Nonlinear Dynamics and Systems Theory, **25** (4) (2025) 445–456

Fractional Nonlinear Reaction-Diffusion System with Gradient Source Terms

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Received: October 14, 2024; Revised: July 21, 2025

Abstract: Over the years, partial reaction-diffusion systems have attracted the attention of numerous researchers due to their application in various fields such as, for example, population dynamics, the dynamics of gas, dynamic systems, fusion process, certain biological models, etc. The aim of this work is to prove the global existence of a solution for an arbitrary-order fractional reaction-diffusion system. The inspiration for this study arises from the research conducted recently by Barrouk and Mesbahi [2].

Keywords: *semigroups; fractional reaction-diffusion systems; local solution; global solution.*

Mathematics Subject Classification (2020): 35R11, 35K57, 35K55, 37L05, 70K99, 93A30.

1 Introduction

In recent years, fractional differential equations have garnered significant attention from researchers because of their extensive applications across various scientific, technological, and medical fields, we can find important applications, for example, in finance [15], mechanics [14], biomedicine [9], pattern formation [8], we find numerous real applications in biology, medicine and ecology, see the works of Djemai and Mesbahi [6], Khayar, Brouri and Ouzahra *[12]* and corresponding references therein, etc.

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Our particular objective in this type of anomalous diffusion problems is to study the following fractional reaction-diffusion system:

$$\begin{cases} \dfrac{\partial \vartheta_1}{\partial t} - d_1 (-\Delta)^{\alpha_1} \vartheta_1 = f_1 (t, x, \vartheta, \nabla \vartheta) & \text{in } \mathbb{R}^+ \times \Omega, \\ \vdots & \vdots \\ \dfrac{\partial \vartheta_m}{\partial t} - d_m (-\Delta)^{\alpha_m} \vartheta_m = f_m (t, x, \vartheta, \nabla \vartheta) & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial \vartheta_i}{\partial \eta} = 0 \quad \text{or} \quad \vartheta_i = 0 \ , \ \text{for all} \ \ 1 \leq i \leq m & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ \vartheta_i (0, .) = \vartheta_{i_0} (.) \ \text{for all} \ \ 1 \leq i \leq m & \text{in } \Omega, \end{cases} \tag{1}$$

where $\vartheta = (\vartheta_1, \ldots, \vartheta_m)$, $\nabla\vartheta = (\nabla\vartheta_1, \ldots, \nabla\vartheta_m)$, $m \geq 2$, Ω is a bounded and regular domain of $\mathbb{R}^N$ with boundary $\partial\Omega$, $N \geq 2$, $\vartheta_i = \vartheta_i (t, x)$, $1 \leq i \leq m$ for $(t, x) \in Q_T = (0, T) \times \Omega$ and f_i are real functions, the presence of the non local operator $(-\Delta)^{\alpha_i}$, $0 < \alpha_i < 1$ for all $1 \leq i \leq m$, which accounts for the anomalous diffusion [11, 16], means that the sub-populations face some obstacles that slow their movement, and the constants of diffusion d_i are assumed to be nonnegative, $f_i : (0, T) \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}^m$ are enough regular, ϑ_{i_0} are nonnegative functions in $L^1 (\Omega)$, for all $1 \leq i \leq m$.

The local existence in time of the solution ϑ_i is classical. The positivity of the solution stems from the positivity of ϑ_{i_0}, which are assumed to be continuous, for all $1 \leq i \leq m$.

Several mathematical researchers have investigated the system derived from (1) by substituting the abnormal diffusion operator with the standard Laplacian operator $(-\Delta)$, employing various methods and techniques. Notable studies include those by Barrouk and Mesbahi [2], Barrouk and Abdelmalek [1], Moumeni and Dehimi [17], and Moumeni and Mebarki [18].

Note that over the past years, very important works have appeared in fractional reaction-diffusion equations. We mention the following.

The work of Hnaien et *al.* [10], is devoted to the study of the fractional systems: an abnormal diffusion system describing the propagation of an epidemic in a confined population of the SIR type, the fractional temporal Brusselator system and a reaction-diffusion system, temporal fractional with an equilibrium law. This study is based on Banach's fixed point theorem, semigroup estimates and Sobolev's integration theorem.

In [3], Besteiro and Rial studied the initial value problem for finite dimensional fractional non-autonomous reaction-diffusion equations. They proved the global existence and the asymptotic behavior of solutions by applying the general time splitting method and the technique of invariant regions.

We emphasize that there are many other references that approach this subject in various analytical and numerical ways.

This paper is organized as follows. In the next section, we provide some results necessary to understand the content of this work. In the next three sections, we give some results concerning the approximate problem. In Section 6, we state our main result and also present its proof in detail. The penultimate section is devoted to an application of the obtained result. Finally, we close with a conclusion.

2 Important Results

2.1 Hypotheses

To study problem (1), we assume that the functions $f_i : (0,T) \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}$, $1 \le i \le m$, satisfy the following simple assumptions, which allow them to be chosen from a wide range.

(A1) We preserve for all time the nonnegativity of the solutions, so we assume that f_i are quasipositive for all $1 \le i \le m$.

(A2) There exists $C \ge 0$ independent of $\vartheta_1, \dots, \vartheta_m$ such that

$$f_i\left(t, x, \vartheta, \nabla\vartheta\right) \le C \sum_{i=1}^{m} \vartheta_i \ , \ \forall \vartheta_i \ge 0, \ 1 \le i \le m. \tag{2}$$

(A3) The functions $f_i : (0,T) \times \Omega \times \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}$ are measurable and $f_i : \mathbb{R}^m \times \mathbb{R}^{mN} \to \mathbb{R}$ are locally Lipschitz continuous for all $1 \le i \le m$.

2.2 Preliminaries

To prove the main result, we need the following results.

Theorem 2.1 *Let Ω be an open bounded domain in $\mathbb{R}^N$. The following system*

$$\begin{cases} \left(-\Delta\right)^{\alpha} \varphi_k = \lambda_k^{\alpha} \varphi_k & in\ \Omega, \\ \dfrac{\partial \varphi_k}{\partial \eta} = 0 & on\ \partial\Omega, \end{cases}$$

where

$$D\left(\left(-\Delta\right)^{\alpha}\right) = \left\{ \vartheta \in L^2\left(\Omega\right), \ \frac{\partial \vartheta}{\partial \eta} = 0, \ \left\| \left(-\Delta\right)^{\alpha} \vartheta \right\|_{L^2(\Omega)} < +\infty \right\},$$

$$\left\| \left(-\Delta\right)^{\alpha} \vartheta \right\|_{L^2(\Omega)}^2 = \sum_{k=1}^{+\infty} \left| \lambda_k^{\alpha} \left\langle \vartheta, \varphi_k \right\rangle \right|^2,$$

has a countable sequence of eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$ and $\lambda_k \to \infty$ as $k \to \infty$, and φ_k are the corresponding eigenvectors for all $k \ge 1$.

So, for $\vartheta \in D\left(\left(-\Delta\right)^{\alpha}\right)$, we have

$$\left(-\Delta\right)^{\alpha} \vartheta = \sum_{k=1}^{+\infty} \lambda_k^{\alpha} \left\langle \vartheta, \varphi_k \right\rangle \varphi_k.$$

Also, we have the formula of integration by parts as follows:

$$\int_{\Omega} \vartheta\left(x\right) \left(-\Delta\right)^{\alpha} \bar{\vartheta}\left(x\right) dx = \int_{\Omega} \bar{\vartheta}\left(x\right) \left(-\Delta\right)^{\alpha} \vartheta\left(x\right) dx, \ for\ \vartheta, \bar{\vartheta} \in D\left(\left(-\Delta\right)^{\alpha}\right). \tag{3}$$

Proof. See Hnaien et *al.* [10] and corresponding references therein.

Lemma 2.1 ([13]) *Let $\theta \in C_0^\infty(Q_T)$, $\theta \geq 0$, then there exists a nonnegative function $\Phi \in C^{1,2}(Q_T)$ being the solution of the system*

$$\begin{cases} -\Phi_t - d\Delta\Phi = \theta & \text{in } Q_T, \\ \Phi(t,x) = 0 & \text{on } \Sigma_T, \\ \Phi(T,x) = 0 & \text{in } \Omega, \end{cases}$$

where $\Sigma_T = (0,T) \times \partial\Omega$, for all $q \in (1,\infty)$, there exists $C \geq 0$, not dependent on θ, such that

$$\|\Phi\|_{L^{q'}(Q_T)} \leq C \|\theta\|_{L^q(Q_T)}.$$

And for all $\vartheta_0 \in L^1(\Omega)$ and $h \in L^1(Q_T)$, we obtain the equalities

$$\int_{Q_T} (S(t)\vartheta_0(x))\,\theta dxdt = \int_\Omega \vartheta_0(x)\,\Phi(0,x)\,dx \tag{4}$$

and

$$\int_{Q_T} \left(\int_0^t S(t-s)\,h(s,x,\vartheta(s),\nabla\vartheta(s))\,ds \right) \theta dxdt = \int_{Q_T} h(s,x,\vartheta(s),\nabla\vartheta(s))\,\Phi(s,x)\,dxds. \tag{5}$$

Proof. *To prove this Lemma, see Bonafede and Schmitt [4].*

3 Local Existence of the Solution

We will transform the system (1) to an abstract system of first order in the Banach space $X = \left(L^1(\Omega)\right)^m$. For this, we define the functions $\vartheta_{i_0}^n$, for all $n > 0$ and $1 \leq i \leq m$, by

$$\vartheta_{i_0}^n = \min\{\vartheta_{i_0}, n\}.$$

Obviously, $\vartheta_{i_0}^n$ satisfies

$$\vartheta_{i_0}^n \in L^1(\Omega) \text{ and } \vartheta_{i_0}^n \geq 0 \text{ for all } 1 \leq i \leq m.$$

Now, consider the problem

$$\begin{cases} \dfrac{\partial \vartheta_{1_n}}{\partial t} - d_1(-\Delta)^{\alpha_1}\vartheta_{1_n} = f_1(t,x,\vartheta_n,\nabla\vartheta_n) & \text{in } Q_T, \\ \vdots & \\ \dfrac{\partial \vartheta_{m_n}}{\partial t} - d_m(-\Delta)^{\alpha_m}\vartheta_{m_n} = f_m(t,x,\vartheta_n,\nabla\vartheta_n) & \text{in } Q_T, \\ \dfrac{\partial \vartheta_{i_n}}{\partial \eta} = 0 \text{ or } \vartheta_{i_n} = 0\ ,\ 1 \leq i \leq m & \text{in } \Sigma_T, \\ \vartheta_{i_n}(0,x) = \vartheta_{i_0}^n(x) \geq 0\ ,\ 1 \leq i \leq m & \text{in } \Omega. \end{cases} \tag{6}$$

Hence, if $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ is a solution of (6), then it satisfies the following integral equation:

$$\vartheta_{i_n}(t) = S_i(t)\,\vartheta_{i_0}^n + \int_0^t S_i(t-s)\,f_i(s,.,\vartheta_n(s),\nabla\vartheta_n(s))\,ds, \tag{7}$$

where $S_i(t)$ is the semigroup which is generated by the operator $d_i(-\Delta)^{\alpha_i}$, $1 \leq i \leq m$. (See Pazy [19]).

Theorem 3.1 *There exists $T_M > 0$ and $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ being a local solution of (6) for all $t \in [0, T_M]$.*

Proof. Note that $S_i(t)$ are contraction semigroups and F is locally Lipschitz, $0 \leq \vartheta_{i_0}^n \leq n$, which ensures the existence of $T_M > 0$ such that $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ becomes a local solution of (6) on $[0, T_M]$.

Theorem 3.2 *Let $\vartheta_{i_0}^n \in L^1(\Omega)$, then there exist a maximal time $T_{\max} > 0$ and a unique solution $(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) \in \left(C\left([0, T_{\max}), L^1(\Omega)\right)\right)^m$ of the system (6), with the alternative:*
- either $T_{\max} = +\infty$,
- or $T_{\max} < +\infty$ and $\lim_{t \to T_{\max}} \left(\sum_{i=1}^m \|\vartheta_{i_n}(t)\|_\infty\right) = +\infty$.

Proof. For $T > 0$, we define the following Banach space:

$$\begin{aligned} E_T := \{&(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) \in \left(C\left([0, T_{\max}), L^1(\Omega)\right)\right)^m, \\ &\|(\vartheta_{1_n}, \ldots, \vartheta_{m_n})\| \leq 2\left\|\left(\vartheta_{1_0}^n, \ldots, \vartheta_{m_0}^n\right)\right\| = R\}, \end{aligned}$$

where $\|.\|_\infty := \|.\|_{L^\infty(\Omega)}$ and $\|.\|$ is the norm of E_T defined by

$$\|(\vartheta_{1_n}, \ldots, \vartheta_{m_n})\| := \sum_{i=1}^m \|\vartheta_{i_n}\|_{L^\infty([0,T], L^\infty(\Omega))}.$$

Next, for every $(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) \in E_T$, we define

$$\Psi(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) := \left(\Psi_1(\vartheta_{1_n}, \ldots, \vartheta_{m_n}), \ldots, \Psi_m(\vartheta_{1_n}, \ldots, \vartheta_{m_n})\right),$$

where for $t \in [0, T]$ and $1 \leq i \leq m$,

$$\Psi_i(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) = S_i(t)\vartheta_{i_0}^n + \int_0^t S_i(t-s) f_i(s, ., \vartheta_n, \nabla\vartheta_n)\, ds.$$

Using the Banach fixed point theorem, we will demonstrate the local existence.
• $\Psi : E_T \to E_T$. Let $(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) \in E_T$, we obtain, by the maximum principle,

$$\|\Psi_i(\vartheta_{1_n}, \ldots, \vartheta_{m_n})\|_\infty \leq \left\|\vartheta_{i_0}^n\right\|_\infty + C\sum_{i=1}^m \|\vartheta_{i_n}\|_\infty T.$$

So, we have

$$\begin{aligned} \|\Psi(\vartheta_{1_n}, \ldots, \vartheta_{m_n})\| &\leq \sum_{i=1}^m \left\|\vartheta_{i_0}^n\right\|_\infty + mC\sum_{i=1}^m \|\vartheta_{i_n}\|_\infty T, \\ &\leq 2\sum_{i=1}^m \left\|\vartheta_{i_0}^n\right\|_\infty, \text{ by choosing } T \text{ such that } T \leq \frac{1}{2mC}. \end{aligned}$$

Then $\Psi(\vartheta_{1_n}, \ldots, \vartheta_{m_n}) \in E_T$ for $T \leq \frac{1}{2mC}$.

Ψ is a contraction mapping for $(\vartheta_{1_n},\dots,\vartheta_{m_n})$, $\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\in E_T$, we have

$$\begin{aligned}&\left\|\Psi_1(\vartheta_{1_n},\dots,\vartheta_{m_n})-\Psi_1\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\right\|_\infty\\ \leq\;& L\int_0^t\left\|(\vartheta_{1_n},\dots,\vartheta_{m_n})-\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\right\|_\infty d\tau,\\ \leq\;& LT\left(\sum_{i=1}^m\left\|\vartheta_{i_n}-\tilde{\vartheta}_{i_n}\right\|_\infty\right).\end{aligned}$$

Similarly, we obtain, for $2\leq k\leq m$,

$$\left\|\Psi_k(\vartheta_{1_n},\dots,\vartheta_{m_n})-\Psi_k\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\right\|_\infty\leq LT\left(\sum_{i=1}^m\left\|\vartheta_{i_n}-\tilde{\vartheta}_{i_n}\right\|_\infty\right).$$

These estimates imply that

$$\begin{aligned}&\left\|\Psi(\vartheta_{1_n},\dots,\vartheta_{m_n})-\Psi\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\right\|_\infty\\ \leq\;& mLT\left(\sum_{i=1}^m\left\|\vartheta_{i_n}-\tilde{\vartheta}_{i_n}\right\|_\infty\right),\\ \leq\;& \frac{1}{2}\left\|(\vartheta_{1_n},\dots,\vartheta_{m_n})-\left(\tilde{\vartheta}_{1_n},\dots,\tilde{\vartheta}_{m_n}\right)\right\|\end{aligned}$$

for $T\leq\max\left(\frac{1}{2mC},\frac{1}{2mL}\right)$.

Consequently, according to the Banach fixed point theorem, the problem (6) has a unique mild solution $(\vartheta_{1_n},\dots,\vartheta_{m_n})\in E_T$.

We can extend the solution on a maximal interval $[0,T_{\max})$, where

$$T_{\max}:=\sup\{T>0,\;\;(\vartheta_{1_n},\dots,\vartheta_{m_n})\text{ is a solution to (6) in }E_T\}.$$

For the global existence, we need the fact that the solutions are positive.

4 Positivity of the Solution

Lemma 4.1 *Let $(\vartheta_{1_n},\dots,\vartheta_{m_n})$ be a solution of the system (6) satisfying*

$$\vartheta_{i_0}^n(x)\geq 0,\quad\forall x\in\Omega.$$

Then

$$\vartheta_{i_n}(t,x)\geq 0,\quad\forall(t,x)\in[0,T)\times\Omega,\quad 1\leq i\leq m.$$

Proof. Let $\bar{\vartheta}_{1_n}(t,x)=0$ in $]0,T[\times\Omega$, then $\dfrac{\partial\bar{\vartheta}_{1_n}}{\partial t}=0$, $\nabla\bar{\vartheta}_{1_n}=0$ and $(-\Delta)^{\alpha_1}\bar{\vartheta}_{1_n}=0$, then according to the hypothesis (A1), we obtain

$$\begin{aligned}0\;&=\;\frac{\partial\vartheta_{1_n}}{\partial t}-d_1(-\Delta)^{\alpha_1}\vartheta_{1_n}-f_1(t,x,\vartheta_n,\nabla\vartheta_n)\\ &\geq\;\frac{\partial\bar{\vartheta}_{1_n}}{\partial t}-d_1(-\Delta)^{\alpha_1}\bar{\vartheta}_{1_n}-f_1\left(t,x,\bar{\vartheta}_{1_n},\dots,\vartheta_{m_n},\nabla\bar{\vartheta}_{1_n},\dots,\nabla\vartheta_{m_n}\right)\end{aligned}$$

and

$$\vartheta_{1_n}(0,x) = \vartheta_{1_0}^n(x) \geq 0 = \bar{\vartheta}_{1_n}(0,x).$$

Therefore, by the comparison theorem ([5] or [7]), we get $\vartheta_{1_n}(t,x) \geq \bar{\vartheta}_{1_n}(t,x)$, where $\vartheta_{1_n}(t,x) \geq 0$.

In the same way, we find

$$\vartheta_{k_n}(t,x) \geq 0 \quad , \quad 2 \leq k \leq m.$$

Then $\vartheta_{i_n}(t,x) \geq 0$ for all $1 \leq i \leq m$.

5 Global Existence of the Solution

To show the global existence of the solution of the problem (6) for all $t \geq 0$, it suffices to find an estimate of the solution for all $t \geq 0$, from the alternative. The following Lemma shows the existence of an estimate of the solution of (6) in $L^1(\Omega)$.

Lemma 5.1 *Consider $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ as the solution of the system (6), then there exists $M(t)$, depending only on t, such that for all $0 \leq t \leq T_M$, we have*

$$\left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(\Omega)} \leq M(t).$$

From this estimate, we conclude that the solution $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ given by Theorem 3.1 is a global solution.

Proof. Adding the equations of system (6), we obtain

$$\frac{\partial}{\partial t}\sum_{i=1}^{m} \vartheta_{i_n} - \sum_{i=1}^{m} d_i(-\Delta)^{\alpha_i}\vartheta_{i_n} = \sum_{i=1}^{m} f_i(t,x,\vartheta_n,\nabla\vartheta_n).$$

Taking into account (2), we get

$$\frac{\partial}{\partial t}\sum_{i=1}^{m} \vartheta_{i_n} - \sum_{i=1}^{m} d_i(-\Delta)^{\alpha_i}\vartheta_{i_n} \leq Cm\sum_{i=1}^{m} \vartheta_{i_n}.$$

Let us integrate on Ω, so by using the formula (3) of integration by parts

$$\int_\Omega (-\Delta)^{\alpha_i}\vartheta_{i_n}(x)\,dx = 0,$$

we obtain

$$\frac{\partial}{\partial t}\int_\Omega \sum_{i=1}^{m} \vartheta_{i_n} dx \leq Cm\int_\Omega \sum_{i=1}^{m} \vartheta_{i_n} dx,$$

so

$$\int_\Omega \sum_{i=1}^{m} \vartheta_{i_n} dx \leq \exp\{Cmt\}\int_\Omega \sum_{i=1}^{m} \vartheta_{i_0}^n dx,$$

and for $\vartheta_{i_0}^n \leq \vartheta_{i_0}$, we have

$$\int_\Omega \sum_{i=1}^{m} \vartheta_{i_n} dx \leq \exp\{Cmt\}\int_\Omega \sum_{i=1}^{m} \vartheta_{i_0} dx.$$

If we put

$$M(t) = \exp\{Cmt\} \left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)},$$

then

$$\left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(\Omega)} \leq M(t) \quad , \quad 0 \leq t \leq T_M.$$

The following Lemma ensures the existence of estimate of the solution $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ of the system (6) in $\left(L^1(Q_T)\right)^m$.

Lemma 5.2 *For any solution $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ of (6), there exists a constant $K(t)$ depending only on t and such that*

$$\left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \leq K(t) \left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)}.$$

Proof. We multiply the first equation of (7) by θ in $C_0^\infty(Q)$ with $\theta \geq 0$ and we integrate on Q_T, by using (4) and (5), we obtain, for all $1 \leq i \leq m$,

$$\begin{aligned} \int_{Q_T} \vartheta_{i_n} \theta dxdt &= \int_\Omega \vartheta_{i_0}^n(x) \Phi(0,x) dx \\ &\quad + \int_{Q_T} f_i(t,x,\vartheta_n,\nabla\vartheta_n) \Phi(s,x) dxds, \end{aligned}$$

therefore

$$\begin{aligned} \int_{Q_T} \sum_{i=1}^{m} \vartheta_{i_n} \theta dxdt &= \int_\Omega \sum_{i=1}^{m} \vartheta_{i_0}^n(x) \Phi(0,x) dx + \\ & \int_{Q_T} \sum_{i=1}^{m} f_i(t,x,\vartheta_n,\nabla\vartheta_n) \Phi(s,x) dxds. \end{aligned}$$

According to (3) and as $\vartheta_{i_0}^n \leq \vartheta_{i_0}$, we have

$$\int_{Q_T} \sum_{i=1}^{m} \vartheta_{i_n} \theta dxdt \leq \int_\Omega \sum_{i=1}^{m} \vartheta_{i_0}(x) \Phi(0,x) dx + Cm \int_{Q_T} \sum_{i=1}^{m} \vartheta_{i_n} \Phi(s,x) dxds.$$

Using the Hölder inequality, we deduce

$$\begin{aligned} \int_{Q_T} \sum_{i=1}^{m} \vartheta_{i_n} \theta dxdt &\leq \left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} \|\Phi(0,.)\|_{L^\infty(Q)} \\ &\quad + Cm \left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \|\Phi\|_{L^\infty(Q)}, \end{aligned}$$

$$\leq \left(\left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} + Cm \left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \right) . \|\Phi\|_{L^\infty(Q)}.$$

$$\leq \max\{1, Cm\} \left(\left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} + \left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \right) . \|\Phi\|_{L^\infty(Q)} ,$$

$$\leq k_1(t) \left(\left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} + \left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \right) . \|\theta\|_{L^\infty(Q)} ,$$

where $k_1(t) \geq \max\{c, cCm\}$.

Since θ is arbitrary in $C_0^\infty(Q_T)$, we get

$$\left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \leq k_1(t) \left(\left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} + \left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \right).$$

Taking $k(t) = \frac{k_1(t)}{1-k_1(t)}$, we find

$$\left\| \sum_{i=1}^{m} \vartheta_{i_n} \right\|_{L^1(Q)} \leq k(t) \left\| \sum_{i=1}^{m} \vartheta_{i_0} \right\|_{L^1(\Omega)} .$$

6 Main Result

Now, we present the main result of this work, which states that the existence of global solutions for the system (1) is equivalent to the existence of ϑ_i for all $1 \leq i \leq m$, it is formulated in the following theorem.

Theorem 6.1 *Under the hypotheses (A1)-(A3), there exists* $(\vartheta_1, \ldots, \vartheta_m)$ *being a solution of the following system:*

$$\begin{cases} \vartheta_i \in C\left([0, +\infty[\, , L^1(\Omega)\right), \\ f_i(t, x, \vartheta, \nabla\vartheta) \in L^1(Q_T), \\ \vartheta_i(t) = S_i(t)\vartheta_{i_0} + \int_0^t S_i(t-s) f_i(s, ., \vartheta(s), \nabla\vartheta(s))\, ds, \ \forall t \geq 0, \end{cases} \tag{8}$$

where $S_i(t)$ *are the semigroups of contractions in* $L^1(\Omega)$ *generated by* $d_i(-\Delta)^{\alpha_i}$*,* $1 \leq i \leq m$*.*

Proof. We define the map L by

$$L : (\vartheta_0, h) \to S_d(t)\vartheta_0 + \int_0^t S_d(t-s)\, h(s, ., \vartheta(s), \nabla\vartheta(s))\, ds,$$

where $S_d(t)$ is the contraction semigroup generated by the operator $-d(-\Delta)^\delta$. According to the compactness of the application L of $\left(L^1(Q_T)\right)^m$ in $L^1(Q_T)$ (see [1, 2]), there is a subsequence $\left(\vartheta_{1_n}^j, \ldots, \vartheta_{m_n}^j\right)$ of $(\vartheta_{1_n}, \ldots, \vartheta_{m_n})$ and ϑ_i of $\left(L^1(Q_T)\right)^m$ such that $\left(\vartheta_{1_n}^j, \ldots, \vartheta_{m_n}^j\right)$ converges towards $(\vartheta_1, \ldots, \vartheta_m)$.

Let us now show that $\left(\vartheta_{1_n}^j, \ldots, \vartheta_{m_n}^j\right)$ is a solution of (7), we have, for all $1 \leq i \leq m$,

$$\vartheta_{i_n}^j(t, x) = S_i(t)\vartheta_{i_0}^j + \int_0^t S_i(t-s) f_i\left(s, ., \vartheta_n^j, \nabla\vartheta_n^j\right) ds. \tag{9}$$

It suffices to show that $(\vartheta_1, \ldots, \vartheta_m)$ verifies (8). Obviously, if $j \to +\infty$, we obtain, for all $1 \leq i \leq m$, the limit as follows:

$$\vartheta_{i_0}^j \to \vartheta_{i_0},$$

and

$$f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right) \to f_i\left(s, ., \vartheta, \nabla \vartheta\right). \tag{10}$$

Thus, to show that $(\vartheta_1, \ldots, \vartheta_m)$ verifies (8), it remains to show that, for all $1 \leq i \leq m$,

$$f_i\left(s, x, \vartheta_n^j, \nabla \vartheta_n^j\right) \to f_i\left(s, x, \vartheta, \nabla \vartheta\right)$$

in $L^1(Q)$ when $j \to +\infty$.

Make the integration by part of (6) on Q_T by taking (3) into consideration, we obtain

$$-d_i \int_{Q_T} (-\Delta)^{\alpha_i} \vartheta_{i_n}^j dxdt = 0.$$

We have

$$\int_\Omega \vartheta_{i_n}^j dx - \int_\Omega \vartheta_{i_0}^j dx = \int_{Q_T} f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right) dxdt,$$

from where

$$-\int_{Q_T} f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right) dxdt \leq \int_\Omega \vartheta_{i_0} dx \quad , \quad 1 \leq i \leq m. \tag{11}$$

We denote

$$N_{i_n} = C\left(\sum_{i=1}^m \vartheta_{i_n}^j\right) - f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right) \quad , \quad 1 \leq i \leq m.$$

It is clear that N_{i_n} are positive according to (2), we obtain

$$\int_{Q_T} N_{i_n} dxdt \leq C \int_{Q_T} \left(\sum_{i=1}^m \vartheta_{i_n}^j\right) dxdt + \int_\Omega \vartheta_{i_0} dx.$$

Lemma 5.2 gives $\int_{Q_T} N_{i_n} dxdt < +\infty$, which implies

$$\int_{Q_T} \left|f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right)\right| dxdt \leq C \int_{Q_T} \left(\sum_{i=1}^m \vartheta_{i_n}^j\right) dxdt + \int_{Q_T} N_{i_n} dxdt < +\infty.$$

Let

$$h_{i_n} = N_{i_n} + C\left(\sum_{i=1}^m \vartheta_{i_n}^j\right) \quad , \quad 1 \leq i \leq m,$$

h_{i_n} are in $L^1(Q)$ and positive. Furthermore,

$$\left|f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right)\right| \leq h_{i_n} \quad \text{a.e. } 1 \leq i \leq m.$$

Combining this result with (10) and by applying Lebesgue's dominated convergence theorem, we obtain

$$f_i\left(s, ., \vartheta_n^j, \nabla \vartheta_n^j\right) \to f_i\left(s, ., \vartheta, \nabla \vartheta\right) \text{ in } L^1(Q).$$

By passage to the limit when $j \to +\infty$ of (9) in $L^1(Q_T)$, we find, for all $1 \leq i \leq m$,

$$\vartheta_i(t) = S_i(t)\vartheta_{i_0} + \int_0^t S_i(t-s) f_i\left(s, ., \vartheta_1(s), \ldots, \vartheta_m(s), \nabla \vartheta_1(s), \ldots, \nabla \vartheta_m(s)\right) ds.$$

Then $(\vartheta_1, \ldots, \vartheta_m)$ verifies (8), consequently, $(\vartheta_1, \ldots, \vartheta_m)$ is the solution of the system (1).

7 Application

The concept of fractional calculus is also found in the study of diffusion phenomena. Numerous studies have shown the presence of abnormal diffusion processes such as the Lévy processes, for instance, in physical models where diffusive phenomena are more accurately represented by the Lévy processes rather than by other processes, reaction-diffusion equations featuring the fractional Laplacian instead of the standard Laplacian are used (see, for example, [16]).

The fractional reaction diffusion systems are systems involving constituents locally transformed into each other by chemical reactions and transported in space by diffusion. They arise in many applications, in chemistry, chemical engineering, physics, and various biological processes including population dynamics and biology. They have been the subject of countless studies in the past few decades. One of the most important aspects of this broad field is proving the global existence of solutions under certain assumptions and restrictions

8 Conclusion

This paper has explained the important factors needed to study the global existence of a solution for fractional nonlinear reaction-diffusion system. We have carried out this study by using the compact semigroup methods coupled with certain mathematical estimates and techniques. By building upon previous works, we have confirmed a global existence of a solution to the fractional system. For attaining our purpose, we have introduced and derived several theoretical results related to the existence theory.

There will be future research and applications on fractional reaction-diffusion system.

Acknowledgment

We thank the referees for the insightful review of our manuscript.

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