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Generalized n-Characteristic, Coincidence and Fixed Point Theorems for a Class of Pairs of Morphisms

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Abstract: This paper is devoted to the construction and study of a topological invariant for a class of pairs of morphisms $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$, where Top denotes the category of Hausdorff topological spaces and continuous single valued maps and X, Y, Z represent subsets of $\mathbb{R}^{n+1}$ such that X, Y contain the sphere S^n. This invariant termed as a generalized n-characteristic of the pair (f, g), is derived using homotopy methods serving as a valuable tool in coincidence point theory. The paper establishes several properties of this invariant, extends it to a class of admissible multivalued mappings, and presents a fixed point theorem among its results.

Keywords: *homotopy; topological invariant; n-connected spaces; multivalued mappings.*

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1 Introduction

The concept of topological invariants, degrees, characteristics, or generalized characteristics, has been extensively explored by numerous authors for various classes of single-valued and multivalued mappings (see, for example, [7, 10, 13]. By exploration of various topological techniques, this concept serves as a powerful tool in analyzing and proving results in fixed point theory. This provides practical applications across diverse fields such as nonlinear analysis [6], economics [3], biology [5] and physics [9]. In some cases, they are very useful to prove the existence of solutions for linear or semi-linear dynamical systems [8].

Moreover, topological invariants are highly instrumental for the study of bifurcations and nonlinear dynamical systems (see [9]). In the case when nonlinearities are not smooth enough, they help to identify fixed points and their stability, which are critical

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in understanding bifurcations. Also, they offer deep insights into the global structure and complexity of systems, making it possible to fully comprehend qualitative transitions as parameters change. With these invariants, researchers can not only detect and classify bifurcations, but also develop predictive theories and models for a wide range of dynamical phenomena.

Let $S^n \subset X \subseteq Y \subset \mathbb{R}^{n+1}$, where S^n is the unit sphere of $\mathbb{R}^{n+1}$ and X is an n-connected space, and let Z be an arbitrary non empty subset of $\mathbb{R}^{n+1}$.

In the present paper, we build a topological invariant for a class of pairs of morphisms $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$ such that the maps f, g do not coincide on the sphere S^n. This class of morphisms is denoted M_Z. This construction can be seen as a natural extension of the characteristic $\chi_{S^n}(f)$ defined on the sphere S^n (see [4], [14]). Indeed, recall briefly that the degree of continuous maps f in $Mor_{Top}(S^n, S^n)$ is closely related to the group $\pi_n(S^n)$ and it is often used in analysis. Consider a continuous map $f : S^n \to S^n$ and let γ_n be a generator of the group $\pi_n(S^n) \simeq \mathbb{Z}$ (see [16]), then $f_*(\gamma_n) = \alpha\gamma_n$, where $\alpha \in \mathbb{Z}$. The number α is called the degree of f and it is denoted by $deg f$.

Since the space $\mathbb{R}^{n+1} \setminus \{\theta\}$ is homotopy equivalent to the sphere S^n, one may define the degree of morphisms f in $Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. It is called the characteristic (or rotation) of the vector field f, and it is denoted by $\chi_{S^n}(f)$. Using algebraic and geometric topological methods, different generalizations for topological degree are given for single valued maps and multivalued maps (see [13, 14]).

This paper is divided into three sections. After the introduction, in Section 2, we build and develop some properties of the generalized n-characteristic, in particular, we show that it is a homotopy invariant. We also show that it can be used in coincidence and fixed points theories. In Section 3, we build and study a generalized n-mult characteristic for a class of multivalued mappings. As application, a fixed point theorem for this class is provided.

2 Generalized n-Characteristic of a Pair of Morphisms

Let Top be the category of T_2-topological spaces and continuous single valued maps and G_d be the category of graded groups and homomorphisms of degree zero. In what follows, $\pi : Top \to G_d$ stands for the covariant functor of the homotopy which assigns to a T_2-topological space X a graded group $\{\pi_n(X)\}_{n\geq 0} \in Obj(G_d)$, and to a given morphism $f \in Mor_{Top}(X, Y)$ a homomorphism of degree zero $\pi_n(f) \in Mor_{G_d}(\{\pi_n(X)\}_{n\geq 0}, \{\pi_n(Y)\}_{n\geq 0})$.

For given topological spaces X, Y in the category Top, the set of all continuous maps from X to Y is denoted by $C(X, Y)$. For the sake of easy reference, we recall some terminology and facts concerning n-connected spaces.

Definition 2.1 (see [12], [4]) A path-connected space X is n-connected if its n-first homotopy groups $\pi_k(X)$ $(0 < k \leq n)$ are trivial.

Thus, 0-connected means path connected and 1-connected means simply connected. The examples of n-connected spaces are:

- The Euclidean space $\mathbb{R}^n$ is 1-connected.
- The unit sphere S^n is (n-1)-connected.
- The unit ball $B^{n+1} = \{x \in \mathbb{R}^{n+1}, \|x\| \leq 1\}$ is an n-connected space.

- Every CW complex X, with exactly one 0 cell and all other cells having dimensions greater than n, is n-connected.

Because of its size (n-connected spaces can also be produced from other spaces) and the intriguing properties and applications that have been developed for it, the class of n-connected spaces is quite significant. Examples include those of Whitehead, Hurewicz, and Frendenthal (refer to [1, 11, 12, 16]), which, when applied to various problems, yield remarkable results in the area of algebraic topology.

Let M_Z be the class of maps defined in the Introduction. Our goal in this section is to define for the elements (f, g) of M_Z a topological invariant.

Let $(f, g) \in M_Z$, which means that $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$ such that $S^n \subset X \subseteq Y \subset \mathbb{R}^{n+1}$, X is an n-connected space and Z is an arbitrary non empty subset of $\mathbb{R}^{n+1}$. We can get the pair of morphisms $(\tilde{f}, \tilde{g}) \in Mor_{Top}(S^n, Z)$, where $\tilde{f}, \tilde{g} : S^n \to Z$ are the restrictions of f, g, respectively, on the sphere S^n. The fact that $\text{Coinc}((f, g), S^n) = \emptyset$ entails $(\tilde{f} - \tilde{g}) \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. This allows us to give the following definition.

Definition 2.2 The generalized n-characteristic of the pair $(f, g) \in M_Z$ is defined as the homomorphism $\xi(f, g) = \pi_n(\tilde{f} - \tilde{g}) = (\tilde{f} - \tilde{g})_{*,n} \in Mor_{G_d}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\}))$.

Let us give some properties of this generalized n-characteristic.

Proposition 2.1 *Let $(f, g) \in M_Z$ so that the generalized n-characteristic $\xi(f, g)$ is not trivial, then the equation $f(x) - g(x) = \theta$ admits at least one solution in $X \setminus S^n$, i.e., there exists a coincidence point of f, g in $X \setminus S^n$.*

Proof. For the reason that $(f, g) \in M_Z$, we have $\text{Coinc}((f, g), S^n) = \emptyset$. Suppose that $f(x) \neq g(x)$ for all $x \in X \setminus S^n$, so $(f - g) \in Mor_{Top}(X, \mathbb{R}^{n+1} \setminus \{\theta\})$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f-g} & \mathbb{R}^{n+1} \setminus \{\theta\} \\ & \underset{i}{\nwarrow} & \uparrow \tilde{f} - \tilde{g} \\ & & S^n. \end{array}$$

Applying the co-variant functor π to the diagram above and using the fact that X is an n-connected space, we deduce that $(\tilde{f} - \tilde{g})_{n,*}$ is a trivial homomorphism.

Corollary 2.1 *Let $X \subseteq Z \subseteq \mathbb{R}^{n+1}$, $i \in Mor_{Top}(X, Z)$ be a canonical injection, and $(i, g) \in M_Z$ so that $\xi(i, g)$ is not trivial, then the equation $x - g(x) = \theta$ admits at least one solution in $X \setminus S^n$.*

Proof. We apply Proposition 2.1 to the pair (i, g).

Let us show that the generalized n-characteristic is a homotopy invariant. First, we introduce the following definition.

Definition 2.3 Two pairs (f, g), (f', g') in M_Z are said to be homotopic if there exists a pair of continuous maps $(F, G) \in Mor_{Top}(X \times [0, 1], Z) \times Mor_{Top}(Y \times [0, 1], Z)$ such that the following conditions are satisfied:

1. $\text{Coinc}((F, G), S^n \times [0, 1]) = \emptyset$.
2. $F(x, 0) = f(x), F(x, 1) = f'(x)$, for all $x \in X$.

3. $G(y,0) = g(y), G(y,1) = g'(y)$, for all $y \in Y$.

In this case, (F,G) is said to be the homotopy between the pairs (f,g) and (f',g').

Proposition 2.2 *If (f,g), (f',g') are some homotopic elements of M_Z, then $\xi(f,g) = \xi(f',g')$.*

Proof. Let (F,G) be the homotopy between (f,g) and (f',g'). One can consider the morphism $\tilde{F} - \tilde{G} \in Mor_{Top}(S^n \times [0,1], \mathbb{R}^{n+1} \setminus \{\theta\})$, where $\tilde{F}, \tilde{G}$ are the restrictions of F, G to $S^n \times [0,1]$. Using the fact that $\tilde{F}$, $\tilde{G}$ are, respectively, homotopies between $\tilde{f}$, $\tilde{f'}$ and $\tilde{g}$, $\tilde{g}'$, one deduces that $\tilde{F} - \tilde{G}$ is continuous and it is a homotopy between the morphisms $(\tilde{f} - \tilde{g}), (\tilde{f'} - \tilde{g}') \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. Therefore, we have

$$\xi(f,g) = (\tilde{f} - \tilde{g})_{n,*} = (\tilde{f'} - \tilde{g}')_{n,*} = \xi(f',g').$$

Proposition 2.3 *Let X_0 be a subset of $\mathbb{R}^{n+1}$ which contains S^n, and $\varphi \in Mor_{Top}(X_0, X)$ so that $\varphi(S^n) = \tilde{\varphi}(S^n) \subseteq S^n$ and $(f,g) \in M_Z$, then $(f,g) \circ \varphi = (f \circ \varphi, g \circ \varphi) \in M_Z$ and $\xi((f,g) \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*}$.*

Proof. Let us consider $f \circ \varphi \in Mor_{Top}(X_0, Z)$ and $\tilde{f} \circ \tilde{\varphi} \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. If we suppose that there exists $x_0 \in S^n$ such that $f \circ \varphi(x_0) = g \circ \varphi(x_0)$, then we obtain $y_0 = \varphi(x_0) \in \text{Coinc}((f,g), S^n)$. It is impossible because $(f,g) \in M_Z$. We deduce that $\text{Coinc}(f \circ \varphi, g \circ \varphi) = \emptyset$, then $(f \circ \varphi, g \circ \varphi) \in M_Z$. Furthermore,

$$\xi(f \circ \varphi, g \circ \varphi) = (\widetilde{f \circ \varphi} - \widetilde{f \circ \varphi})_{n,*} = ((\tilde{f} - \tilde{g}) \circ \tilde{\varphi})_{n,*} = (\tilde{f} - \tilde{g})_{n,*} \circ \tilde{\varphi}_{n,*}.$$

Thus, $\xi(f \circ \varphi, g \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*}$.

Corollary 2.2 *Let $S^n \subset X_0 \subset \mathbb{R}^{n+1}$, $\varphi \in Mor_{Top}(X_0, X)$ with $\varphi(S^n) = S^n$, and the restriction $\tilde{\varphi} : S^n \to S^n$ induces an epimorphism in the n-th group of homotopy, then if $(f,g) \in M_Z$, we have the equivalence*

$$\xi(f,g) \textit{ is not trivial if only if } \xi((f,g) \circ \varphi) \textit{ is not trivial.}$$

Proof. Suppose that $\xi((f,g) \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*} = 0$. Using the fact that $\varphi_{n,*}$ is an epimorphism, we can deduce that

$$\xi(f,g) \circ \tilde{\varphi}_{n,*}(\pi_n(S^n)) = \xi(f,g)(\pi_n(S^n)) = 0_{\pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})}.$$

Then $\xi(f,g) = 0$. On the other hand, suppose that $\xi(f,g) = 0$, then $\xi(f,g) \circ \tilde{\varphi}_{n,*} = 0$, so $\xi((f,g) \circ \varphi) = 0$.

Proposition 2.4 *Let $(f',g') \in Mor_{Top}(X',Z') \times Mor_{Top}(Y',Z')$ satisfying the conditions*

1. $S^n \subset X' \subseteq Y' \subset \mathbb{R}^{n+1}$ *and* $Z' \subseteq \mathbb{R}^{n+1}$,
2. $\text{Coinc}((f',g'), S^n) = \emptyset$.

Let (f,g) be an element of M_Z, then the product pair $(f \times f', g \times g') \in M_{Z \times Z'}$ and $\xi(f \times f', g \times g') = J \circ (\xi(f,g) \times \xi(f',g')) \circ I$, where I, J stand for the product isomorphisms.

Proof. Using the fact that (f,g) and (f',g') are in M_Z, we can check that $(f\times f', g\times g')\in M_{Z\times Z'}$. Furthermore, denote by P_r the continuous projection, we can built the following diagram:

$$\begin{array}{ccc} S^n & \overset{\tilde{f}'-\tilde{g}'}{\longrightarrow} & \mathbb{R}^{n+1}\setminus\{\theta\} \\ Pr_{r_2}^{X'}\uparrow & (I) & \uparrow Pr_{r2}^{Y'} \\ S^n\times S^n & \overset{\widetilde{f\times f'}-\widetilde{g\times g'}}{\longrightarrow} & \mathbb{R}^{n+1}\setminus\{\theta\}\times\mathbb{R}^{n+1}\setminus\{\theta\} \\ Pr_{r1}^{X}\downarrow & (II) & \downarrow Pr_{r1}^{Y} \\ S^n & \overset{\tilde{f}-\tilde{g}}{\longrightarrow} & \mathbb{R}^{n+1}\setminus\{\theta\}. \end{array}$$

From the commutativity of the two squares (I), (II) of the diagram above, one can deduce the equality $j\circ(\widetilde{f\times f'}-\widetilde{g\times g'})=((\tilde{f}-\tilde{g})\circ\ (\tilde{f}'-\tilde{g}'))\circ i$, where $i=Pr_{r1}^{X}\times Pr_{r2}^{X'}$ and $j=Pr_{r1}^{Y}\times Pr_{r2}^{Y'}$. Since the induced homomorphisms I,J of i,j, respectively, are both isomorphisms (see [15]), we deduce the equality

$$\xi(f\times f', g\times g') = J^{-1}\circ(\xi(f,g)\times\xi(f',g'))\circ I.$$

Let us consider the case $X=Y=Z=B^{n+1}=\{x\in\mathbb{R}^{n+1}/\|x\|\le 1\}$ and let $C_\theta : B^{n+1}\to B^{n+1}$ be the constant map, where $C_\theta(x)=\theta$ for every element $x\in B^{n+1}$.

Proposition 2.5 *Let $(Id_{B^{n+1}},g)$ be an element of $M_{B^{n+1}}$. Then for C_θ as defined above, we have $\xi(Id_{B^{n+1}},g)=\xi(Id_{B^{n+1}},C_\theta)$ and it is not trivial.*

Proof. Let $F: B^{n+1}\times[0,1]\to B^{n+1}$ and $G: B^{n+1}\times[0,1]\to B^{n+1}$ be given by the rules $F(x,t)=x$ and $G(x,t)=tg(x)$ for every $(x,t)\in B^{n+1}\times[0,1]$. The pair (F,G) satisfies the conditions of Definition 2.3, then we deduce that it is a homotopy between $(Id_{B^{n+1}},C_\theta)$ and $(Id_{B^{n+1}},g)$. Hence, from Proposition 2.2, one gets $\xi(Id_{B^{n+1}},C_\theta)=\xi(Id_{B^{n+1}},g)$. We conclude the proof by remarking that $\xi(Id_{B^{n+1}},C_\theta)=i_*$, where $i: S^n\to\mathbb{R}^{n+1}\setminus\{\theta\}$ is a homotopy equivalence (see [4]).

Corollary 2.3 *If $(Id_{B^{n+1}},g)$ is an element of $M_{B^{n+1}}$, then the equation $x-g(x)=\theta$ admits at least one solution in the interior of B^{n+1}.*

Proof. From Proposition 2.5, we have $\xi(Id_{B^{n+1}},g)$ is not trivial. The result is obtained by applying Proposition 2.1 to the pair $(Id_{B^{n+1}},g)$.

Proposition 2.6 *If (f,g) in $M_{B^{n+1}}$ is such that $f(x)-g(x)$ and x are not in the opposite sense for all vector x in S^n, then $\xi(f,g)$ is not trivial.*

Proof. First, we can verify that the pair $(Id, g-f)$ is in $M_{B^{n+1}}$. Let $F: B^{n+1}\times[0,1]\to B^{n+1}$ and $G: B^{n+1}\times[0,1]\to B^{n+1}$ be continuous maps defined by $F(x,t)=(1-t)f(x)+tId_{B^{n+1}}(x)$ and $G(x,t)=g(x)-tf(x)$ for every $(x,t)\in B^{n+1}\times[0,1]$. Since $f(x)-g(x)$ and x are not in the opposite sense for all vectors $x\in S^n$, one can show that $\mathrm{Coinc}((F,G),S^n\times[0,1])=\emptyset$. Furthermore, we verify that the pair (F,G) is a homotopy between the pairs (f,g) and $(Id,g-f)$. Thus, $\xi(f,g)=\xi(Id,g-f)$. We end the proof by remarking that $\xi(f,g)$ is not trivial because $\xi(Id,g-f)$ is not trivial.

Corollary 2.4 *Let $(f,g)\in M_{B^{n+1}}$ so that $f(x)-g(x)$ and x are not in opposite sense for all $x\in S^n$, then f and g have at least in $B^{n+1}\setminus S^n$ a coincidence point.*

Proof. It is a consequence of Proposition 2.6 and Proposition 2.1.

3 A Generalized n-Mult Characteristic for a Class of Multivalued Mapping

Let Z_1, Z_2 be two arbitrary subsets of $\mathbb{R}^{n+1}$ such that $Z_1 \subseteq Z_2 \subset \mathbb{R}^{n+1}$ and $F : Z_1 \to Z_2$ is an upper semi continuous multivalued mapping.

Definition 3.1 F is said to be admissible (respectively, strongly admissible) if there exist an n-connected space X such that $S^n \subset X \subset \mathbb{R}^{n+1}$ and a pair $(p, q) \in Mor_{Top}(X, Z_1) \times Mor_{Top}(X, Z_2)$ such that p is surjective and $q \circ p^{-1}(x) \subseteq F(x)$ (respectively, $q \circ p^{-1}(x) = F(x)$) for all $x \in Z_1$. In this case, the pair (p, q) is called a representation of the multivalued mapping F on the n-connected space X and will be denoted by $(p, q, X) \subseteq F$, or $(p, q) \subseteq F$ if we do not to specify X.
The class of admissible multivalued mappings from Z_1 to Z_2 is denoted by $\mathcal{A}(Z_1, Z_2)$.

Let us consider the set $\text{Coinc}(p, q) = \{x \in X/p(x) = q(x)\}$ and the set

$$\mathcal{AM}(Z_1, Z_2) = \{F \in \mathcal{A}(Z_1, Z_2)/\text{Coinc}((p, q), S^n) = \emptyset, \forall (p, q) \subseteq F\}.$$

In the case where $Z_1 = Z_2 = Z$, the space $\mathcal{AM}(Z_1, Z_2)$ will be denoted by $\mathcal{AM}(Z)$.
For any representation (p, q) of F, the restrictions of p, q on the sphere S^n are denoted by $\tilde{p}, \tilde{q}$.

Definition 3.2 The generalized n-mult characteristic of $F \in \mathcal{AM}(Z_1, Z_2)$ is denoted and defined by

$$\chi_{\mathcal{AM}}(F) = \{\pi_n(\tilde{p} - \tilde{q}) \in Mor_{Top}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})), \forall (p, q) \subseteq F\}.$$

Lemma 3.1 *For every admissible representation (p, q) of F, we have*

$$p(\text{Coinc}(p, q)) = \{z \in Z_1/z \in q \circ p{-}1(z)\}.$$

Proof. The proof is obvious.
We have the following theorem.

Theorem 3.1 *Let F be a multivalued mapping satisfying the following conditions:*

1. *$F \in \mathcal{AM}(Z)$,*
2. *$\chi_{\mathcal{AM}}(F) \neq \{0\}$, where 0 is the zero homomorphism,*

so there exists at least a representation (p, q) of F and an element $z \in Z \setminus p(S^n)$ such that $z \in F(z)$, i.e., a fixed point of F.

Proof. Because $\chi_{\mathcal{AM}}(F) \neq \{0\}$, then by definition, there exists at least a representation (p, q) of F on an n-connected space X such that $\pi_n(\tilde{p} - \tilde{q}) \neq \{0\}$. By Proposition 2.1, the equation $p(x) - q(x) = \theta$ has at least one solution x_0 in $X \setminus S^n$. We put $z = p(x_0)$, so we obtain $z \in q(p^{-1}(z)) = F(z)$.

Definition 3.3 Let F, G be two elements of $\mathcal{AM}(Z)$. F, G are said to be homotopic if there exist two representations (p, q), (p', q') on the same n-connected space X which contains S^n, respectively, and a pair of continuous maps

$$(H, H') \in Mor_{Top}(X \times [0, 1], Z) \times Mor_{Top}(X \times [0, 1], Z)$$

such that

1. $Coin((H,H'), S^n \times [0,1]) = \emptyset$,
2. $H(x,0) = p(x), H(x,1) = p'(x), \forall x \in X$,
3. $H'(x,0) = q(x), H'(x,1) = q'(x), \forall x \in X$.

In this case, the pair $H = (H, H')$ is called a homotopy between F and G.

We have the following proposition.

Proposition 3.1 *If F, G are two homotopic elements of $\mathcal{AM}(Z)$, then $\chi_{\mathcal{AM}}(F) \cap \chi_{\mathcal{AM}}(G) \neq \emptyset$.*

Proof. Since F, G are homotopic, there exist representations (p,q), (p',q') on an n-connected space X such that they are homotopic in the sense of Definition 2.3. So, by Proposition 2.2, $\xi(p,q) = \xi(p',q')$. But $\xi(p',q') \in \chi_{\mathcal{AM}}(G)$, so $\chi_{\mathcal{AM}}(F) \cap \chi_{\mathcal{AM}}(G) \neq \emptyset$.

Remark 3.1 The proposition above can be formulated as follows. If F, G are two homotopic elements of $\mathcal{AM}(Z_1, Z_2)$, then there exist two representations (p,q), (p',q') of F and G, respectively, on the same n-connected space X which contains S^n such that $\pi_n(\tilde{p} - \tilde{q}) = \pi_n(\tilde{p}' - \tilde{q}')$.

Proposition 3.2 *Assume that $F, G : Z_1 \to Z_2$ are two upper semi continuous multivalued mappings such that the following conditions are satisfied:*

1. *$F(x) \subset G(x)$, $\forall x \in Z_1$,*
2. *$F \in \mathcal{AM}(Z_1, Z_2)$.*

Then $G \in \mathcal{AM}(Z_1, Z_2)$ and $\chi_{\mathcal{AM}}(F) \subset \chi_{\mathcal{AM}}(G)$.

Proof. Indeed, $\forall (p,q) \subseteq F$, we have $(p,q) \subseteq G$.

Let us consider the case $Z_1 = Z_2 = B^{n+1}$ and let $\mathcal{AM}(B^{n+1})$. Consider the set $\mathcal{AM}_{B^{n+1}}(B^{n+1}) \subset \mathcal{AM}(B^{n+1})$, where $\mathcal{AM}_{B^{n+1}}(B^{n+1})$ contains all admissible multivalued mappings F of $\mathcal{AM}(B^{n+1})$ which have at least a representation (p,q) on the unit ball B^{n+1}. Denote

$$\chi_{\mathcal{AM}}(F, B^{n+1}) = \{\xi(p,q) = \pi_n(\tilde{p} - \tilde{q}) \in Mor_{Top}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})), \forall (p,q,B^{n+1}) \subseteq F\}.$$

It is easy to see that $\chi_{\mathcal{AM}}(F, B^{n+1}) \subset \chi_{\mathcal{AM}}(F)$.

Proposition 3.3 *If $F \in \mathcal{AM}(B^{n+1})$ is a multivalued mapping defined by $F(x) = \{f(x)\}$ for every $x \in B^{n+1}$, then $\chi_{\mathcal{AM}}(F) \neq \{0\}$.*

Proof. In this case, we take $p = Id_{B^{n+1}}$, $q = f$ and $X = B^{n+1}$, we get $(Id_{B^{n+1}}, f)$ is one representation of F. Because $F \in \mathcal{AM}(B^{n+1})$, so $\mathrm{Coinc}(Id_{B^{n+1}}, f) = \emptyset$ on the sphere S^n. By Proposition 2.5, we have $\xi(Id_{B^{n+1}}, f) = \xi(Id_{B^{n+1}}, C_\theta)$ and it is not trivial. But $\xi(Id_{B^{n+1}}, f) \in \chi_{\mathcal{AM}}(F)$, so $\chi_{\mathcal{AM}}(F) \neq \{0\}$.

Proposition 3.4 *If $F \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ is such that there exists a representation (p,q) of f satisfying the condition $p(x) - q(x)$ and x are not in opposite sense for every $x \in S^n$, then $\chi_{\mathcal{AM}}(F) \neq \{0\}$.*

Proof. We apply Proposition 2.6 for the pair (p,q), we obtain $\xi(p,q) \neq 0$. But $\xi(p,q) \in \chi_{\mathcal{AM}}(F)$, so $\chi_{\mathcal{AM}}(F) \neq \{0\}$.

Corollary 3.1 *Let $F \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ be such that there exists a representation $(p,q) \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ of F satisfying the condition $p(x)-q(x)$ and x are not in opposite sense for every $x \in S^n$, then there exists $z \in B^{n+1} \backslash p(S^n)$ such that $z \in F(z) = q \circ p^{-1}(z)$.*

Proof. By Proposition 3.4, we have $\xi(p,q) \neq 0$. But $\xi(p,q) \in \chi_{\mathcal{AM}}(F)$, so by Theorem 3.1, we obtain the result.

4 Conclusion

This paper introduces a topological invariant designed for the pairs of mappings defined on n-connected spaces containing the sphere S^n, under the condition that they differ on this sphere. The invariant's definition indicates that it is a group homomorphism, establishes its novelty, and extends the definition given for the sphere S^n. Among the properties that are provided, we show its homotopic invariance and utilize it to establish theorems in coincidence and fixed point theories when these properties are not verified on S^n. In Section 3, leveraging the aforementioned construction, we define a generalized n-mult characteristic for a class of admissible multivalued mappings. Various properties of this n-mult characteristic are delineated, notably including the fixed point theorem for this class of multivalued mappings. In conclusion, recognizing the effectiveness of topological invariants in solving diverse nonlinear problems, we anticipate that the introduced topological invariant will also find applications, serving as a potent tool for addressing various mathematical and scientific challenges across different domains.

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