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Double-Phase System with Neumann Boundary Condition

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Abstract: This paper investigates the existence of multiple solutions for double-phase systems subject to Neumann conditions. The study is conducted within the framework of Sobolev spaces featuring variable exponents. Assuming appropriate conditions on the given data, we establish the existence of at least two weak solutions, each characterized by distinct energy signs. We employ the Nehari manifold and variational method as the foundation for our approach.

Keywords: *Neumann boundary; two-phase operator; weak solutions; convex-concave source.*

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1 Introduction

Let a bounded domain $\mathcal{U} \subseteq \mathbb{R}^N$, $N \geq 2$, with the Lipschitz boundary $\partial\mathcal{U}$ be given. Consider the following double-phase system:

$$\begin{cases} \mathcal{L}^{\mu_1(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = \lambda_1|\mathrm{u}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{u}(\mathrm{y}) + \dfrac{2r(\mathrm{y})}{r(\mathrm{y})+s(\mathrm{y})}|\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})-2}\mathrm{u}(\mathrm{y})|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} & \text{in} \quad \mathcal{U}, \\ \mathcal{L}^{\mu_2(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{v} = \lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{v}(\mathrm{y}) + \frac{2s(\mathrm{y})}{r(\mathrm{y})+s(\mathrm{y})}|\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})-2}\mathrm{v}(\mathrm{y}) & \text{in} \quad \mathcal{U}, \\ \left(\mathrm{Du}(\mathrm{y})\,|^{p(\mathrm{y})-2}\mathrm{Du} + \mu_1(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q(\mathrm{y})-2}\,\mathrm{Du}\right).\nu = h_1(\mathrm{y},\mathrm{u}(\mathrm{y})) & \text{on} \quad \partial\mathcal{U}, \\ \left(|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})-2}\mathrm{Dv} + \mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{Dv}\right).\nu = h_2(\mathrm{y},\mathrm{v}(\mathrm{y})) & \text{on} \quad \partial\mathcal{U}, \end{cases} \tag{1}$$

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where ν the outer unit normal parameters λ_1, λ_1 are positive, $\mu_1, \mu_2 : \overline{\mathcal{U}} \to (0, \infty)$ are Lipschitz continuous, and s, r, q, $p : \mathcal{U} \to \mathbb{R}$ are continuous functions that satisfy the following conditions:

$$1 < q^- \leq q^+ < p^- \leq p^+ < s^- + r^- = \min_{\mathrm{y}\in\mathcal{U}} s(\mathrm{y})) + \min_{\mathrm{y}\in\mathcal{U}} r(\mathrm{y})) \leq s^+ + r^+ < \infty, \tag{2}$$

$$\frac{p^-}{s^+ + r^+} < \left(\frac{p^- - q^+}{s^+ + r^+ - q^+}\right)\left(\frac{s^- + r^- - q^+}{p^+ - q^-}\right), \tag{3}$$

$$\mathcal{L}^{\mu_i(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = -\mathrm{div}\left(|\,\mathrm{Du}(\mathrm{y})\,|^{p(\mathrm{y})-2}\mathrm{Du} + \mu_i(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q(\mathrm{y})-2}\,\mathrm{Du}\right) \quad \text{for all } i = 1, 2. \tag{4}$$

We assume $h_i : \partial\mathcal{U}\times\mathbb{R} \to \mathbb{R}$ are Carathéodry functions satisfying the following conditions:

($\mathcal{H}_1$) There exists $\beta > r^+ + s^+$ for some B such that for each $|\theta| > B$, we get

$$0 < \int_{\partial\mathcal{U}} H_i(\mathrm{y},\theta)d\nu \leq \int_{\partial\mathcal{U}} h_i(\mathrm{y},\theta)\frac{\theta}{\beta}d\nu \quad a.e\, x \in \partial\mathcal{U} \quad \text{with } H_i(\mathrm{y},\theta) = \int_0^\theta h_i(\mathrm{y},t)dt,$$

($\mathcal{H}_2$) $h_i(\mathrm{y}, 0) = 0$,

($\mathcal{H}_3$) $\lim\limits_{\varepsilon\to 0} \dfrac{h_i(\mathrm{y},\varepsilon)}{|\varepsilon|^{q(\mathrm{y})-2}\varepsilon} = 0$ and $\lim\limits_{\varepsilon\to\pm\infty} \dfrac{h_i(\mathrm{y},\varepsilon)}{|\varepsilon|^{q(\mathrm{y})-2}\varepsilon} = +\infty$ uniformly a.e.y $\in \partial\mathcal{U}$.

Due to the presence of two different elliptic growths $p(\mathrm{y})$ and $q(\mathrm{y})$, the problem (1) is said to be the double-phase type system. Recently, great attention has been devoted to treating the functional

$$\mathrm{v} \mapsto \int_{\mathcal{U}} (|D\mathrm{v}|^p + a(\mathrm{y})|D\mathrm{v}|^q)\,\mathrm{dy} \quad \text{with } 1 < p < q \quad \text{and } a(\mathrm{y}) \geq 0, \quad \text{for all y} \in \mathcal{U} \subset \mathbb{R}^N. \tag{5}$$

This kind of functional was introduced by Zhikov [19–21]. The functional (5) has been used in various scientific fields. Zhikov [20, 21] discusses its numerous applications in the study of duality theory. In recent works, several researchers have focused on studying equations or systems involving the double-phase operator. The reader may consult the work of Liu and Dai [11] who studied the following equations:

$$\begin{cases} \mathcal{L}^{\mu(\mathrm{y})}_{p,q}\mathrm{u} &= f(\mathrm{y},\mathrm{u}) \quad \text{in } \mathcal{U}, \\ \mathrm{u} \,=\, 0 & \quad\quad\quad\quad\;\; \text{on } \partial\mathcal{U}, \end{cases}$$

where $\mathcal{L}^{\mu(\mathrm{y})}_{p,q}\mathrm{u} = -\mathrm{div}(\,|\,\mathrm{Du}(\mathrm{y})\,|^p\mathrm{Du} + \mu(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^q\,\mathrm{D}u)$. For the performed comparable processing via Nehari's manifold method, see Gasiński et al. in [10], see also Arora et al. in [3]. Along the same lines, refer to [1, 4, 14–18]. In the same context, Gasinski and Winkert [9] studied the double-phase equation with a non-linear boundary condition of the following form:

$$\begin{cases} \mathcal{L}^{\mu(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = f(x,\mathrm{u}) - |\,\mathrm{u}(\mathrm{y})\,|^{p-2}\,\mathrm{u}(\mathrm{y}) - \mu(\mathrm{y})|\,\mathrm{u}(\mathrm{y})\,|^{q-2}\,\mathrm{u}(\mathrm{y}) & \text{in } \mathcal{U}, \\ (\,|\,\mathrm{Du}(\mathrm{y})\,|^{p-2}\mathrm{Du} + \mu_2(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q-2}\,\mathrm{Dv}).\nu = h(x,\mathrm{u}(\mathrm{y})) & \text{on } \partial\mathcal{U}. \end{cases}$$

Choudhuri, Repovš and Saudi [6] proved the existence of solutions to a double-phase problem with a nonlinear boundary condition that is specified and nonlinear. Concerning the analysis of nonlinear systems with Dirichlet-type boundaries, Aberqi et al. [2] demonstrated the existence of at least two nonnegative nontrivial solutions to a $p(z)$-Laplacian system with critical nonlinearity. Furthermore, Winkert et al. [13] investigated a system involving the double-phase operator with sources that depend on Neumann boundary conditions, and the gradient structured as follows:

$$\begin{cases} \mathcal{L}^{\mu_1(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = f_1(x,\mathrm{u},\mathrm{v},\mathrm{Du},\mathrm{Dv}) & \text{in } \mathcal{U}, \\ \mathcal{L}^{\mu_2(\mathrm{y})}_{p_1(x),q_1(x)}(v) = f_2(x,\mathrm{u},\mathrm{v},\mathrm{Du},\mathrm{Dv}) & \text{in } \mathcal{U}, \\ (\,|\,\mathrm{Du}(\mathrm{y})\,|^{p_1(x)-2}\mathrm{Du} + \mu_1(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q_1(x)-2}\,Du).\nu = h_1(x,\mathrm{u},\mathrm{v}) & \text{on } \partial\mathcal{U}, \\ (\,|\,\mathrm{Du}(\mathrm{y})\,|^{p_2(x)-2}\mathrm{Du} + \mu_2(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q_2(x)-2}\,\mathrm{Du}).\nu = h_2(x,\mathrm{u},\mathrm{v}) & \text{on } \partial\mathcal{U}. \end{cases}$$

More recently, Marino and Winkert [13] showed the existence of at least one weak solution of such systems. Some research related to our contribution are: Marano, Marino and Moussaoui [12], Chen and Wu [5].

Remark 1.1
- *Neumann boundary conditions may cause non-uniqueness unless the solution is normalized.*
- *Neumann conditions often stabilize the system compared to Dirichlet conditions.*
- *They allow global attractors or stationary solutions to emerge depending on the nonlinearities.*

Motivated by the aforementioned works and the remarks above, in this study, we will employ a variational approach to investigate the existence of at least two positive nontrivial solutions to the system (1) under conditions $(\mathcal{H}_1)-(\mathcal{H}_3)$, (2)-(3).
The theorem presented below constitutes the primary result of this paper.

Theorem 1.1 *If hypotheses* $(\mathcal{H}_1)$ *to* $(\mathcal{H}_3)$ *are satisfied, and there exists a positive constant* $K > 0$ *such that* $\lambda_1 + \lambda_2 \in (0, K)$, *then the system* (1) *possesses at least two non-negative, non-trivial solutions.*

The paper is structured as follows. In Section 2, we review established properties of the variable exponent spaces $L^{p(.)}(\mathcal{U})$ and Musielak-Orlicz Sobolev space $W^{1,p(.)}(\mathcal{U})$ that are compatible with the variable exponent double-phase operator, along with other technical tools that will be employed later. We present our main results in Section 3.

2 Background Results

This section presents the essential definitions and properties of the Sobolev-Orlicz space featuring variable exponents. To delve deeper into the theory of Sobolev-Orlicz spaces, refer to [2,7,8].

Consider the following sets: $P(\mathcal{U}) = \{w : \mathcal{U} \to \mathbb{R}, w \text{ measurable}\}$ and $C_+(\bar{\mathcal{U}}) = \{p : \mathcal{U} \to (1,\infty)) \text{ continuous } : p_- > 1\}$.

Definition 2.1 (see [7]) Let $q \in C_+(\bar{\mathcal{U}})$. We denote by $L^{q(.)}(\mathcal{U})$ the Lebesgue space with variable exponent, that is,

$$L^{q(.)}(\mathcal{U}) = \left\{ w \in P(\mathcal{U}) : \int_{\mathcal{U}} |w|^{q(\mathrm{y})} \mathrm{dy} < \infty \right\},$$

whose modular is given by $\varrho_{q(.)}(w) = \int_{\mathcal{U}} |w|^{q(\mathrm{y})} \mathrm{dy}$ and which is endowed with its corresponding Luxemburg norm

$$\|w\|_{q(.)} = \inf\left\{\lambda > 0 : \int_{\mathcal{U}} \left(\frac{|w|}{\lambda}\right)^{q(y)} \mathrm{dy} \leqslant 1\right\}.$$

Definition 2.2 (see [7]) Let $w : \mathcal{U} \to \mathbb{R}$, we define the Sobolev space by

$$W^{1,q(\mathrm{y})}(\mathcal{U}) = \left\{w \in L^{q(\mathrm{y})}(\mathcal{U}) : |\nabla w| \in L^{q(\mathrm{y})}(\mathcal{U})\right\},$$

it is equipped with the norm $\|w\|_{1,q(.)} = \|w\|\|_{q(.)} + \|\nabla w\|_{q(.)}$, and let $W_0^{1,q(\mathrm{y})}(\mathcal{U}) = \overline{C^{\infty}(\mathcal{U})}^{W^{1,q(\mathrm{y})}(\mathcal{U})}$.

Proposition 2.1 *(see [7]) Let $q \in C_+(\bar{\mathcal{U}})$ be such that there exists a vector $X \in \mathbb{R}^N \setminus \{0\}$ with the property that for all $x \in \mathcal{U}$, the function*

$$g_x(t) = q(x + tX) \ \ with \ \ K_x = \{t \in \mathbb{R} : x + tX \in \mathcal{U}\}$$

is monotone. Then there exsits a constant $C > 0$ such that

$$\varrho_{q(.)}(w) \leqslant C.\varrho_{q(.)}(\nabla w) \quad for\ all \ \ w \in W^{1,q(.)}(\mathcal{U}),\ where\, \varrho_{q(.)}(\nabla w) = \varrho_{q(.)}(|\nabla w|).$$

Proposition 2.2 *(See [8]) Let $q \in C_+(\bar{\mathcal{U}})$, then there exists $C_0 > 0$ such that*

$$\|w\|_{q(.)} \leqslant C_0 \|\nabla w\|_{q(.)},$$

thus, we can define the equivalent norm on $W_0^{1,q(.)}(\mathcal{U})$, $\|u\|_{1,q(.),0} = \|\nabla u\|_{q(.)}$.

Proposition 2.3 *(See [8]) Let $u \in L^{r(.)}(\mathcal{U})$, $v \in L^{r'(.)}(\mathcal{U})$. Then we have*

$$\int_{\mathcal{U}} |u(\mathrm{y})\mathrm{v}(\mathrm{y})| d\mathrm{y} \leq \left(\frac{1}{r^-} + \frac{1}{r'^-}\right) \|u\|_{L^{r(.)}(\mathcal{U})}.\|v\|_{L^{r'(.)}(\mathcal{U})},$$

where $\frac{1}{r'(\mathrm{y})} + \frac{1}{r(\mathrm{y})} = 1$.

Proposition 2.4 *(See [2]) Let $w \in L^{r(.)}(\mathcal{U})$, $\{w_j\} \in L^{r(.)}(\mathcal{U})$, $j \in \mathbb{N}$. Then we have*
(i) $\|w\|_{r(x)} < 1\, (resp. = 1, > 1) \Longleftrightarrow \rho_{r(.)} < 1\, (resp. = 1, > 1)$,
(ii) $\|w\|_{r(x)} < 1 \Rightarrow \|w\|^{r^-}_{r(x)} \leq \rho_{r(.)} \leq \|w\|^{r^+}_{r(x)}$,
(iii) $\lim\limits_{j\to\infty} \|w_j - w\|_{r(x)} = 0 \Longleftrightarrow \lim\limits_{j\to\infty} \rho_{r(x)}(w_j - w) = 0.$

$$min\left\{\rho_{r(x)}(w)^{\frac{1}{r^-}}; \rho_{r(x)}(w)^{\frac{1}{r^+}}\right\} \leq \|w\|_{r(x)} \leq max\left\{\rho_{r(x)}(w)^{\frac{1}{r^-}}; \rho_{r(x)}(w)^{\frac{1}{r^+}}\right\}.$$

Theorem 2.1 *(See [2]) Let $q(\mathrm{y}), p(\mathrm{y}), r(\mathrm{y}) + s(\mathrm{y}) \in L^{\infty}(\mathcal{U}) \cap C(\bar{\mathcal{U}})$. If*

$$q(\mathrm{y}) < r(\mathrm{y}), \quad p(\mathrm{y}) < r(\mathrm{y}) + s(\mathrm{y}) < q^* = \frac{Nq(\mathrm{y})}{N - q(\mathrm{y})},$$

then

$$W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow\hookrightarrow L^{p(\mathrm{y})}(\mathcal{U}) and\ W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow\hookrightarrow L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}).$$

In addition, we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ on $\partial\mathcal{U}$. Introduce the space $(L^{r(.)}(\partial\mathcal{U}), \|.\|_{r(.),\partial\mathcal{U}})$ and the continuous trace map $\gamma : W^{1,r(.)}(\mathcal{U}) \to L^{\bar{r}}(\partial\mathcal{U})$ with $\bar{r} < r_*$, such that

$$\gamma(u) = u|_{\partial\mathcal{U}} \quad for\ all\ u \in W^{1,p} \cap C^0(\bar{\mathcal{U}}),$$

and

$$r_* = \frac{(N-1)r}{N-1}\ if\, r < N;\ r* = \quad \text{any}\, l \in (r, +\infty)\ if\ r \geq N.$$

According to the trace embedding theorem, γ is compact for any $\bar{r} < r_*$. So, we understand all restrictions of Sobolev functions to $\partial\mathcal{U}$ in the sense of traces.

Lemma 2.1 *Let* $(\mathrm{u}, \mathrm{v}) \in W^{1.q(\mathrm{y})}(\mathcal{U}) \times W^{1.q(\mathrm{y})}(\mathcal{U})$, *then we have*

$$\begin{aligned}
&\int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\right) d\mathrm{y} \\
&\leq c_2(\lambda_1 + \lambda_2) \max\left(\|u\|^{q^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{q^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|u\|^{q^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{q^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}\right),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\int_{\mathcal{U}} |u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} d\mathrm{y} \\
&\leq c_3 \max\left(\|u\|^{r^-+s^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{r^-+s^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|u\|^{r^++s^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{r^++s^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}\right).
\end{aligned} \tag{7}$$

Proof. The proof is similar to the proof in [2], we will omit it.
The weighted variable exponent Lebesgue space $L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$ is defined as follows:

$$L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U}) = \left\{ w : \mathcal{U} \to \mathbb{R}\ is\ measurable\ ; \int_{\mathcal{U}} \mu(\mathrm{y}) |w|^{q(\mathrm{y})} \mathrm{dy} < \infty \right\}$$

and endowed with

$$\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} = \inf\left\{ \eta > 0 : \int_{\mathcal{U}} |\frac{w}{\eta}|^{q(\mathrm{y})} \mu(\mathrm{y}) \mathrm{dy} \leq 1 \right\}.$$

Moreover, the weighted modular on $L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$ is $\rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w) = \displaystyle\int_{\mathcal{U}} \mu(\mathrm{y}) |w(x)|^{q(\mathrm{y})} dv_g(\mathrm{y})$.

Proposition 2.5 *(See [2]) Let* w *and* $\{w_n\} \subset L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$, *then we get*
(i) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} < 1\, (resp. = 1, > 1) \Longleftrightarrow \rho_{q(\mathrm{y}),\mu(\mathrm{y})} < 1\, (resp. = 1, > 1)$,
(ii) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} < 1 \Rightarrow \|w\|^{q^-}_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \rho_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \|w\|^{q^+}_{q(\mathrm{y}),\mu(\mathrm{y})}$,
(iii) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} > 1 \Rightarrow \|w\|^{q^+}_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \rho_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \|w\|^{q^-}_{q(\mathrm{y}),\mu(\mathrm{y})}$,
(iv) $\lim_{n\to\infty} \|w_n\|_{q(\mathrm{y}),\mu(\mathrm{y})} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w_n) = 0$,
(v) $\lim_{n\to\infty} \|w_n\|_{q(\mathrm{y}),\mu(\mathrm{y})} = \infty \Longleftrightarrow \lim_{n\to\infty} \rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w_n) = \infty$.
It should be noted that the non-negative weighted function $\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}^+_*$ *satisfies the following condition:*
$\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}^+_*$ *such that* $\mu(.) \in L^{\varsigma(x)}(\mathcal{U})$ *with*

$$\frac{Np(\mathrm{y})}{Np(\mathrm{y}) - q(\mathrm{y})(N - p(\mathrm{y}))} < \varsigma(x) < \frac{p(\mathrm{y})}{p(\mathrm{y}) - q(\mathrm{y})}. \tag{8}$$

In fact, because $\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}_*^+$, then there exists $\mu_0 > 0$, and for all $x \in \mathcal{U}$, we get that $\mu(\mathrm{y}) > \mu_0$.

Theorem 2.2 *(See [2]) Assume that $q(\mathrm{y}) \in C(\bar{\mathcal{U}}) \cap L^\infty(\mathcal{U})$ and $\mathcal{U}$ is a bounded domain with smooth boundaries. Suppose that the assumption* (8) *is verified. Then we have the compact embedding*

$$W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U}).$$

3 Proof of Theorem 1.1

Let us denote $W = W^{1.q(\mathrm{y})}(\mathcal{U}) \times W^{1.q(\mathrm{y})}(\mathcal{U})$, endowed with norm $\|(\mathrm{u},\mathrm{v})\| = \|\mathrm{u}\| + \|\mathrm{v}\|$, and $D(\mathcal{U})$ is the space of C_c^∞ functions with compact support in $\mathcal{U}$.

3.1 Nehari analysis for system (1)

First, we define the weak solution of the system (1) as follows.

Definition 3.1 We say that $(\mathrm{u},\mathrm{v}) \in W$ is a weak solution of the system (1) if

$$\begin{aligned}
&\int_{\mathcal{U}} \Big(|Du(\mathrm{y})|^{p(\mathrm{y})-2} g(Du(\mathrm{y}), D\phi(\mathrm{y})) + |Dv(\mathrm{y})|^{p(\mathrm{y})-2} g(Dv(\mathrm{y}), D\psi(\mathrm{y}))\Big)\,\mathrm{dy} \\
&+ \int_{\mathcal{U}} \Big(\mu_1(\mathrm{y})|Du(\mathrm{y})|^{q(\mathrm{y})-2} g(Du(\mathrm{y}), D\phi(\mathrm{y})) + \mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})-2} g(Dv(\mathrm{y}) D\psi(\mathrm{y}))\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (H_1(\mathrm{y}, u(\mathrm{y}))\phi(x) + h_2(\mathrm{y}, \mathrm{v}(\mathrm{y}))\psi(x))\,\mathrm{d}\nu \\
&= \int_{\mathcal{U}} \Big(\lambda_1|\, u(\mathrm{y})|^{q(\mathrm{y})-2} u(\mathrm{y})\phi(\mathrm{y}) + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})-2} v(\mathrm{y})\psi(\mathrm{y})\Big)\,\mathrm{dy} \\
&+ \int_{\mathcal{U}} \left(\frac{2r(\mathrm{y})}{r(\mathrm{y}) + s(\mathrm{y})}|u(\mathrm{y})|^{n(\mathrm{y})-2} u(\mathrm{y})\phi(\mathrm{y}) + \frac{2s(\mathrm{y})}{r(\mathrm{y}) + s(\mathrm{y})}|v(\mathrm{y})|^{m(\mathrm{y})-2} v(\mathrm{y})\psi(\mathrm{y})\right)\mathrm{dy}
\end{aligned}$$

for all $(\phi, \psi) \in D(\mathcal{U}) \times D(\mathcal{U})$.

Let $J_{\lambda_1,\lambda_2} : W \to \mathbb{R}$ be the energy functional defined by

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) =& \int_{\mathcal{U}} \frac{1}{p(\mathrm{y})} \Big(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} \Big(\mu_1 |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + |\mu_2 Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (H_1(\mathrm{y}, u(\mathrm{y})) + H_2(\mathrm{y}, \mathrm{v}(\mathrm{y})))\mathrm{d}\nu \\
&- \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} \Big(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} - \int_{\mathcal{U}} \frac{2}{s(\mathrm{y}) + r(\mathrm{y})} |u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}$$

By a direct calculation, we have $J_{\lambda_1,\lambda_2} \in C^1(W, \mathbb{R})$. Consider the Nehari manifold $N_{\lambda_1,\lambda_2} = \Big\{(\mathrm{u},\mathrm{v}) \in W \backslash \{(0,0)\} : \langle J'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}), (\mathrm{u},\mathrm{v})\rangle = 0\Big\}$. Then, $(\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2}$ equivalent:

$$\begin{aligned}
&\int_{\mathcal{U}} \Big(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |Dv(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\mathcal{U}} \Big(\mu_1 |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2 |Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (h_1(\mathrm{y}, u(\mathrm{y})).u(\mathrm{y}) + h_2(\mathrm{y}, \mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}))\,\mathrm{d}\nu - \int_{\mathcal{U}} \Big(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\mathcal{U}} 2|u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy} = 0.
\end{aligned}$$

Lemma 3.1 *Suppose that the hypotheses $(\mathcal{H}_1)-(\mathcal{H}_3)$ are satisfied, then the energy functionals J_{λ_1,λ_2} are bounded and coercive on W.*

Proof. For $(\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2}$, according to (3), (4) and Proposition 2.2

$$
\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) &= \int_{\mathcal{U}} \left(\frac{1}{p(\mathrm{y})} - \frac{1}{r(\mathrm{y})+s(\mathrm{y})}\right)\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
&+ \int_{\mathcal{U}} \left(\frac{1}{q(\mathrm{y})} - \frac{1}{r(\mathrm{y})+s(\mathrm{y})}\right)\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&+ \int_{\partial\mathcal{U}} \left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})}h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y}) - H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu\\
&+ \int_{\partial\mathcal{U}} \left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})}h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}) - H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+ \int_{\mathcal{U}} \left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})} - \frac{1}{q(\mathrm{y})}\right)\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2|v(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&\geq c_0\left(\frac{1}{p^+} - \frac{1}{r^- + s^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-} + \frac{\mu_0}{k^{p+}(c+1)^{p^+}q^+}\left(\frac{1}{q^+} - \frac{1}{r^-+s^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}\\
&+ c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^++s^+} - \frac{1}{q^-}\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}.
\end{aligned}
$$

As $\frac{1}{p^+} > \frac{1}{r^++s^+} > \frac{1}{\beta}$ and $p^- > q^+$, then $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \to \infty$ as $\|(\mathrm{u},\mathrm{v})\| \to \infty$. As a result J_{λ_1,λ_2} is coercive and bounded below on N_{λ_1,λ_2}. Then we look at the energy $\xi_{\lambda_1,\lambda_2} : N_{\lambda_1,\lambda_2} \to \mathbb{R}$ given by

$$\xi_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \langle J'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \text{ for all } (\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2}.$$

Therefore, we divide N_{λ_1,λ_2} into

$$
\begin{aligned}
N^+_{\lambda_1,\lambda_2} &= \left\{(\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2} : \langle \xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle > 0\right\},\\
N^0_{\lambda_1,\lambda_2} &= \left\{(\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2} : \langle \xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle = 0\right\},\\
N^-_{\lambda_1,\lambda_2} &= \left\{(\mathrm{u},\mathrm{v}) \in N_{\lambda_1,\lambda_2} : \langle \xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle < 0\right\}.
\end{aligned}
$$

Lemma 3.2 *For each $(\lambda_1,\lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, there exists a constant $K_1 > 0$ such that for all $0 < \lambda_1 + \lambda_2 < K_1$, we have $N^0_{\lambda_1,\lambda_2} = \emptyset$.*

Proof. Suppose $N^0_{\lambda_1,\lambda_2} \neq \emptyset$ for all $(\lambda_1,\lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Take $(\mathrm{u},\mathrm{v}) \in N^0_{\lambda_1,\lambda_2}$ such that $\|(\mathrm{u},\mathrm{v})\| > 1$, then the definition of $N^0_{\lambda_1,\lambda_2}$, $(\mathcal{H}_1)-(\mathcal{H}_3)$ and (4), lead to

$$
\begin{aligned}
0 &= \langle \xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle\\
&\geq (p^- - q^+)\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy} + (q^- - q^+)\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&+ \int_{\partial\mathcal{U}}\left(q^+h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y}) - H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu + \int_{\partial\mathcal{U}}\left(q^+h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}) - H_1(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+ 2\left(q^+ - (r^+ + s^+)\right)\int_{\mathcal{U}}\|u(\mathrm{y})\|^{r(\mathrm{y})}\|\mathrm{v}(\mathrm{y})\|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}
$$

Since $q^+ > \frac{1}{q^+} > \frac{1}{\beta}$, by $(\mathcal{H}_1)$, we obtain

$$\begin{aligned}
0 &= \langle \xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \\
&\geq (p^- - q^+)\int_{\mathcal{U}}\Big(|\mathrm{Du(y)}|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + (q^- - q^+)\int_{\mathcal{U}}\Big(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&+ 2\left(q^+ - (r^+ + s^+)\right)\int_{\mathcal{U}}\|u(\mathrm{y})\|^{r(\mathrm{y})}\|\mathrm{v(y)}\|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}$$

Therefore, from Theorem 2.2, Poincare's inequality, and Lemma 2.1, we have

$$0 \geq \frac{1}{c}\left(p^- - q^+\right)\|(\mathrm{u},\mathrm{v})\|^{p^-} + c_1\left(q^- - q^+\right)\|(\mathrm{u},\mathrm{v})\|^{q^+} + 2\left(q^+ - (r^+ + s^+)\right)c_3\|(\mathrm{u},\mathrm{v})\|^{r^+ + s^+}.$$

As $p^- > q^+$, then

$$0 \geq \left(\frac{1}{c}\left(p^- - q^+\right) + c_1\left(q^- - q^+\right)\right)\|(\mathrm{u},\mathrm{v})\|^{q^+} + 2\left(q^+ - (r^+ + s^+)\right)c_3\|(\mathrm{u},\mathrm{v})\|^{r^+ + s^+}.$$

$$\text{Then, } \|(\mathrm{u},\mathrm{v})\| \geq \left(\frac{\frac{1}{c}(p^- - q^+) + c_1(q^- - q^+)}{2\left(q^+ - (r^+ + s^+)\right)c_3}\right)^{\frac{1}{r^+ + s^+ - q^+}}. \tag{9}$$

Analogously,

$$\begin{aligned}
0 &= \langle \xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \\
&\leq (p^+ - r^- - s^-)\int_{\mathcal{U}}\Big(|\mathrm{Du(y)}|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\partial\mathcal{U}}\left((r^- + s^-)h_2(\mathrm{y},\mathrm{v(y)}).\mathrm{v(y)} - H_2(\mathrm{y},\mathrm{v(y)})\right)\mathrm{d}\nu \\
&+ (r^- + s^- - q^-)\int_{\mathcal{U}}\Big(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v(y)}|^{q(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\partial\mathcal{U}}\left((r^- + s^-)h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y}) - H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu \\
&+ (q^+ - r^- - s^-)\int_{\mathcal{U}}\Big(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy}.
\end{aligned}$$

According to Theorem 2.1, (3) and $(\mathcal{H}_1)$, we have

$$\begin{aligned}
0 \leq& (p^+ - r^- - s^-)\int_{\mathcal{U}}\Big(|\mathrm{Du(y)}|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + (r^- - s^- - q^+)\int_{\mathcal{U}}\Big(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v(y)}|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&+ (r^- + s^-)\int_{\partial\mathcal{U}}\left(h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y}) + h_2(\mathrm{y},\mathrm{v(y)}).\mathrm{v(y)}\right)\mathrm{d}\nu \\
\leq& \frac{1}{c}(p^+ - r^- - s^-)\|(\mathrm{u},\mathrm{v})\|^{p^-} + c_2(\lambda_1+\lambda_2)(r^- - s^- - q^+)\|(\mathrm{u},\mathrm{v})\|^{q^-} + (r^- + s^-)\|(\mathrm{u},\mathrm{v})\|^{q^-}.
\end{aligned}$$

$$\text{Then } \|(\mathrm{u},\mathrm{v})\| \leq \left(\frac{c(c_2(\lambda_1+\lambda_2)+1)(r^- + s^-)}{r^- + s^- - p^+}\right)^{\frac{1}{p^- - q^-}}. \tag{10}$$

According to (9) and (10), we deduce that

$$\left(\frac{\frac{1}{c}(p^- - q^+) + c_1(q^- - q^+)}{2\left(q^+ - (r^+ + s^+)\right)c_3}\right)^{\frac{1}{r^+ + s^+ - q^+}} \leq \left(\frac{c(c_2(\lambda_1+\lambda_2)+1)(r^- + s^-)}{r^- + s^- - p^+}\right)^{\frac{1}{p^- - q^-}}.$$

Then $\lambda_1 + \lambda_2 > K_1$, which is a contradiction, hence we can conclude that for any $0 < \lambda_1 + \lambda_2 < K_1$, we have $N^0_{\lambda_1,\lambda_2} = \emptyset$ for all $(\lambda_1,\lambda_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Remark 3.1 As a conclusion of Lemma 3.1, we can write $N_{\lambda_1,\lambda_2} = N^+_{\lambda_1,\lambda_2} \cup N^-_{\lambda_1,\lambda_2}$ and we define

$$\gamma^+_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v})\in r^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \text{ and } \gamma^-_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}).$$

Lemma 3.3 *Suppose that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *are satisfied. If* $0<\lambda_1+\lambda_2<K_2$ *then for all* $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})<0$.

Proof. Suppose $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, then the definition of J_{λ_1,λ_2} leads to

$$
\begin{aligned}
&J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \leq \frac{1}{p^-}\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
&+\frac{1}{q^-}\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\int_{\partial\mathcal{U}}\left(H_1(\mathrm{y},u(\mathrm{y}))+H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu-\frac{1}{q^-}\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\frac{2}{r^++s^+}\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}\\
&\leq\left(\frac{1}{p^-}-\frac{1}{q^-}\right)\int_{\mathcal{U}}\left[|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}+\mu_1|u(\mathrm{y})|^{q(\mathrm{y})}+\mu_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right]\mathrm{dy}\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{q^+}h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})-H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{q^+}h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})-H_1(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+2\left(\frac{1}{q^+}-\frac{1}{r^++s^+}\right)\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}\\
&\leq\frac{1}{c}\left(\frac{1}{p^-}-\frac{1}{q^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}+\left(c_1(\frac{1}{p^-}-\frac{1}{q^+})+\frac{1}{q^+}\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}\\
&+2c_3\left(\frac{1}{q^+}-\frac{1}{r^++s^+}\right)\|(\mathrm{u},\mathrm{v})\|^{r^-+s^-},
\end{aligned}
\tag{11}
$$

with (c_1,c_3) being the embedding constant of (6), (7). As $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, we get

$$
\begin{aligned}
&p^+\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+q^+\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\int_{\partial\mathcal{U}}\left(H_1(\mathrm{y},u(\mathrm{y}))+H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu-q^-\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-2(r^-+s^-)\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}>0.
\end{aligned}
\tag{12}
$$

Multiplying (3.1) by $-(r^- + s^-)$, by a direct calculation

$$\begin{aligned}
&- (r^- + s^-) \int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} \\
&- (r^- + s^-) \left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+ (r^- + s^-) \int_{\partial\mathcal{U}} (h_1(\mathrm{y}, u(\mathrm{y})).u(\mathrm{y}) + h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)}) \, \mathrm{d}\nu \\
&+ (r^- + s^-) \int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+ 2(r^- + s^-) \int_{\mathcal{U}} |\mathrm{u(y)}|^{r(\mathrm{y})} |\mathrm{v(y)}|^{s(\mathrm{y})} \mathrm{dy} = 0.
\end{aligned} \tag{13}$$

Combining (12) with (13), we get that

$$\begin{aligned}
&\left(p^+ - (r^- + s^-)\right) \int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} + \left(q^+ - (r^- + s^-)\right) \\
&\times \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+ \int_{\partial\mathcal{U}} \left((r^- + s^-) h_1(\mathrm{y}, \mathrm{u(y)}).\mathrm{u(y)} - H_1(\mathrm{y}, \mathrm{u(y)})\right) \mathrm{d}\nu \\
&+ \int_{\partial\mathcal{U}} \left((r^- + s^-) h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)} - H_2(\mathrm{y}, \mathrm{v(y)})\right) \mathrm{d}\nu \\
&(r^- + s^- - q^+) \int_{\mathcal{U}} \left(\lambda_1 |\mathrm{u(y)}|^{q(\mathrm{y})} + \lambda_2 |\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} > 0.
\end{aligned}$$

Combining $(\mathcal{H}_1)$ with Poincare's inequality and Lemma 2.1, we have

$$\begin{aligned}
&(r^- + s^- - q^+) \int_{\mathcal{U}} \left(\lambda_1 |\mathrm{u(y)}|^{q(\mathrm{y})} + \lambda_2 |\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+ (r^- + s^-) \int_{\partial\mathcal{U}} (h_1(\mathrm{y}, \mathrm{u(y)}).\mathrm{u(y)} + h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)}) \, \mathrm{d}\nu \\
&> \left((r^- + s^-) - p^+\right) \int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} + \left((r^- + s^-) - q^+\right) \\
&\times \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&> \frac{1}{c} \left((r^- + s^-) - p^+\right) \|(\mathrm{u}, \mathrm{v})\|^{p^-} + \frac{\mu_0 \left((r^- + s^-) - q^+\right)}{D^{p^+} (c+1)^{p^+}} \|(\mathrm{u}, \mathrm{v})\|^{p^-},
\end{aligned}$$

then

$$\begin{aligned}
&\left(c_2(r^- + s^- - q^+) - q^+(\lambda_1 + \lambda_2) + (r^- + s^-)\right) \|(\mathrm{u}, \mathrm{v})\|^{q^+} \\
&> \left((r^- + s^-) - p^+\right) \left[\frac{1}{c} + \frac{\mu_0}{D^{p^+} (c+1)^{p^+}}\right] \|(\mathrm{u}, \mathrm{v})\|^{p^-}
\end{aligned}$$

and $\|(\mathrm{u}, \mathrm{v})\|^{p^-} < \dfrac{[c_2(r^- + s^- - q^+)(\lambda_1 + \lambda_2) + (r^- + s^-)]}{((r^- + s^-) - p^+) \left[\frac{1}{c} + \frac{\mu_0}{D^{p^+} (c+1)^{p^+}}\right]} \|(\mathrm{u}, \mathrm{v})\|^{q^+}.$

As $(r^- + s^-) > q^+$, using (11), we get

$$J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) < \left[-(\frac{p^- - q^+}{p^- q^+}) \times \frac{[c_2(r^- + s^- - q^+)(\lambda_1 + \lambda_2) + (r^- + s^-)]}{c\left((r^- + s^-) - p^+\right)\left[\frac{1}{c} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\right]} \right] \|(\mathrm{u},\mathrm{v})\|^{q^+}$$
$$+ \left[\left(c_1(\frac{1}{p^-} - \frac{1}{q^+}) + \frac{1}{q^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^+}$$
$$+ \left[2c_3 \left(\frac{1}{q^+} - \frac{1}{r^+ + s^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^+}.$$

Finally, for $\lambda_1 + \lambda_2$ sufficiently large, we get $\gamma^+_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v}) \in N^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) < 0$.

Lemma 3.4 *Under assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$, *if* $0 < \lambda_1 + \lambda_2 < K_3$, *then for all* $(\mathrm{u},\mathrm{v}) \in N^-_{\lambda_1,\lambda_2}$, $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$.

Proof. Let $(\mathrm{u},\mathrm{v}) \in N^-_{\lambda_1,\lambda_2}$. By definition of J_{λ_1,λ_2}, (4), $(\mathcal{H}_1)$, and (3.1), we get

$$J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq \frac{1}{p^+} \int_{\mathcal{U}} \left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy}$$
$$+ \frac{1}{q^+} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y}) |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y}) |\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy}$$
$$- \int_{\partial\mathcal{U}} (H_1(\mathrm{y}, \mathrm{u}(\mathrm{y})) + H_2(\mathrm{y}, \mathrm{v}(\mathrm{y})))\, \mathrm{d}\nu - \frac{1}{q^-} \int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy}$$
$$- \frac{2}{r^- + s^-} \int_{\mathcal{U}} |\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} \mathrm{dy}$$
$$\geq \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) \int_{\mathcal{U}} \left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy}$$
$$+ \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \int_{\mathcal{U}} \left(\mu_1(\mathrm{y}) |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y}) |\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy}$$
$$- \int_{\partial\mathcal{U}} \left(\frac{1}{r^- + s^-} h_1(\mathrm{y}, \mathrm{u}(\mathrm{y})).\mathrm{u}(\mathrm{y}) - H_1(\mathrm{y}, \mathrm{u}(\mathrm{y})) \right) \mathrm{d}\nu$$
$$- \int_{\partial\mathcal{U}} \left(\frac{1}{r^- + s^-} h_2(\mathrm{y}, \mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}) - H_2(\mathrm{y}, \mathrm{v}(\mathrm{y})) \right) \mathrm{d}\nu$$
$$+ \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy}.$$

Since $\beta > r^- + s^-$, we get $\frac{1}{r^- + s^-} > \frac{1}{\beta}$, and by $(\mathcal{H}_1)$, we have

$$J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq \frac{1}{c} \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) \|(\mathrm{u},\mathrm{v})\|^{p^-} + \mu_0 \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \|(\mathrm{u},\mathrm{v})\|^{q^-}$$
$$+ c_2(\lambda_1 + \lambda_2) \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \|(\mathrm{u},\mathrm{v})\|^{q^+}.$$

Since $q^- \leq q^+ < p^-$, we have

$$J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq \left[\frac{1}{c} \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) + \mu_0 \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^-}$$
$$+ \left[\frac{1}{c} c_2(\lambda_1 + \lambda_2) \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^-}.$$

So, if we take $\lambda_1+\lambda_2 \le \left[\frac{1}{cc_2}(\frac{1}{p^+}-\frac{1}{r^-+s^-})+\frac{\mu_0}{c_2}(\frac{1}{q^+}-\frac{1}{r^-+s^-})\right]\left[\frac{q^+(s^++r^+)}{s^-+r^--q^+}\right]=K_3$, we obtain that $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})>0$, this implies that $\gamma^-_{\lambda_1,\lambda_2}=\inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})>0$. Hence, $N_{\lambda_1,\lambda_2}=N^-_{\lambda_1,\lambda_2}\cup N^+_{\lambda_1,\lambda_2},\ N^-_{\lambda_1,\lambda_2}\cap N^+_{\lambda_1,\lambda_2}=\emptyset$, by the above lemma, we must have $(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}$.

4 Minimizer on $N^+_{\lambda_1,\lambda_2}$ and $N^-_{\lambda_1,\lambda_2}$.

We will show that there are two nonnegative solutions to the system.

Theorem 4.1 *Under assumptions $(\mathcal{H}_1)-(\mathcal{H}_3)$, there exists a minimizer $(\mathrm{u}_0^+,\mathrm{v}_0^+)$ of $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})$ on $N^+_{\lambda_1,\lambda_2}$, for every $\lambda_1+\lambda_2<K=\min(K_1,K_2)$, such that $J_{\lambda_1,\lambda_2}(\mathrm{u}_0^+,\mathrm{v}_0^+)=\gamma^+_{\lambda_1,\lambda_2}$.*

Proof. Lemma 3.1 implies J_{λ_1,λ_2} is bounded below on N_{λ_1,λ_2}, so it is bounded below in $N^+_{\lambda_1,\lambda_2}$, then there exists a minimizing sequence $\{\mathrm{u}_n^+,\mathrm{v}_n^+\}\in r^+_{\lambda_1,\lambda_2}$ such that

$$\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^+,\mathrm{v}_n^+)=\inf_{(\mathrm{u},\mathrm{v})\in r^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})=\gamma^+_{\lambda_1,\lambda_2}<0. \tag{14}$$

Note that J_{λ_1,λ_2} is bounded on W. Hence, without loss of generality, we suppose $(\mathrm{u}_n^+,\mathrm{v}_n^+)\to(\mathrm{u}_0^+,\mathrm{v}_0^+)$ on W; and by the compact embedding, we get

$$\begin{aligned}
&\mathrm{u}_n^+\to\mathrm{u}_0^+ \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{\alpha(x)}(\mathcal{U}) \text{ and } L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n\to\infty,\\
&\mathrm{v}_n^+\to\mathrm{v}_0^+ \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{\alpha(x)}(\mathcal{U}) \text{ and } L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n\to\infty,\\
&\mathrm{u}_n^+\to u_0^+ \text{ and } \mathrm{v}_n^+\to\mathrm{v}_0^+ \text{ a.e in } \mathcal{U} \text{ as } n\to\infty.
\end{aligned} \tag{15}$$

Next, we will prove that $\mathrm{u}_n^+\to\mathrm{u}_0^+$ and $\mathrm{v}_n^+\to\mathrm{v}_0^+$ on $W^{1,p(\mathrm{y})}(\mathcal{U})$ as $n\to\infty$. Otherwise, let $\mathrm{u}_n^+\to u_0^+$ and $\mathrm{v}_n^+\to\mathrm{v}_0^+$ on $W^{1,p(\mathrm{y})}(\mathcal{U})$ as $n\to\infty$, then we have

$$\rho_{q(\mathrm{y})}(\mathrm{u}_0^+)\le\lim_{n\to\infty}\inf\rho_{q(\mathrm{y})}(\mathrm{u}_n^+),\ \text{and } \rho_{q(\mathrm{y})}(\mathrm{v}_0^+)\le\lim_{n\to\infty}\inf\rho_{q(\mathrm{y})}(\mathrm{v}_n^+).$$

Since $\langle J_{\lambda_1,\lambda_2}(\mathrm{u}_n^+,\mathrm{v}_n^+),(\mathrm{u}_n^+,\mathrm{v}_n^+)\rangle=0$, we get

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u}_n^+,\mathrm{v}_n^+)\ge{}&\frac{1}{c}\left(\frac{1}{p^+}-\frac{1}{r^-+s^-}\right)\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{p^-}+\mu_0\left(\frac{1}{q^+}-\frac{1}{r^-+s^-}\right)\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{q^-}\\
&+c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^++s^+}-\frac{1}{q^+}\right)\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{q^+}.
\end{aligned}$$

That is,

$$\begin{aligned}
\lim_{n\to\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^+,\mathrm{v}_n^+)\ge{}&\frac{1}{c}\left(\frac{1}{p^+}-\frac{1}{r^-+s^-}\right)\lim_{n\to\infty}\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{p^-}\\
&+\mu_0\left(\frac{1}{q^+}-\frac{1}{r^-+s^-}\right)\lim_{n\to\infty}\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{q^-}\\
&+c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^++s^+}-\frac{1}{q^+}\right)\lim_{n\to\infty}\|(\mathrm{u}_n^+,\mathrm{v}_n^+)\|^{q^+}.
\end{aligned}$$

By (15) and (4), we have

$$\sigma^{+}_{\lambda_1,\lambda_2} > \frac{1}{c}\left(\frac{1}{p^{+}} - \frac{1}{r^{-}+s^{-}}\right)\|(u_0^{+}, \mathrm{v}_0^{+})\|^{p^{-}} + \mu_0\left(\frac{1}{q^{+}} - \frac{1}{r^{-}+s^{-}}\right)\|(\mathrm{u}_0^{+}, \mathrm{v}_0^{+})\|^{q^{-}}$$
$$+ c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^{+}+s^{+}} - \frac{1}{q^{+}}\right)\|(\mathrm{u}_0^{+}, \mathrm{v}_0^{+})\|^{q^{+}}.$$

Since $r^{-} + s^{-} > p^{-} > q^{+}$ for $\|(\mathrm{u},\mathrm{v})\| > 1$, we deduce that $\sigma^{+}_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v})\in r^{+}_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$, which is in contradiction with Lemma 3.3, hence

$$\mathrm{u}_n^{+} \to \mathrm{u}_0^{+} \text{ and } \mathrm{v}_n^{+} \to \mathrm{v}_0^{+} \text{ on } W_0^{1,p(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty.$$

Consequently, $(\mathrm{u}_0^{+}, \mathrm{v}_0^{+})$ is a minimizer of J_{λ_1,λ_2} on $N^{+}_{\lambda_1,\lambda_2}$.

Theorem 4.2 *Suppose that conditions* $(\mathcal{H}_1)$- $(\mathcal{H}_3)$ *are true. Then there exists a minimizer* $(\mathrm{u}_0^{-}, \mathrm{v}_0^{-})$ *of* J_{λ_1,λ_2} *on* $N^{-}_{\lambda_1,\lambda_2}$ *for all* $0 < \lambda_1 + \lambda_2 < K = \min\{K_1, K_2\}$ *such that* $J_{\lambda_1,\lambda_2}(u_0^{-}, \mathrm{v}_0^{-}) = \sigma^{-}_{\lambda_1,\lambda_2}$.

Proof. J_{λ_1,λ_2} is bounded below on N_{λ_1,λ_2}, and so on $r^{-}_{\lambda_1,\lambda_2}$, then there exists a minimizing sequence $\{\mathrm{u}_n^{-}, \mathrm{v}_n^{-}\} \in N^{-}_{\lambda_1,\lambda_2}$ such that

$$\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^{-}, \mathrm{v}_n^{-}) = \inf_{(\mathrm{u},\mathrm{v})\in N^{-}_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma^{-}_{\lambda_1,\lambda_2} > 0.$$

So, the sequence$\{\mathrm{u}_n^{-}, \mathrm{v}_n^{-}\}_{n\in\mathbb{N}}$ is bounded in W. There exists $(u_0^{-}, \mathrm{v}_0^{-}) \in W$ such that up to a subsequence $(\mathrm{u}_n^{-}, \mathrm{v}_n^{-}) \rightharpoonup (u_0^{+}, \mathrm{v}_0^{+})$ in W. Thanks to Theorem 2.1, we obtain

$$\begin{cases} \mathrm{u}_n^{-} \to u_0^{-} \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty, \\ \mathrm{v}_n^{-} \to \mathrm{v}_0^{-} \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty, \\ \mathrm{u}_n^{-} \to u_0^{-} \text{ and } \mathrm{v}_n^{-} \to \mathrm{v}_0^{-} \text{ a.e in } \mathcal{U} \text{ as } n \to \infty. \end{cases} \quad (16)$$

Hence, $(\mathrm{u}_0^{-}, \mathrm{v}_0^{-}) \in N^{-}_{\lambda_1,\lambda_2}$, $\exists t > 0$, such that $(t\mathrm{u}_0^{-}, t\mathrm{v}_0^{-}) \in N^{-}_{\lambda_1,\lambda_2}$ and $J_{\lambda_1,\lambda_2}(\mathrm{u}_0^{-}, \mathrm{v}_0^{-}) \geq J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^{-}, t\mathrm{v}_0^{-})$. According to $(\mathcal{H}_1)$ and the definition of $\xi'_{\lambda_1,\lambda_2}$, we have

$$
\begin{aligned}
\left\langle \xi^{'}_{\lambda_1,\lambda_2}(tu_0^-, t\mathrm{v}_0^-), (tu_0^-, t\mathrm{v}_0^-) \right\rangle &= \int_{\mathcal{U}} p(\mathrm{y}) \left(|Dtu_0^-(\mathrm{y})|^{p(\mathrm{y})} + |Dt\mathrm{v}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\
&- 2\int_{\mathcal{U}} (r(\mathrm{y}) + s(\mathrm{y})|tu_0^-(x)|^{r(\mathrm{y})}|t\mathrm{v}_0^-(x)|^{s(\mathrm{y})}\mathrm{dy} \\
&+ \int_{\mathcal{U}} q(\mathrm{y}) \left(\mu_1(\mathrm{y})|Dtu_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|Dt\mathrm{v}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- \int_{\partial\mathcal{U}} \left(H_1(x, t\mathrm{u}_0^-(x)) + H_2(x, t\mathrm{v}_0^-(x)) \right) \mathrm{d}\nu \\
&- \int_{\mathcal{U}} q(\mathrm{y}) \left(\lambda_1 |t\mathrm{u}_0^-(x)|^{q(\mathrm{y})} + \lambda_2 |t\mathrm{v}_0^-(x)|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- 2\int_{\mathcal{U}} (r(\mathrm{y}) + s(\mathrm{y})|tu_0^-(x)|^{r(\mathrm{y})}|t\mathrm{v}_0^-(x)|^{s(\mathrm{y})}\mathrm{dy} \\
&\le p^+ t^{p^+} \int_{\mathcal{U}} \left(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\
&+ q^+ t^{q^-} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- q^- t^{q^-} \int_{\mathcal{U}} \left(\lambda_1 |\mathrm{u}_0^-(x)|^{q(\mathrm{y})} + \lambda_2 |\mathrm{v}_0^-(x)|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- 2(r^- + s^-) t^{r^- + s^-} \int_{\mathcal{U}} |\mathrm{u}_0^-(x)|^{r(\mathrm{y})} |\mathrm{v}_0^-(x)|^{s(\mathrm{y})} \mathrm{dy}.
\end{aligned}
$$

Due to $q^- \le q^+ < p^+ < s^- + r^-$, and by Propositions 2.2, 2.3, it follows that $\langle \xi^{'}_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-), (t\mathrm{u}_0^-, t\mathrm{v}_0^-) \rangle < 0$. Hence, by definition of $N^-_{\lambda_1,\lambda_2}$, $(t\mathrm{u}_0^-, t\mathrm{v}_0^-) \in N^-_{\lambda_1,\lambda_2}$. Next, we show that $(\mathrm{u}_n^-, \mathrm{v}_n^-) \to (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W(\mathcal{U})$. Assume that $(\mathrm{u}_n^-, \mathrm{v}_n^-) \nrightarrow (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W$, by Fatou's Lemma, we have

$$
\begin{aligned}
&\int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&\le \lim_{n\to+\infty} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_n^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_n^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy},
\end{aligned}
$$

by (16), we get

$$
\int_{\mathcal{U}} \left(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \le \lim_{n\to+\infty} \int_{\mathcal{U}} \left(|\mathrm{Du}_n^-(\mathrm{y})|^{p(\mathrm{y})} |\mathrm{Dv}_n^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy}.
$$

Then, by $(\mathcal{H}_1)$, we have

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-) \leq{}& \frac{t^{p^+}}{p^-}\int_{\mathcal{U}}\Big(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} \\
&+ \frac{t^{q^+}}{q^-}\int_{\mathcal{U}}\Big(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}}\big(H_1(x, t\mathrm{u}_0^-(x)) + H_2(x, t\mathrm{v}_0^-(\mathrm{y}))\big)\,\mathrm{d}\nu \\
&- \frac{t^{q^-}}{q^+}\int_{\mathcal{U}}\Big(\lambda_1|\mathrm{u}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2|\mathrm{v}_0^-(x)|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- 2\frac{t^{r^-+s^-}}{r^+ + s^+}\int_{\mathcal{U}}|\mathrm{u}_0^-(x)|^{r(\mathrm{y})}|\mathrm{v}_0^-(x)|^{s(\mathrm{y})}\mathrm{dy} \\
&\leq \lim_{n\to+\infty}\frac{t^{p^+}}{p^-}\int_{\mathcal{U}}\Big(|\mathrm{Du}_n^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_n^-(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} \\
&+ \lim_{n\to+\infty}\frac{t^{q^+}}{q^-}\int_{\mathcal{U}}\Big(\mu_1(\mathrm{y})|\mathrm{Du}_n^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_n^-(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \lim_{n\to+\infty}\int_{\partial\mathcal{U}}\big(H_1(x, t\mathrm{u}_n^-(x)) + H_2(x, t\mathrm{v}_n^-(x))\big)\,\mathrm{d}\nu \\
&- \lim_{n\to+\infty}\frac{t^{q^-}}{q^+}\int_{\mathcal{U}}\Big(\lambda_1|\mathrm{u}_n^-(x)|^{q(\mathrm{y})} + \lambda_2|\mathrm{v}_n^-(x)|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \lim_{n\to+\infty}\frac{2t^{r^-+s^-}}{r^+ + s^+}\int_{\mathcal{U}}|\mathrm{u}_n^-(x)|^{r(\mathrm{y})}|\mathrm{v}_n^-(x)|^{s(\mathrm{y})}\mathrm{dy} \\
&\leq \lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(t\mathrm{u}_n^-, t\mathrm{v}_n^-) \\
&< \lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^-, \mathrm{v}_n^-) = \inf_{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}^-} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma_{\lambda_1,\lambda_2}^-.
\end{aligned}$$

Hence

$$J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-) < \inf_{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma_{\lambda_1,\lambda_2}^-.$$

This a contradiction, consequently, $(\mathrm{u}_n^-, \mathrm{v}_n^-) \to (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W(\mathcal{U})$, and $\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^-, \mathrm{v}_n^-) = J_{\lambda_1,\lambda_2}(\mathrm{u}_0^-, \mathrm{v}_0^-) = \gamma_{\lambda_1,\lambda_2}^-$. After that, we infer that $(\mathrm{u}_0^-, \mathrm{v}_0^-)$ is a minimization of J_{λ_1,λ_2} on $N_{\lambda_1,\lambda_2}^-$.

Proof of Theorem 1.1 From Theorem 4.1 and Theorem 4.2, there are $(u^+, v^+) \in N_{\lambda_1,\lambda_2}^+$ and $(u^-, v^-) \in N_{\lambda_1,\lambda_2}^-$ for every $\lambda_1 + \lambda_2 \in (0, min\,\{K_1, K_2\})$ such that

$$J_{\lambda_1,\lambda_2}(\mathrm{u}_0^-, \mathrm{v}_0^-) = \inf_{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}^-}(\mathrm{u},\mathrm{v}) \text{ and } J_{\lambda_1,\lambda_2}(\mathrm{u}_0^+, \mathrm{v}_0^+) = \inf_{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}^+}(\mathrm{u},\mathrm{v}).$$

Then the system (1) admits $(\mathrm{u}_0^-, \mathrm{v}_0^-) \in N_{\lambda_1,\lambda_2}^-$ and $(\mathrm{u}_0^+, \mathrm{v}_0^+) \in N_{\lambda_1,\lambda_2}^+$ as two solutions in $W(\mathcal{U})$; thanks to Lemma (3.2), it follows that $N_{\lambda_1,\lambda_2}^- \cap N_{\lambda_1,\lambda_2}^+ = \emptyset$. Then $(\mathrm{u}_0^-, \mathrm{v}_0^-) \neq (\mathrm{u}_0^+, \mathrm{v}_0^+)$. Following this, we show that $(\mathrm{u}_0^\pm, \mathrm{v}_0^\pm)$ are non-negative in $\mathcal{U}$. For that, we introduce the truncation function $h_{+,i}(\mathrm{y}, s) : \partial\mathcal{U}\times\mathbb{R}\to\mathbb{R}$ defined by

$$h_{+,i}(\mathrm{y}, s) = h_i(\mathrm{y}, s) \text{ if } s > 0, \text{ and } h_{+,i}(\mathrm{y}, s) = 0, \text{ if } s < 0, \text{ with } i = 1, 2.$$

We set $H_{+,i}(\mathrm{y}, s) = \int_0^s h_i(\mathrm{y}, t)dt$, and the C^1-functional $J^+_{\lambda_1,\lambda_2} : W \to \mathbb{R}$ is given by

$$
\begin{aligned}
&J_{\lambda_1,\lambda_2}(\mathrm{u}, \mathrm{v}) \\
&= \int_{\mathcal{U}} \frac{1}{p(\mathrm{y})} \Big(|\, \mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})} \Big) \mathrm{dy} + \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} \mu_1(\mathrm{y}) |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} \\
&+ \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} |\mu_2(\mathrm{y}) \mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})} \mathrm{dy} - \int_{\partial\mathcal{U}} H_{+,1}(\mathrm{y}, \mathrm{u}(\mathrm{y})) + H_{+,2}(\mathrm{y}, \mathrm{v}(\mathrm{y})) \mathrm{d}\nu.
\end{aligned}
$$

Then by Proposition 2.3, we get that for all $(\mathrm{u}_0^-, \mathrm{v}_0^-) = \min(0, (\mathrm{u}, \mathrm{v}))$,

$$
\begin{aligned}
0 &= \Big\langle J^+_{\lambda_1,\lambda_2}(u_-, v_-), (u_-, v_-) \Big\rangle \\
&\geq p^- \rho_{p(.)}(Du_-, Dv_-) + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}} \rho_{q(.)}(u_-, v_-) \\
&\geq \rho_{p(.)}(u_-, v_-) \\
&\geq \|(u_-, v_-)\|^{p^-}.
\end{aligned}
$$

Hence $\|(u_-, v_-)\|^{p^-} = 0$ and thus, $(\mathrm{u}, \mathrm{v}) = (u_+, v_+)$, then, choosing $(\mathrm{u}, \mathrm{v}) = (\mathrm{u}_0^-, \mathrm{v}_0^-)$ and $(\mathrm{u}, \mathrm{v}) = (\mathrm{u}_0^+, \mathrm{v}_0^+)$, we conclude $(\mathrm{u}_0^\pm, \mathrm{v}_0^\pm)$ is a non-negative solution of system (1).

5 Conclusion

The Nehari manifold method is a powerful variational tool for proving the existence (and sometimes multiplicity) of solutions to nonlinear coupled elliptic systems under Neumann boundary conditions. Its effectiveness arises from converting the PDE problem into a constrained minimization problem in an appropriate Sobolev space. The method filters out trivial or non-physical solutions by exploiting the geometry of the energy functional and the nonlinearity.

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