpha-1}h(s)ds\\&=f(t,u(t))+\omega\gamma(t)+\int_0^1 G(t,s)\phi'(s)h(s)ds.\end{aligned}$$

The converse can be easily inferred from Lemma 2.1.

Lemma 3.2 *The following estimates are satisfied by the Green function G defined by equation* (6)*:*
(i) $G(t,s) \le \dfrac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)}$ *for all* $t, s \in J$.
(ii) $G(t,s) \ge 0$ *for all* $t, s \in J$.

Proof.
(i) Since ϕ is a strictly increasing function, we have $\phi(t)-\phi(0) \le \phi(1)-\phi(0)$ whenever $t \in J$,
which implies that $\gamma(t) \le 1$. For $0 \le t \le s \le 1$, we can easily conclude that

$$\frac{\gamma(t)}{\Gamma(\alpha)}(\phi(1)-\phi(s))^{\alpha-1} \le \frac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)},$$

and for $0 \le s \le t \le 1$,

$$\begin{aligned}\frac{\gamma(t)}{\Gamma(\alpha)}\left((\phi(1)-\phi(s))^{\alpha-1} - \frac{1}{\gamma(t)}(\phi(t)-\phi(s))^{\alpha-1}\right) &\le \frac{1}{\Gamma(\alpha)}\Bigg((\phi(1)-\phi(0))^{\alpha-1} \\ &\quad - \frac{(\phi(1)-\phi(0))^{\alpha-1}(\phi(t)-\phi(s))^{\alpha-1}}{(\phi(t)-\phi(0))^{\alpha-1}}\Bigg) \\ &\le \frac{1}{\Gamma(\alpha)}(\phi(1)-\phi(0))^{\alpha-1} \\ &\quad \left(1-\frac{(\phi(t)-\phi(s))^{\alpha-1}}{(\phi(t)-\phi(0))^{\alpha-1}}\right) \\ &\le \frac{1}{\Gamma(\alpha)}(\phi(1)-\phi(0))^{\alpha-1}.\end{aligned}$$

Hence, $G(t,s) \le \dfrac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)}$ for $t, s \in J$.
(ii) By a similar calculation, we can prove that $G(t,s) \ge 0$ for all $t, s \in J$. This completes the proof.

Let us define the operator $\mathcal{T} : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by

$$\mathcal{T}(u(t)) = f(t,u(t)) + \omega\gamma(t) + \int_0^1 G(t,s)\phi'(s)\sigma_u(s)ds$$

with $\sigma_u(t) = g\left(t, u(t), D_{0^+}^{\alpha,\phi}(u(t)-f(t,u(t)))\right)$.
Here, $C(J,\mathbb{R})$ is equipped with the norm

$$\|u\|_\infty = \max_{t\in J}|u(t)|.$$

We note that any fixed point of this operator is a solution to the problem (P).

3.1 Existence results

Assume that the functions $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous and satisfy the following conditions:

(H_1) There exists a constant $\Lambda_g \in \mathbb{R}_+^*$ such that for all $u, v \in \mathbb{R}$ and $t \in J$,

$$|g(t,u,v)| \leq \Lambda_g,$$

(H_2) There exists a constant $\Lambda_f \in \mathbb{R}_+^*$ such that for all $u \in \mathbb{R}$ and $t \in J$,

$$|f(t,u)| \leq \Lambda_f.$$

Theorem 3.1 *We assume that the conditions* $(H_1) - (H_2)$ *are satisfied. Then the problem* (P) *has at least one solution.*

Proof. The proof will be given in four steps.
Step one: $\mathcal{T}$ is continuous. Let (u_n) be a convergent sequence towards $u \in C(J,\mathbb{R})$. Therefore, for all $t \in J$, we have

$$\begin{aligned}
|\mathcal{T}(u_n(t)) - \mathcal{T}(u(t))| &= \Big| f(t,u_n(t)) - f(t,u(t)) + \omega\gamma(t) - \omega\gamma(t) \\
&\quad + \int_0^t G(t,s)\phi'(s)(\sigma_{u_n}(s) - \sigma_u(s))ds \Big| \\
&\leq \Big| f(t,u_n(t)) - f(t,u(t)) \Big| + \int_0^1 G(t,s)\phi'(s) \Big| \sigma_{u_n}(s) - \sigma_u(s) \Big| ds \\
&\leq \Big| f(t,u_n(t)) - f(t,u(t)) \Big| + \frac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \phi'(s) \Big| \sigma_{u_n}(s) - \sigma_u(s) \Big| ds \\
&\leq \|f(t,u_n(.)) - f(t,u(.))\|_\infty + \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)} \|\sigma_{u_n}(.) - \sigma_u(.)\|_\infty.
\end{aligned}$$

Since the functions f and g are continuous, we get

$$\lim_{n\to\infty} \|\mathcal{T}(u_n(.)) - \mathcal{T}(u(.))\|_\infty = 0.$$

Hence, $\mathcal{T}$ is continuous.
Step two: The image of every bounded set of $C(J,\mathbb{R})$ under $\mathcal{T}$ is uniformly bounded in $C(J,\mathbb{R})$. To establish this, it suffices to demonstrate that for any given $r > 0$, there exists a positive constant $l > 0$. Therefore, for every $u \in B_r$, we have $\|\mathcal{T}u(.)\|_\infty \leq l$ with

$$B_r = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq r\}.$$

For every $t \in J$ and by using the conditions (H1) and (H2), we get

$$\begin{aligned}
|\mathcal{T}(u(t))| &\leq |\omega|\gamma(t) + |f(t,u(t))| + \int_0^1 G(t,s)\phi'(s)|\sigma_u(s)|ds \\
&\leq |\omega|\gamma(t) + \Lambda_f + \frac{(\phi(1)-\phi(0))^{\alpha-1}}{\Gamma(\alpha)}\Lambda_g \int_0^1 \phi'(s)ds \\
&\leq |\omega| + \Lambda_f + \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\Lambda_g = l.
\end{aligned}$$

Hence, $\mathcal{T}(B_r)$ is uniformly bounded.
Step three: The image of every bounded set of $C(J,\mathbb{R})$ under $\mathcal{T}$ is an equicontinuous

set in $C(J,\mathbb{R})$. For each $u \in B_r$ and $t_1, t_2 \in J, t_1 < t_2$, we have

$$\begin{aligned}|\mathcal{T}(u(t_2)) - \mathcal{T}(u(t_1))| &= |f(t_2, u(t_2)) - f(t_1, u(t_1)) + \omega(\gamma(t_2) - \gamma(t_1)) \\ &+ \int_0^1 (G(t_2, s) - G(t_1, s))\phi'(s)\sigma_u(s)ds| \\ &\leq |f(t_2, u(t_2)) - f(t_1, u(t_1))| + |\omega||\gamma(t_2) - \gamma(t_1)| \\ &+ \int_0^1 |G(t_2, s) - G(t_1, s)|\phi'(s)\sigma_u(s)ds\end{aligned}$$

and

$$|G(t_2, s) - G(t_1, s)| = \left| \frac{(\phi(1) - \phi(s))^{\alpha-1}}{(\phi(1) - \phi(0))^{\alpha-1}\Gamma(\alpha)} \left[(\phi(t_2) - \phi(0))^{\alpha-1} - (\phi(t_1) - \phi(0))^{\alpha-1}\right] \right.$$
$$\left. + \frac{1}{\Gamma(\alpha)} \left[(\phi(t_1) - \phi(s))^{\alpha-1} - (\phi(t_2) - \phi(s))^{\alpha-1}\right] \right|.$$

By applying the mean value theorem [8], we obtain

$$|G(t_2, s) - G(t_1, s)| = |t_2 - t_1| \left[\frac{(\phi(1) - \phi(s))^{\alpha-1}}{(\phi(1) - \phi(0))^{\alpha-1}\Gamma(\alpha)} h_1(\xi) + \frac{1}{\Gamma(\alpha)} h_2(\theta) \right]$$

with

$$h_1(\xi) = (\alpha - 1)\phi'(\xi)(\phi(\xi) - \phi(0))^{\alpha-2},$$
$$h_2(\theta) = (\alpha - 1)\phi'(\theta)(\phi(\theta) - \phi(s))^{\alpha-2},$$

where $t_1 < \theta, \xi < t_2$. Therefore, as $t_1 \longrightarrow t_2$, $|\mathcal{T}(u(t_2)) - \mathcal{T}(u(t_1))| \longrightarrow 0$.
Hence, by the Arzela-Ascoli theorem, $\mathcal{T}$ is completely continuous.
Step four: We will prove that the set χ is bounded, where

$$\chi = \Big\{u \in C(J,\mathbb{R}) : u(t) = \lambda\mathcal{T}(u(t)), 0 < \lambda < 1\Big\}.$$

Let $u \in \chi$. For all $t \in J$, we have

$$\begin{aligned}u(t) &= \lambda \left[\omega\gamma(t) + f(t, u(t)) + \int_0^1 G(t, s)\phi'(s)\sigma_u(s)ds\right] \\ |u(t)| &< |\omega|\gamma(t) + \Lambda_f + \Lambda_g \int_0^1 G(t, s)\phi'(s)ds \\ &\leq |\omega| + \Lambda_f + \frac{(\phi(1) - \phi(0))^{\alpha}}{\Gamma(\alpha)}\Lambda_g = L.\end{aligned}$$

Hence, χ is bounded. By using Schaefer's fixed point theorem, we found that the problem (P) has at least one solution.

Example 3.1 Consider the problem with the following general fractional differential equations:

$$\begin{cases} D_{0^+}^{\frac{7}{4};\frac{e^t}{7}}(u(t) - f(t, u(t))) + g\left(t, u(t), D_{0^+}^{\frac{7}{4};\frac{e^t}{7}}(u(t) - f(t, u(t)))\right) = 0, t \in J, \\ \lim_{t\to 0}(\phi(t) - \phi(0))^{2-\frac{7}{4}}(u(t) - f(t, u(t))) = 0, \\ u(1) = 1 + f(1, u(1)), \end{cases} \quad \text{(Q)}$$

where

$$f(t,u(t)) = (\frac{1}{2}+t)\cos(u(t)),$$

$$g\left(t,u(t),D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}(u(t)-f(t,u(t)))\right) = (\frac{1}{3}+t)\cos(u(t))+\frac{1}{9}\sin(D_{0^+}^{\frac{7}{4},\frac{e^t}{7}}(u(t)-f(t,u(t)))).$$

Let us put $f(t,u)=\frac{t}{2}\cos(u)$ and $g(t,u,v)=\frac{1}{3}(1+t)\cos(u)+\dfrac{1}{9}\sin(v)$. For $u,v\in\mathbb{R}$ and $t\in J$, we have

$$|f(t,u)|\leq \frac{3}{2},\quad |g(t,u,v)|\leq \frac{7}{9}.$$

We can easily verify all conditions of Theorem 3.1 with $\Lambda_f=\frac{3}{2},\Lambda_g=\frac{7}{9}$. Therefore, we conclude that the problem (Q) has at least one solution.

3.2 Uniqueness results

In what follows, we will establish the existence of a unique solution to the problem (P) using the Banach fixed point theorem under certain conditions imposed on the functions f and g. We impose the following conditions:

(H_3) There exist constants $k_1,k_3\in\mathbb{R}_+^*$ and $k_2\in(0,1)$ such that

$$|g(t,u,v)-g(t,\bar{u},\bar{v})|\leq k_1|u-\bar{u}|+k_2|v-\bar{v}|,$$

$$|f(t,u)-f(t,\bar{u})|\leq k_3|u-\bar{u}|$$

for every $u,v,\bar{u},\bar{v}\in\mathbb{R}$ and $t\in J$.

Theorem 3.2 *We assume that the condition (H_3) is satisfied. If*

$$\Upsilon = k_3+\frac{(\phi(1)-\phi(0))^\alpha k_1}{\Gamma(\alpha)(1-k_2)}<1, \tag{7}$$

then the problem (P) *admits a unique solution in $C(J,\mathbb{R})$.*

Proof. We consider the previously defined operator $\mathcal{T}$ for all $x,y\in C(J,\mathbb{R})$ and $t\in J$. By the condition (H_3), we have

$$\begin{aligned}|\mathcal{T}(x(t))-\mathcal{T}(y(t))| &= |f(t,x(t))-f(t,y(t))+\omega\gamma(t)-\omega\gamma(t)\\ &\quad+\int_0^1 G(t,s)\phi'(s)\left(\sigma_x(s)-\sigma_y(s)\right)ds|\\ &\leq |f(t,x(t))-f(t,y(t))|+\int_0^1 G(t,s)\phi'(s)|\left(\sigma_x(s)-\sigma_y(s)\right)|ds.\end{aligned}$$

Then

$$|\mathcal{T}(x(t))-\mathcal{T}(y(t))|\leq k_3|x(t)-y(t)|+\frac{(\phi(1)-\phi(0))^\alpha}{\Gamma(\alpha)}\int_0^1|\sigma_x(s)-\sigma_y(s)|\,ds. \tag{8}$$

On the other hand,

$$\begin{aligned}|\sigma_x(t)-\sigma_y(t)| &= |g\left(t,x(t),D^{\alpha,\phi}\left(x(t)-f(t,x(t))\right)-g\left(t,y(t),D^{\alpha,\phi}\left(y(t)-f(t,y(t))\right)\right|\\ &\leq k_1|x(t)-y(t)|+k_2|D^{\alpha,\phi}\left(x(t)-f(t,x(t)\right)-D^{\alpha,\phi}\left(y(t)-f(t,y(t)\right)|\\ &\leq k_1|x(t)-y(t)|+k_2\left|\sigma_x(s)-\sigma_y(s)\right|.\end{aligned}$$

Then

$$|\sigma_x(s) - \sigma_y(s)| \leq \frac{k_1}{(1-k_2)}|x(t) - y(t)|. \tag{9}$$

By substituting (9) in (8), we get

$$\begin{aligned} |\mathcal{T}(x(t)) - \mathcal{T}(y(t))| &\leq k_3|x(t) - y(t)| + \frac{(\phi(1) - \phi(0))^\alpha k_1}{\Gamma(\alpha)(1-k_2)}|x(t) - y(t)| \\ &\leq \left[k_3 + \frac{(\phi(1) - \phi(0))^\alpha k_1}{\Gamma(\alpha)(1-k_2)}\right]|x(t) - y(t)|. \end{aligned}$$

Thus,

$$\|\mathcal{T}x(.) - \mathcal{T}y(.)\|_\infty \leq \Upsilon\|x - y\|_\infty.$$

According to (7), the operator $\mathcal{T}$ is a contraction. Then, by Banach's fixed point theorem, it admits a unique fixed point, and it is the unique solution of the problem (P).

Example 3.2 Consider the problem (Q). According to the condition (H_3), we have for $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$,

$$|f(t,u) - f(t,\bar{u})| \leq \frac{1}{2}|u - \bar{u}|,$$

$$|g(t,u,v) - g(t,\bar{u},\bar{v})| \leq \frac{2}{3}|u - \bar{u}| + \frac{1}{9}|v - \bar{v}|.$$

Hence, the satisfaction of the conditions of Theorem (3.2) can be easily checked, and $\Upsilon = 0.5698569 < 1$ with $k_1 = \frac{2}{3}, k_2 = \frac{1}{9}, k_3 = \frac{1}{2}$. Therefore, there exists a unique solution of the problem (Q).

3.3 Ulam-Hyers stability results

Lemma 3.3 *If y is a solution for the following fractional differential inequality:*

$$D^{\alpha,\phi}\left(y(t) - f(t,y(t))\right) + g\left(t, y(t), D^{\alpha,\phi}\left(y(t) - f(t,y(t))\right)\right) < \varepsilon \tag{10}$$

for $\varepsilon > 0$, then y is a solution of the following inequality:

$$|y(t) - \mathcal{T}(y(t))| \leq \frac{(\phi(1) - \phi(0))^\alpha}{\Gamma(\alpha)}\varepsilon. \tag{11}$$

Proof. Let y be a solution of the inequality (11). For $\varepsilon > 0$ and by using Lemma 3.1 and Remark 2.1, $|\psi(t)| < \varepsilon$, $t \in J$, and according to (10), we have

$$y(t) = \omega\gamma(t) + f(t,y(t)) + \int_0^1 G(t,s)\phi'(s)\left[\sigma_y(s) + \psi(s)\right]ds.$$

Then

$$
\begin{aligned}
|y(t)-\mathcal{T}y(t)| =& \Big|\omega\gamma(t)+f(t,y(t))+\int_0^1 G(t,s)\phi'(s)\left[\sigma_y(s)+\psi(s)\right]ds \\
& -\omega\gamma(t)-f(t,y(t))-\int_0^1 G(t,s)\phi'(s)\sigma_y(s)ds\Big| \\
&= \Big|\int_0^1 G(t,s)\phi'(s)\psi(s)ds\Big| \\
&\leq \int_0^1 G(t,s)\phi'(s)|\psi(s)|ds \\
&\leq \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\varepsilon.
\end{aligned}
$$

Theorem 3.3 *We assume that the conditions* (H_3) *and the inequality* (7) *are satisfied. Then the problem* (P) *is Ulam-Hyers stable.*

Proof. Under the condition (H_3) and the inequality (7), there exists a unique solution for the problem (P) in $C(J,\mathbb{R})$. Let $y \in C(J,\mathbb{R})$ be a solution to the inequality (11). Therefore, for $t \in J$, we have

$$
\begin{aligned}
|y(t)-u(t)| =& |y(t)-\omega\gamma(t)-f(t,u(t))-\int_0^1 G(t,s)\phi'(s)\sigma_u(s)ds| \\
&\leq |y(t)-\mathcal{T}(y(t))+\mathcal{T}(y(t))-\mathcal{T}(u(t))| \\
&\leq |y(t)-\mathcal{T}(y(t))|+|\mathcal{T}(y(t))-\mathcal{T}(u(t))| \\
&\leq \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)}\varepsilon+\Upsilon|y(t)-u(t)|.
\end{aligned}
$$

Thus,

$$|y(t)-u(t)| \leq \frac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)(1-\Upsilon)}\varepsilon.$$

We put $\Theta = \dfrac{(\phi(1)-\phi(0))^{\alpha}}{\Gamma(\alpha)(1-\Upsilon)}$, then we get

$$|y(t)-u(t)| \leq \Theta\varepsilon.$$

Therefore, the problem (P) is stable according to Ulam-Hyers.

Example 3.3 *Consider the problem* (Q). *All conditions of Theorem 3.3 hold with* $\Theta = 0.2165387$. *Then the unique solution of the problem* (Q) *is Ulam-Hyers stable.*

4 Concluding Remarks

In this paper, the authors provided some sufficient conditions guaranteeing the existence of solutions for a class of second-type hybrid fractional differential equations involving generalized Riemann-Liouville fractional derivatives of order $1 < \alpha < 2$. We have developed some adequate conditions for the uniqueness of solution. Also, this paper constitutes a successful application of the Ulam-Hyers stability concept to investigate the stability of solutions to this class of problems. The respective results have been verified by providing a suitable example.

References

[1] S. Abbas, M. Benchohra and J. R. Graef. Upper and Lower Solutions for Fractional q-Difference Inclusions. *Nonlinear Dynamics and Systems Theory* **21** (1) (2021) 1–12.

[2] N. Adjimi, M. Benbachir and K. Guerbati. Existence results for (ψ)-Caputo hybrid fractional integro-differential equations. *Malaya Journal of Matematics* **9** (2) (2021) 46–54.

[3] H. Afshari and E. A. Karapinar. Discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces. *Adv. Differ. Equ* **2020** 616 (2020). https://doi.org/10.1186/s13662-020-03076-z.

[4] R. Almeida and A. Caputo. Fractional derivative of a function with respect to an other function. *Communications in Nonlinear Science and Numerical Simulation* **44** (2017) 460–481.

[5] G. Andrzej and D. James. *Fixed Point Theory*. Springer Monographs in Mathematics. New York, Springer-Verlag, 2003.

[6] Z. Bai. On positive solutions of a non-local fractional boundary value problem. *Nonlinear Analysis, Theory, Methods and Applications* **72** (2) (2010) 916–924.

[7] R. Barkat and T. Menacer. Dynamical Analysis, Stabilization and Discretization of a Chaotic Financial Model with Fractional Order. *Nonlinear Dynamics and Systems Theory* **20** (3) (2020) 253–266.

[8] T. M. Flett. "2742. a mean value theorem". *Mathematical Gazette* **42** (339) (1958) 38–39.

[9] R (ed). Hilfer. *Applications of Fractional Calculus in Physics*. World Scientific Publishing Co., Inc., River Edge, NJ. 2000.

[10] R. W. Ibrahim. Generalized Ulam-Hyers stability for fractional differential equations. *Int. J. Math.* **23** (2012) 1250056.

[11] E. Kenef, I. Merzoug and A. Guezane-Lakoud. Existence, uniqueness and Ulam stability results for a mixed-type fractional differential equations with p-Laplacian operator. *Arabian Journal of Mathematics* **12** (3) (2023) 633–645.

[12] A. Samadi, S. K. Ntouyas and J. Tariboon. Nonlocal fractional hybrid boundary value problems involving mixed fractional derivatives and integrals via a generalization of Darboís theorem. Journal of Mathematics **2021** (1) (2021) 6690049.

[13] A. Seemab, M. Ur Rehman, J. Alzabut and A. Hamdi. On the existence of positive solutions for generalized fractional boundary value problems. *Boundary Value Problems* **2019** (2019) 1–20.