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Limit Cycles for a Class of Generalized Liénard Polynomial Differential Systems via the First-Order Averaging Method

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Abstract: In this paper, using the averaging method of first order, we compute the maximum number of limit cycles that can bifurcate from the periodic orbits of the center $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2q-1}$ with p and q being positive integers, under perturbation in the particular class of the generalized Liénard polynomial differential systems.

Keywords: *limit cycle; averaging method; periodic orbit; polynomial differential system.*

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1 Introduction

The second part of the Hilbert's 16th problem [9,17] aims to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree. The limit cycles problem and the center problem are concentrated on specific classes of systems. For instance, much has been written on Kukles systems, Duffing systems, Mathieu differential equations, Kolmogorov systems (see for example, [5,10,11,15]) and Liénard systems, that is, systems of the form

$$\dot{x} = y,\ \dot{y} = -x - f(x),$$

where $f(x)$ is a polynomial in the variable x of degree m. The motivation in the Liénard family is because it is one of the most important families related to the Hilbert's

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16th problem. Moreover, some classes of Liénard systems appear in applied sciences. Bifurcation of limit cycles in Liénard systems have been tackled by several authors and by using different approaches. See, for example, [1, 4, 7].

In [13], Lliber et al. proved that the generalized Liénard polynomial differential system

$$\dot{x} = y,\ \dot{y} = -g(x) - f(x)y,$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m, respectively, can have $\left[\frac{n+m-1}{2}\right]$ limit cycles, where [.] denotes the integer part function. In [14], Llibre and Makhlouf proved that the maximum number of limit cycles of the following generalized Liénard polynomial differential system:

$$\dot{x} = -y^{2p-1},\ \dot{y} = x^{2q-1} - \varepsilon f(x)y^{2m-1},$$

is at most $\left[\frac{n}{2}\right]$, where p, q and m are positive integers, ε is a small parameter and $f(x)$ is a polynomial of degree n. In [2], Benterki et al. studied the maximum number of crossing limit cycles of planar piecewise differential systems formed by linear Hamiltonian saddles.

In this paper, we want to study the maximum number of limit cycles of the following generalized Liénard polynomial differential system:

$$\dot{x} = -y^{2p-1},\ \dot{y} = x^{2q-1} - \varepsilon f(x,y)y^{2m-1}, \tag{1}$$

where p, q and m are positive integers, ε is a small parameter and $f(x,y)$ is a polynomial of degree n. Clearly, system (1) with $\varepsilon = 0$ is a Hamiltonian system with the Hamiltonian

$$H(x,y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}.$$

More precisely, our main results are as follows.

Theorem 1.1 *For the sufficiently small $|\varepsilon|$, system (1) has at most*

$$\mu = \left[\frac{n}{2}\right] \max\{p, q\}$$

limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2q-1}$, using averaging theory of first order, where [.] denotes the integer part function.

The proof of Theorem 1.1 is given in Section 3.

Theorem 1.2 *Consider system (1) with $q = lp$, l is a positive integer, then for $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized Liénard polynomial differential system (1) bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}, \dot{y} = x^{2lp-1}$, using the averaging theory of first order, is*

$$\textbf{(a)}\ \mu_1 = \frac{1}{2}\left(\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+3\right)\right) \quad \textit{if } \left[\frac{n}{2}\right] \leq l-1,$$

$$\textbf{(b)}\ \mu_2 = l\left[\frac{n}{2}\right] - \frac{l(l-3)+2}{2} \quad \textit{if } \left[\frac{n}{2}\right] \geq l.$$

The proof of Theorem 1.2 is given in Section 4.

2 Preliminaries

2.1 First order averaging method

The averaging theory is also an interesting method to research the limit cycles. Essentially, we have to look for the zeros of some specific function associated to the initial system.

Theorem 2.1 *Consider the following two initial value problems:*

$$\dot{x} = \varepsilon R(t,x) + \varepsilon^2 G(t,x,\varepsilon), x(0) = x_0 \tag{2}$$

and

$$\dot{y} = \varepsilon f^{\,0}(y), y(0) = x_0, \tag{3}$$

where x, y *and* $x_0 \in D$ *which is an open domain of* $\mathbb{R}$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, R *and* G *are periodic functions with their period* T *with its variable* t, *and* $f^0(y)$ *is the average function of* $R(t,y)$ *with respect to* t, *i.e.,*

$$f^0(y) = \tfrac{1}{T}\int_0^T R(t,y)dt.$$

Assume that

(i) R, $\dfrac{\partial R}{\partial x}$, $\dfrac{\partial^2 R}{\partial x^2}$, G *and* $\dfrac{\partial G}{\partial x}$ *are well defined, continuous and bounded by a constant independent of* $\varepsilon \in (0,\varepsilon_0]$ *in* $[0,\infty) \times D$.

(ii) T *is a constant independent of* ε.

(iii) $y(t)$ *belongs to* D *on the time scale* $1/\varepsilon$. *Then the following statements hold:*

(a) *On the time scale* $\dfrac{1}{\varepsilon}$, *we have*

$$x(t) - y(t) = O(\varepsilon), \text{ as } \varepsilon \longrightarrow 0.$$

(b) *If* p *is an equilibrium point of the averaged system* (3) *such that*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \neq 0, \tag{4}$$

then system (2) *has a* T*-periodic solution* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \longrightarrow 0$.

(c) *If* (4) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *of equation* (2) *according to* (t,x) *is asymptotically stable for all* ε *sufficiently small; if* (4) *is positive, then it is unstable.*

For more information about the averaging theory, see [6, 16, 18].

2.2 (p.q)-trigonometric functions

Following Lyapunov [12], let $u(\theta) = Cs\theta$ and $v(\theta) = Sn\theta$ be the solutions of the following initial value problem:

$$\begin{aligned} \dot{u} &= -v^{2p-1}, \dot{v} = u^{2q-1}, \\ u(0) &= \sqrt[2q]{\frac{1}{p}} \text{ and } v(0) = 0. \end{aligned}$$

Moreover, they satisfy the following properties:

(a) The functions $Cs\theta$ and $Sn\theta$ are T–periodic with

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p} + \frac{1}{2q})},$$

where Γ is the gamma function.

(b) For $p = q = 1$, we have $Cs\theta = \cos\theta$ and $Sn\theta = \sin\theta$.

(c) $pCs^{2p}\theta + qSn^{2q}\theta = 1$.

(d) Let $Cs\theta$ and $Sn\theta$ be the $(1.q)$-trigonometrical functions, for i and j being both even (see [8]),

$$\int_0^T Cs^i\theta Sn^j\theta d\theta = 2q^{-\frac{j+1}{2}} \frac{\Gamma(\frac{i+1}{2q})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{i+1}{2q} + \frac{j+1}{2})}. \tag{5}$$

2.3 Descartes theorem

The purpose of the Descartes theorem is to provide an insight into how many real roots a polynomial $P(x)$ may have.

Theorem 2.2 *[3] Consider the real polynomial*

$$p(x) = a_{l_1}x^{l_1} + a_{l_2}x^{l_2} + ... + a_{l_k}x^{l_k}$$

with $0 \leq l_1 < l_2 < ... < l_k$ *and* $a_{l_i} \neq 0$ *being real constants for* $i \in \{1, 2, 3, ..., k\}$. *When* $a_{l_i}a_{l_{i+1}} < 0$, *we say that* a_{l_i} *and* $a_{l_{i+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $k-1$ *positive real roots.*

3 Proof of Theorem 1.1

We shall need the first order averaging theory to prove Theorem 1.1. We write system (1) in (p, q)-polar coordinates (r, θ), where $x = r^pCs\theta$ and $y = r^qSn\theta$. In this way, system

(1) will be written in the standard form for applying the averaging theory. If we write $f(x,y) = \sum_{i+j=0}^{n} a_{i,j}x^i y^j$, then system (1) becomes

$$\begin{cases} \dot{r} = -\varepsilon r^{2q(m-1)+1} \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)} \\ \dot{\theta} = r^{pq-p-q} - \varepsilon r^{2q(m-1)} p \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^{i+1} \left(Sn\theta\right)^{j+2m-1}. \end{cases} \tag{6}$$

Treating θ as the independent variable, we get from system (6) the following:

$$\frac{dr}{d\theta} = \varepsilon R(r,\theta) + O(\varepsilon^2),$$

where

$$R(r,\theta) = -r^{-2pq+q(2m-1)+p+1} \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)}.$$

Using the notation introduced in Section 2, we have

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{i+j=0}^{n} \left(a_{i,j} r^{pi+qj} \int_0^T \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)} \right),$$

we write

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{i+j=0}^{n} a_{i,j} I_{i,j+2(m+p-1)} r^{pi+qj},$$

where

$$I_{i,j} = \int_0^T Cs^i\theta Sn^j\theta d\theta.$$

It is known that

$$\begin{aligned} I_{i,j} &= 0 \text{ if } i \text{ or } j \text{ is odd,} \\ I_{i,j} &> 0 \text{ if } i \text{ and } j \text{ are even.} \end{aligned}$$

Hence

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} a_{2s,2k} I_{2s,2k+2(m+p-1)} r^{2(ps+qk)}. \tag{7}$$

For the simplicity of calculation, let $\lambda_{s,k} = a_{2s,2k} I_{2s,2k+2(m+p-1)}$, therefore, (7) can be reduced to

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2(ps+qk)}. \tag{8}$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$N(r) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2(ps+qk)},$$

then, to find the real positive roots of $N(r)$, we must find the zeros of a polynomial in the variable $t = r^2$,

$$M(t) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} t^{ps+qk}. \tag{9}$$

Now, we expand the polynomial (9) as follows:

$$\begin{aligned} M(t) \;=\; & \lambda_{0,0} \\ & +\lambda_{1,0}t^{p} + \lambda_{0,1}t^{q} \\ & +\lambda_{2,0}t^{2p} + \lambda_{1,1}t^{p+q} + \lambda_{0,2}t^{2q} \\ & +... \\ & +\lambda_{d,0}t^{dp} + \lambda_{d-1,1}t^{(d-1)p+q} + \lambda_{d-2,2}t^{(d-2)p+2q} \\ & +... + \lambda_{2,d-2}t^{2p+(d-2)q} + \lambda_{1,d-1}t^{p+(d-1)q} + \lambda_{0,d}t^{qd} \\ & +... \\ & +\lambda_{\left[\frac{n}{2}\right],0}t^{\left[\frac{n}{2}\right]p} + \lambda_{\left[\frac{n}{2}\right]-1,1}t^{(\left[\frac{n}{2}\right]-1)p+q} + \lambda_{\left[\frac{n}{2}\right]-2,2}t^{(\left[\frac{n}{2}\right]-2)p+2q} \\ & +... + \lambda_{2,\left[\frac{n}{2}\right]-2}t^{2p+(\left[\frac{n}{2}\right]-2)q} + \lambda_{1,\left[\frac{n}{2}\right]-1}t^{p+(\left[\frac{n}{2}\right]-1)q} + \lambda_{0,\left[\frac{n}{2}\right]}t^{\left[\frac{n}{2}\right]q}. \end{aligned}$$

So, the degree of $M(t)$ is bounded by $\mu = \left[\frac{n}{2}\right] \max\{p, q\}$, we conclude that $f^0(r)$ has at most μ positive roots r. Hence, Theorem 1.1 is proved.

4 Proof of Theorem 1.2

Consider the polynomial differential system (1) with $q = lp$, from equation (8), we obtain

$$f^0(r) = -\frac{r^{lp(-2p+2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2p(s+lk)}. \tag{10}$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$G(r) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2p(s+lk)}. \tag{11}$$

We write (11) as follows:

$$\begin{aligned} G(r) &= \lambda_{0,0} + (\lambda_{1,0}r^{2p} + \lambda_{0,1}r^{2pl}) + (\lambda_{2,0}r^{4p} + \lambda_{1,1}r^{(l+1)2p} + \lambda_{0,2}r^{4lp}) \\ &+(\lambda_{3,0}r^{6p} + \lambda_{2,1}r^{(l+2)2p} + \lambda_{1,2}r^{(1+2l)2p} + \lambda_{0,3}r^{6lp}) \\ &+(\lambda_{4,0}r^{8p} + \lambda_{3,1}r^{(l+3)2p} + \lambda_{2,2}r^{(2+2l)2p} + \lambda_{1,3}r^{(1+3l)2p} + \lambda_{0,4}r^{8lp}) \\ &+... \\ &+[\lambda_{[\frac{n}{2}]-2,0}r^{([\frac{n}{2}]-2)2p} + \lambda_{[\frac{n}{2}]-3,1}r^{([\frac{n}{2}]+l-3)2p} + \lambda_{[\frac{n}{2}]-4,2}r^{([\frac{n}{2}]+2l-4)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-3}r^{(1+([\frac{n}{2}]-3)l)2p} + \lambda_{0,([\frac{n}{2}]-2)}r^{([\frac{n}{2}]-2)2lp}] \\ &+[\lambda_{[\frac{n}{2}]-1,0}r^{([\frac{n}{2}]-1)2p} + \lambda_{[\frac{n}{2}]-2,1}r^{([\frac{n}{2}]+l-2)2p} + \lambda_{[\frac{n}{2}]-3,2}r^{([\frac{n}{2}]+2l-3)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-2}r^{(1+([\frac{n}{2}]-2)l)2p} + \lambda_{0,[\frac{n}{2}]-1}r^{([\frac{n}{2}]-1)2lp}] \\ &+[\lambda_{[\frac{n}{2}],0}r^{[\frac{n}{2}]2p} + \lambda_{[\frac{n}{2}]-1,1}r^{([\frac{n}{2}]+l-1)2p} + \lambda_{[\frac{n}{2}]-2,2}r^{([\frac{n}{2}]+2l-2)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-1}r^{(1+([\frac{n}{2}]-1)l)2p} + \lambda_{0,[\frac{n}{2}]}r^{[\frac{n}{2}]2lp}]. \end{aligned} \tag{12}$$

Let us write (12) as

$$\begin{aligned} G(r) &= [\lambda_{0,0} + \lambda_{1,0}r^{2p} + \lambda_{2,0}r^{4p} + \\ &\quad ... + \lambda_{[\frac{n}{2}]-2,0}r^{([\frac{n}{2}]-2)2p} + \lambda_{[\frac{n}{2}]-1,0}r^{([\frac{n}{2}]-1)2p} + \lambda_{[\frac{n}{2}],0}r^{[\frac{n}{2}]2p}] \\ &+[\lambda_{0,1}r^{2lp} + \lambda_{1,1}r^{(l+1)2p} + \lambda_{2,1}r^{(l+2)2p} + \\ &\quad ... + \lambda_{[\frac{n}{2}]-2,1}r^{(l+[\frac{n}{2}]-2)2p} + \lambda_{[\frac{n}{2}]-1,1}r^{(l+[\frac{n}{2}]-1)2p}] \\ &+[\lambda_{0,2}r^{4lp} + \lambda_{1,2}r^{(2l+1)2p} + \lambda_{2,2}r^{(2l+2)2p} + \\ &\quad ... + \lambda_{[\frac{n}{2}]-3,2}r^{(2l+[\frac{n}{2}]-3)2p} + \lambda_{[\frac{n}{2}]-2,2}r^{(2l+[\frac{n}{2}]-2)2p}] \\ &+... + [\lambda_{0,([\frac{n}{2}]-2)}r^{(([\frac{n}{2}]-2)l)2p} + \lambda_{1,[\frac{n}{2}]-2}r^{(1+([\frac{n}{2}]-2)l)2p} \\ &+\lambda_{2,[\frac{n}{2}]-2}r^{(2+([\frac{n}{2}]-2)l)2p}] \\ &+[\lambda_{0,[\frac{n}{2}]-1}r^{([\frac{n}{2}]-1)2lp} + \lambda_{1,[\frac{n}{2}]-1}r^{(1+([\frac{n}{2}]-1)l)2p}] \\ &+\lambda_{0,[\frac{n}{2}]}r^{[\frac{n}{2}]l}. \end{aligned} \tag{13}$$

To find the number of positive roots of polynomials $G(r)$, we distinguish two cases.

Case 1: For $\left[\frac{n}{2}\right] \leq l-1$, the number of terms in polynomial (13) is

$$\left(\left[\frac{n}{2}\right]+1\right) + \left[\frac{n}{2}\right] + \left(\left[\frac{n}{2}\right]-1\right) + ... + 2 + 1 = \frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right).$$

Now, we shall apply the Descartes theorem introduced in Section 2, we can choose the appropriate coefficients $\lambda_{i,j}$ in order that the simple positive root number of $G(r)$ is at most

$$\begin{aligned}\mu_1 &= \frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right)-1\\ &= \frac{1}{2}\left(\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+3\right)\right).\end{aligned}$$

Hence (a) of Theorem 1.2 is proved.

Case 2: For $\left[\frac{n}{2}\right] \geq l$, the number of terms in polynomial (13) is

$$\begin{aligned}&\left(\left[\frac{n}{2}\right]+1\right)+\left[\frac{n}{2}\right]+\left(\left[\frac{n}{2}\right]-1\right)+...+2+1\\ &-\left(\left[\frac{n}{2}\right]-l+1\right)-\left(\left[\frac{n}{2}\right]-l\right)-\left(\left[\frac{n}{2}\right]-l-1\right)-...-2-1\\ =\ &\frac{1}{2}[\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right)-\left(\left[\frac{n}{2}\right]-l+1\right)\left(\left[\frac{n}{2}\right]-l+2\right)]\\ =\ &l\left[\frac{n}{2}\right]-\frac{l(l-3)}{2},\end{aligned}$$

by the Descartes theorem introduced in Section 2, we can choose the appropriate coefficients $\lambda_{i,j}$ in order that the simple positive root number of $G(r)$ is at most

$$\begin{aligned}\mu_2 &= l\left[\frac{n}{2}\right]-\frac{l(l-3)}{2}-1\\ &= l\left[\frac{n}{2}\right]-\frac{l(l-3)+2}{2}.\end{aligned}$$

Hence (b) of Theorem 1.2 is proved.

Example 4.1 We consider system (1), where $p = 1$, $q = 3$, $m = 2$ and

$$f(x, y) = -3.6x^2 + 2.4xy + 0.635y^2 + 0.5.$$

In this case, $n = 2$, $l = 3$ and $Cs\theta$ and $Sn\theta$ are T-periodic functions with period $T = 8.413\,1$. From equation (10), we obtain

$$f^0(r) = -\frac{r^5}{T}(\lambda_{0.0} + \lambda_{1.0}r^2 + \lambda_{0.1}r^6),$$

where $\lambda_{s.k} = a_{2s,2k}I_{2s,2k+4}$.

Using (5), we get

$$I_{0,4} = 0.63098, I_{2,4} = 0.15115 \text{ and } I_{0,6} = 0.19718.$$

So

$$f^0(r) = -\frac{r^5}{8.413\,1}\left(0.315\,49 - 0.544\,14r^2 + 0.125\,21r^6\right).$$

This polynomial has two positive real roots, $r_1 = 0.8$ and $r_2 = 1.3$. According to statement (a) of Theorem 1.2, the system has exactly two limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x^5$, using the averaging theory of first order.

Example 4.2 We consider system (1), where $p = 1$, $q = 2$, $m = 3$ and

$$f(x, y) = 1.5x^5 + 2xy^4 - 56.095x^4 + 13.575x^2y^2 - 0.46834y^4 + 21.227x^2 + y^2 + 2.7x - 1.$$

In this case, $n = 5$, $l = 2$ and $Cs\theta$ and $Sn\theta$ are T-periodic functions with period $T = 7.4163$. From equation (10), we obtain

$$f^0(r) = -\frac{r^8}{T}(\lambda_{0.0} + \lambda_{1.0}r^2 + (\lambda_{0.1} + \lambda_{2.0})\, r^4 + \lambda_{1.1}r^6 + \lambda_{0.2}r^8),$$

where $\lambda_{s,k} = a_{2s,2k}I_{2s,2k+6}$.

Using (5), we get

$$\begin{aligned} I_{0.6} &= 0.48158, I_{2.6} = 8.6894 \times 10^{-2}, I_{0.8} = 0.22474, \\ I_{4.6} &= 3.2105 \times 10^{-2}, I_{2.8} = 3.5780 \times 10^{-2} \text{ and } I_{0.10} = 0.10645. \end{aligned}$$

So

$$f^0(r) = -\frac{r^8}{7.4163}(-0.48158 + 1.8445r^2 - 1.5762r^4 + 0.48571r^6 - 4.9855 \times 10^{-2}r^8).$$

This polynomial has four positive real roots, $r_1 = 0.6, r_2 = 1.4, r_3 = 1.85$ and $r_4 = 2$. According to statement (b) of Theorem 1.2, the system has exactly four limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x^3$, using the averaging theory of first order.

5 Concluding Remarks

The second part of the Hilbert's 16th problem concerns the maximum number of limit cycles of all planar polynomial vector fields of degree n. One way to produce limit cycles is by perturbing a Hamiltonian system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits in the original system. In this work, by using the averaging theory of the first order, we have proved upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of the Hamiltonian system with the Hamiltonian $H(x, y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}$, where p and q are positive integers. We will continue our research on the maximum number of limit cycles for differential systems that model phenomena in biology, physics, etc, using the higher-order averaging theory.

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