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On Existence and Uniqueness of Solution of Heat Equations in Quasi-Metric Spaces

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Abstract: In this paper, we prove the existence and uniqueness of solutions to heat equations in quasi-metric spaces by applying the $\phi G-$contraction in this setting. This type of contraction is analogous to the ψF-contraction introduced by Secelean et al. in 2019. In the ψF-contraction, we have $F : \mathbb{R}^+ \to \mathbb{R}$ as an increasing mapping and $\psi : (-\infty, \mu) \to \mathbb{R}$ for some μ in $\mathbb{R}^+ \cup \{\infty\}$ as an increasing and continuous function such that $\psi(t) < t$ for every t in $(-\infty, \mu)$. Meanwhile, in the ϕG-contraction, we have G as a strictly increasing mapping from $\mathbb{R}^+ \cup \{0\}$ to $\mathbb{R}^+ \cup \{0\}$. Also $\phi : (-\infty, \mu) \to \mathbb{R}^+ \cup \{0\}$ as a strictly increasing and continuous function satisfying $\phi(t) < t$ for all t and $\phi(0) = 0$. This approach also provides a framework for solving nonlinear equations.

Keywords: *fixed point theories; heat equations; quasi-metric spaces.*

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1 Introduction

In the last century, fixed points have become an important topic in pure and applied mathematics, as well as in nonlinear dynamics, see, for example, [1–6]. The concept of fixed points and their associated mappings is crucial in investigating the existence and uniqueness of solutions to various mathematical models. The study of fixed points began with the Banach Contraction Principle in complete metric spaces in 1922 (see [7]). Afterward, many researchers introduced other types of fixed points in complete metric spaces and found their applications both in pure and applied mathematics.

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Many authors have also studied fixed point results in other spaces, including quasi-metric spaces which were first introduced by Wilson in 1931 (see [8]). In the last decade, many authors have found applications of fixed points in quasi-metric spaces, for example, in software engineering [9], the problems of existence and uniqueness of solution of boundary value problems [10–15], nonlinear fractional differential equations [16], and stability analysis of a solution for the fractional-order models on rabies transmission dynamics [17]. More recently, Secelean et al. [18, 19] introduced the ψF-contraction on quasi-metric spaces and obtained the fixed point results on that space. They also found its applications in fractals theory. Following this, Zakiyudin and Fahim [20] introduced the ϕG-contraction on quasi-metric space which is analogous to the ψF-contraction and obtained similar fixed point result. They also found its application to the nonhomogeneous Cauchy equation (see [20]). Other fixed point results and their applications can be seen in [1–5, 21–24].

Our purpose in the present paper is to find an application of ϕG-contraction, specifically for heat equations. For this purpose, let $\Omega \subset \mathbb{R}^n$ be a closed set and let $H = L^2(\Omega)$, where $L^2(\Omega)$ is the space of measurable functions for which the square of the absolute value is Lebesgue integrable. Additionally, let $T \in (0, +\infty)$ and $w = w(t)$ be a function valued in the Banach space H. Consider a Lipschitz function $f : H \to H$, meaning there exists a constant $C \in \mathbb{R}^+$ such that for all $x, y \in H$, one has

$$\|f(x) - f(y)\|_H \leq C\|x - y\|_H.$$

Now, consider the heat equation

$$\begin{cases} \dfrac{dw}{dt} = \Delta w(t) + f(w(t)), & 0 \leq t \leq T, \\ w(0) = w_0, \end{cases} \tag{1}$$

where $w_0 \neq 0, w_0 \in H$, and Δ satisfies the Neumann or Dirichlet Boundary Condition. Then, for $T > 0$ and $p \in [1, +\infty)$, we define $L^p_\lambda(0, T; H)$ as the space

$$L^p_\lambda(0, T; H) = \{f : [0, T] \to H : \int_0^T e^{-\lambda t}\|f(t)\|^p_H \, dt < \infty\}$$

equipped with the norm

$$\|f\|_{L^p_{\lambda^*}(0,T;H)} = \left(\int_0^T e^{-\lambda t}\|f(t)\|^p_H \, dt\right)^{1/p}.$$

In this work, we study the existence and uniqueness of solutions for the heat equation (1) in the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ defined below:

$$X^K_{p,\lambda} = \{g \in L^p_\lambda(0, T; H) : g(\mathbf{0}) = w_0 \text{ and } \|g\|_{L^p_\lambda(0,T;H)} \leq K\}$$

with the mapping

$$\rho^K_{p,\lambda}(g, h) = \begin{cases} K, & \text{for } \|g\|_{L^p_\lambda(0,T;H)} = K \text{ and } \|g - h\|_{L^p_\lambda(0,T;H)} > K, \\ \|g - h\|_{L^p_\lambda(0,T;H)}, & \text{for other } g \text{ and } h, \end{cases}$$

where $K \in (0, +\infty)$ and $p \in [1, +\infty)$.

In order to do this, some concepts on quasi-metric spaces, such as the definition and some properties of this space, for example, forward convergence, forward Cauchy sequence, forward completeness, forward Picard operator, and their analogs for the backward are presented in Section 2. Then we provide the definition of ϕG-contraction in quasi-metric spaces, and the fixed point result is presented in Theorem 2.1. In addition, we give the required conditions for the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ in Lemmas 2.1 and 2.2. In Section 3, we present our result for the heat equation, it begins with Lemma 3.1 which states that there exists a C_0-contractive semigroup with Δ as its infinitesimal generator. Furthermore, we introduce the definition of the mild solution of (1) in Definition 3.1. Afterward, we define a mapping Υ and state that Υ is a self-mapping in Lemma 3.2. Additionally, we prove an inequality that will be used to establish the existence and uniqueness of solution of (1) in Lemma 3.3. Finally, using the provided lemmas, we state Theorem 3.1 which asserts that the mild solution of the heat equation (1) exists and is unique. We end this paper with the conclusion in Section 4.

2 Preliminaries

In this section, we present some preliminaries on quasi-metric spaces and provide the result of our previous work about the ϕG-contraction in such spaces. In addition, we introduce the auxiliary spaces.

2.1 Quasi-metric spaces

Let X be a non-empty set. These preliminaries on quasi-metric spaces are discussed in [25].

Definition 2.1 A quasi-metric $\rho : X \times X \to [0, +\infty)$ is a mapping satisfying the following conditions:

(ρ_1) $\rho(x, y) = 0$ if and only if $x = y$;

(ρ_2) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality).

A pair (X, ρ) denotes a quasi-metric space.

Example 2.1 *For $a > 0$, let $X := \mathbb{R}$ and $\rho : X \times X \to [0, +\infty)$ be defined as*

$$\rho(x, y) := \begin{cases} x - y, & x \geq y, \\ a(y - x), & x < y. \end{cases}$$

Then (X, ρ) is a quasi-metric space.

Example 2.2 *Let $a > 0$ and consider a decreasing function $g : \mathbb{R} \to \mathbb{R}$, let $X = \mathbb{R}$ and $\rho : X \times X \to [0, +\infty)$ be defined as*

$$\rho(x, y) := \begin{cases} x - y, & x \geq y, \\ a\left(f(x) - f(y)\right), & x < y. \end{cases}$$

Then (X, ρ) is a quasi-metric space.

Definition 2.2 If (X, ρ) is a quasi-metric space and (x_n) is a sequence on X, then

1. The sequence (x_n) is forward convergent (for short, f-convergent) to $x \in X$ if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x, x_n) < \varepsilon$ for all $n \geq k$.
2. The sequence (x_n) is backward convergent (for short, b-convergent) to $x \in X$ if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon$ for all $n \geq k$.

Definition 2.3 If (X, ρ) is a quasi-metric space and (x_n) is a sequence on X, then

1. The sequence (x_n) is forward Cauchy (for short, f-Cauchy) if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ for all $m \geq n \geq k$.
2. The sequence (x_n) is backward Cauchy (for short, b-Cauchy) if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m \geq n \geq k$.

Definition 2.4 If (X, ρ) is a quasi-metric space, then (X, ρ) is forward complete (for short, f-complete) if every f-Cauchy sequence is f-convergent, and (X, ρ) is backward complete (for short, b-complete) if every b-Cauchy sequence is b-convergent.

Definition 2.5 If (X, ρ) is a quasi-metric space and $T : X \to X$ is a mapping, then

1. T is a forward Picard Operator (for short, f-P.O.) if there exists a unique $\eta \in X$ such that $T\eta = \eta$ and the sequence $x_n := T^n x_0$ for $x_0 \in X, n \in \mathbb{N}$ is f-convergent to η.
2. T is a backward Picard Operator (for short, b-P.O.) if there exists a unique $\eta \in X$ such that $T\eta = \eta$ and the sequence $x_n := T^n x_0$ for $x_0 \in X, n \in \mathbb{N}$ is b-convergent to η.

2.2 The ϕG-contraction

In the following, we present a fixed point result concerning ϕG-contraction obtained by Zakiyudin and Fahim in [20].

First, we denote by $\mathcal{G}$ a family of all strictly increasing functions $G : [0, +\infty) \to [0, +\infty)$. Let $\mu \in (0, \infty]$, and we denote by Φ_μ a family of all strictly increasing and continuous functions $\phi : [0, \mu) \to [0, +\infty)$ that satisfy $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. We write Φ_μ as Φ_G if $\mu \geq \sup_{x \in [0,+\infty)} G(x)$.

Definition 2.6 Let (X, ρ) be a quasi-metric space, $G \in \mathcal{G}$, and $\phi \in \Phi_G$. A mapping $\Upsilon : X \to X$ is called

(i) forward $\phi G-$contraction if

$$\Upsilon x \neq \Upsilon y \quad \Longrightarrow \quad G(\rho(\Upsilon x, \Upsilon y)) \leq \phi(G(\rho(x, y))),$$

(ii) backward $\phi G-$contraction if

$$\Upsilon x \neq \Upsilon y \quad \Longrightarrow \quad G(\rho(\Upsilon x, \Upsilon y)) \leq \phi(G(\rho(y, x))).$$

Theorem 2.1 *Let $G \in \mathcal{G}$ and $\phi \in \Phi_G$. Let (X, ρ) be a quasi-metric space such that*

1. *(X, ρ) is $f-$complete;*
2. *Every f-convergent sequence in quasi-metric space (X, ρ) is also $b-$convergent.*

If $\Upsilon : X \to X$ is a forward $\phi G-$contraction, then Υ is a forward Picard operator.

2.3 The auxiliary spaces

In the following, we present some properties of the space $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$ obtained by Zakiyudin and Fahim in [20].

Lemma 2.1 *The space $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$ is an f-complete quasi-metric space.*

Proof. First, we prove that $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$ is a quasi-metric space. It can be observed that $\rho_{p,\lambda}^K : X_{p,\lambda}^K \to [0, +\infty)$ and for $x, y \in X_{p,\lambda}^K$, we have $\rho_{p,\lambda}^K(x, y) = 0$ if and only if $x = y$. Next, to prove the triangle inequality, consider the following conditions:

1. When $\|x\|_{L_\lambda^p(0,T;H)} = K$ and $\|y\|_{L_\lambda^p(0,T;H)} = K$.
   We obtain that $\rho_{p,\lambda}^K(x, y) = K$ and $\rho_{p,\lambda}^K(x, z) = \|x - z\|_{L_\lambda^p(0,T;H)}$. Since $\rho_{p,\lambda}^K(x, z)$ and $\rho_{p,\lambda}^K(z, y)$ are non-negative, one has
$$\rho_{p,\lambda}^K(x, y) \le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y).$$
2. When $x \neq 0$, $y = 0$, and $z = 0$.
   We obtain that $\rho_{p,\lambda}^K(z, y) = 0$, $\rho_{p,\lambda}^K(x, y) = \rho_{p,\lambda}^K(x, z)$, then
$$\rho_{p,\lambda}^K(x, y) \le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y).$$
3. When $x = 0$, $y \neq 0$, and $z = 0$.
   We obtain that $\rho_{p,\lambda}^K(x, z) = 0$, $\rho_{p,\lambda}^K(x, y) = \|y\|_{L_\lambda^p(0,T;H)}$, and $\rho_{p,\lambda}^K(z, y) = \|y\|_{L_\lambda^p(0,T;H)}$, then
$$\rho_{p,\lambda}^K(x, y) \le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y).$$
4. When $x = 0$, $y \neq 0$, and $z \neq 0$.
   We obtain that $\rho_{p,\lambda}^K(x, y) = \|y\|_{L_\lambda^p(0,T;H)}$, $\rho_{p,\lambda}^K(x, z) = \|z\|_{L_\lambda^p(0,T;H)}$, and $\rho_{p,\lambda}^K(z, y) = \|z - y\|_{L_\lambda^p(0,T;H)}$. Then, by using the triangle inequality in the norm, one has
$$\|y\|_{L_\lambda^p(0,T;H)} = \|y - z + z\|_{L_\lambda^p(0,T;H)} \le \|y - z\|_{L_\lambda^p(0,T;H)} + \|z\|_{L_\lambda^p(0,T;H)},$$
   so
$$\rho_{p,\lambda}^K(x, y) \le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y).$$
5. When $x \neq 0$, $y = 0$, and $z \neq 0$.
   We obtain that $\rho_{p,\lambda}^K(x, y) \le \|x\|_{L_\lambda^p(0,T;H)}$ and $\rho_{p,\lambda}^K(x, z) = \|x - z\|_{L_\lambda^p(0,T;H)}$.
   If $\|z\|_{L_\lambda^p(0,T;H)} > \frac{1}{2}K$, then
$$\rho_{p,\lambda}^K(x, y) \le \frac{1}{2}K = \rho_{p,\lambda}^K(z, y) \le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y)$$
   and if $\|z\|_{L_\lambda^p(0,T;H)} \le \frac{1}{2}K$, then
$$\begin{aligned}\rho_{p,\lambda}^K(x, y) \le \|x\|_{L_\lambda^p(0,T;H)} &\le \|x - z\|_{L_\lambda^p(0,T;H)} + \|z\|_{L_\lambda^p(0,T;H)} \\ &\le \rho_{p,\lambda}^K(x, z) + \rho_{p,\lambda}^K(z, y).\end{aligned}$$

6. When $x \neq 0$, $y \neq 0$, and $z = 0$.
   We obtain that $\rho_{p,\lambda}^K(x,y) = \|x-y\|_{L_\lambda^p(0,T;H)}$, $\rho_{p,\lambda}^K(x,z) = K$, and $\rho_{p,\lambda}^K(z,y) = \|y\|_{L_\lambda^p(0,T;H)}$. Then, by using the triangle inequality in the norm, one has

$$\|x-y\|_{L_\lambda^p(0,T;H)} \leq \|x\|_{L_\lambda^p(0,T;H)} + \|y\|_{L_\lambda^p(0,T;H)} \leq K + \|y\|_{L_\lambda^p(0,T;H)},$$

   so

$$\rho_{p,\lambda}^K(x,y) \leq \rho_{p,\lambda}^K(x,z) + \rho_{p,\lambda}^K(z,y).$$

7. When $x \neq 0$, $y \neq 0$, and $z \neq 0$.
   We obtain that $\rho_{p,\lambda}^K(x,y) = \|x-y\|_{L_\lambda^p(0,T;H)}$, $\rho_{p,\lambda}^K(x,z) = \|x-z\|_{L_\lambda^p(0,T;H)}$, and $\rho_{p,\lambda}^K(z,y) = \|z-y\|_{L_\lambda^p(0,T;H)}$. Then, by using the triangle inequality in the norm, one has

$$\|x-y\|_{L_\lambda^p(0,T;H)} \leq \|x-z\|_{L_\lambda^p(0,T;H)} + \|z-y\|_{L_\lambda^p(0,T;H)},$$

   so

$$\rho_{p,\lambda}^K(x,y) \leq \rho_{p,\lambda}^K(x,z) + \rho_{p,\lambda}^K(z,y).$$

Therefore, the triangle inequality holds. Next, we prove that $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$ is f-complete. Take any f-Cauchy sequence in $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$, then for every $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\rho_{p,\lambda}^K(x_n, x_m) < \varepsilon, \quad \forall m \geq n \geq M(\varepsilon).$$

Then for every $0 < \varepsilon \leq \frac{1}{2}K$, there exists $M(\varepsilon)$ such that

$$\|x_m - x_n\|_{L_\lambda^p(0,T;H)} < \varepsilon, \quad \forall m \geq n \geq M(\varepsilon).$$

This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M\left(\frac{1}{2}K\right)$ for $\varepsilon > \frac{1}{2}K$. Thus, (x_n) is a Cauchy sequence in $(X_{p,\lambda}^K, \|\cdot\|_{L_\lambda^p(0,T;H)})$. Since this space is complete, (x_n) is also convergent in this space. Therefore, there exists $x \in X_{p,\lambda}^K$ such that for all $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$, where

$$\|x - x_n\|_{L_\lambda^p(0,T;H)} < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

This implies that for every $0 < \varepsilon \leq P$, where

$$P = \begin{cases} \frac{1}{2}K & \text{if } x = 0, \\ \min\left\{\frac{1}{2}K, \frac{1}{2}\|x\|_{L_\lambda^p(0,T;H)}\right\} & \text{if } x \neq 0, \end{cases}$$

there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\rho_{p,\lambda}^K(x, x_n) < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M(P)$ so that (x_n) is f-convergent in the quasi-metric space $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$. Therefore, $(X_{p,\lambda}^K, \rho_{p,\lambda}^K)$ is an f-complete quasi-metric space.

Lemma 2.2 *Every f-convergent sequence in the quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is also b-convergent.*

Proof. Take any f-convergent sequence (x_n) in quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$. Then there exists $x \in X^K_{p,\lambda}$ such that for every $0 < \varepsilon \leq \frac{1}{2}K$, there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\|x - x_n\|_{L^p_\lambda(0,T;H)} = \rho^K_{p,\lambda}(x, x_n) < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

Since $0 < \varepsilon \leq \frac{1}{2}K$, one has $\rho^K_{p,\lambda}(x_n, x) = \|x - x_n\|_{L^p_\lambda(0,T;H)} < \varepsilon$, $\forall n \geq M(\varepsilon)$. This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M\left(\frac{1}{2}K\right)$ for $\varepsilon > \frac{1}{2}K$. Consequently, (x_n) is b-convergent in quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$.

3 The Existence and Uniqueness of Solution of the Heat Equation

In the following section, we establish the existence and uniqueness of solution of the heat equation (1).

First, we define the resolvent set $\sigma(A)$ of a linear operator A as the set of all complex numbers λ for which $\lambda I - A$ is invertible, that is, $(\lambda I - A)^{-1}$ is a bounded linear operator. For $\lambda \in \sigma(A)$, the bounded linear operator $R(\lambda : A) := (\lambda I - A)^{-1}$ is called the resolvent of A. Now, we prove that $\{\mathfrak{R}(t)\}_{t\geq 0}$ is a C_0-contractive semigroup, where Δ is its infinitesimal generator, by using the following lemma.

Lemma 3.1 *There exists a C_0-contractive semigroup $\{\mathfrak{R}(t)\}_{t\geq 0}$, where Δ is its infinitesimal generator.*

Proof. Let $x_n \to x$ in H, where $x_k \in D(\Delta)$ for all $k \in \mathbb{N}$ and $\Delta x_n \to y$ in Y. Since $D(\Delta) = H^2(\Omega)$ is a Hilbert space, $x \in D(\Delta)$ and it follows that

$$\|\Delta x_n - \Delta x\|_H = \|x_n - x\|_{D(\Delta)} \to 0.$$

Therefore, $\Delta x_n \to \Delta x$ in H. We get $y = \Delta x$ since the limit of a sequence in the Hilbert space is unique. Thus Δ is a closed operator and we have $\overline{D(\Delta)} = H$. Then, from [26], for $\lambda \in (0, +\infty)$, one has

$$\|R(\lambda : \Delta)\| \leq \frac{1}{\lambda}|\sin^2(\arg(\lambda))| \leq \frac{1}{\lambda}.$$

By the Hille-Yosida Theorem (see [27]), the operator Δ is the infinitesimal generator of the C_0-contractive semigroup $\{\mathfrak{R}(t)\}_{t\geq 0}$. The properties of C_0-semigroup imply that $u(t) = \mathfrak{R}(t)u_0$ is a unique solution to the differential equation

$$\frac{du}{dt} = \Delta u(t), \quad u(0) = u_0.$$

Next, using the Fourier Transform, we have

$$(\mathcal{F}[u(t)])(\xi) = e^{-4\pi^2|\xi|^2 t}\mathcal{F}[u_0], \quad \xi \in \mathbb{R}^n.$$

Therefore, for all $v \in H$,

$$\mathcal{F}(\mathfrak{R}(t)v) = e^{-4\pi^2|\cdot|^2 t}F[v],$$

and

$$\|\mathfrak{R}(t)v\|_H = \|\mathcal{F}(\mathfrak{R}(t)v)\|_H = \left\|e^{-4\pi^2|\cdot|^2 t}F[v]\right\|_H \leq \|F[v]\|_H = \|v\|_H.$$

This is the definition of a mild solution to the heat equation (1).

Definition 3.1 Let $\{\mathfrak{R}(t)\}_{t\geq 0}$ be a C_0-contractive semigroup and let Δ be its infinitesimal generator. A function $u : [0,T] \to H$ is called the mild solution of equation (1) if there exists $\lambda^* = \lambda^*(\rho, u_0, T) > 0$ such that $u \in L^p_{\lambda^*}(0,T;H)$ and satisfies

$$u(t) = \mathfrak{R}(t)u_0 + \int_0^t \mathfrak{R}(t)f(u(s))\,\mathrm{d}s.$$

Next, we define a mapping Υ such that

$$\Upsilon(v)(t) = \mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t)f(v(s))\,\mathrm{d}s. \tag{2}$$

Thus, we specify some properties of Υ below.

Lemma 3.2 *Let $\lambda^* > 0$ such that*

$$\lambda^* \geq \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\,\|v_0\|_H^p + 2^{2p-2}T^p\,\|f(v_0)\|_H^p}{K^p} + 2^{3p-3}C^pT^{p-1},$$

then the mapping Υ is a self-mapping on X^K_{p,λ^*}.

Proof. Let $v \in X^K_{p,\lambda}$, then $\Upsilon(v)(0) = v_0 = w_0$. Since $\mathfrak{R}$ is a C_0-contractive semigroup of $\{\mathfrak{R}(t)\}_{t\geq 0}$ and by applying Hölder's inequality, we have

$$\begin{aligned}
\|\Upsilon(v)(t)\|^p_{L^p_\lambda(0,T;H)} &= \left\|\mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t)f(v(s))\;ds\right\|^p_{L^p_\lambda(0,T;H)} \\
&\leq 2^{p-1}\left[\|\mathfrak{R}(t)v_0\|^p_{L^p_\lambda(0,T;H)} + \left\|\int_0^t \mathfrak{R}(t)f(v(s))\;ds\right\|^p_{L^p_\lambda(0,T;H)}\right] \\
&= 2^{p-1}\int_0^T e^{-\lambda t}\,\|\mathfrak{R}(t)v_0\|_H^p\;dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t}\left\|\int_0^t \mathfrak{R}(t)f(v(s))\;ds\right\|_H^p\;dt \\
&\leq 2^{p-1}\int_0^T e^{-\lambda t}\,\|v_0\|_H^p\;dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t}\left(\int_0^t \|f(v(s))\|_H\;ds\right)^p\;dt \\
&\leq \frac{2^{p-1}\,\|v_0\|_H^p}{\lambda} + 2^{p-1}\int_0^T e^{-\lambda t}2^{p-1}\left(\int_0^T \|f(v(0))\|_H\;ds\right)^p\;dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t}2^{p-1}\left(\int_0^t C\,\|v(s) - v(0)\|_H\;ds\right)^p dt \\
&\leq \frac{2^{p-1}\,\|v_0\|_H^p}{\lambda} + 2^{2p-2}C^p\int_0^T e^{-\lambda t}2^{p-1}\left(\int_0^t \|v(s)\|_H\;ds)\right)^p\;dt
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{2p-2}C^p \int_0^T e^{-\lambda t} 2^{p-1}\left(\int_0^t \|v(0)\|_H \; ds\right)^p dt \\
&+ 2^{2p-2} \int_0^T e^{-\lambda t}\left(\int_0^T \|f(v(0))\|_H \; ds\right)^p \; dt \\
\le\; & \frac{2^{p-1}\|v_0\|_H^p}{\lambda} + 2^{3p-3}C^p \int_0^T e^{-\lambda t}\left(\int_0^t \|v(s)\|_H \; ds\right)^p dt \\
&+ \left(2^{3p-3}C^p \|v(0)\|_H^p + 2^{2p-2}\|f(v(0))\|_H^p\right)\frac{T^p}{\lambda} \\
\le\; & \frac{2^{p-1}\|v_0\|_H^p}{\lambda} + 2^{3p-3}L^p \int_0^T \int_s^T e^{-\lambda t} t^{p-1}\left(\|v(s)\|_H\right)^p \; dt \; ds \\
&+ \left(2^{3p-3}C^p \|v(0)\|_H^p + 2^{2p-2}\|f(v(0))\|_H^p\right)\frac{T^p}{\lambda} \\
\le\; & \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\|v(0)\|_H^p}{\lambda} + \frac{2^{2p-2}T^p\|f(v(0))\|_H^p}{\lambda} \\
&+ \frac{2^{3p-3}C^pT^{p-1}K^p}{\lambda}.
\end{aligned}
$$

Now, let

$$
\lambda = \lambda^* \ge \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\|v_0\|_H^p + 2^{2p-2}T^p\|f(v_0)\|_H^p}{K^p} + 2^{3p-3}C^pT^{p-1},
$$

and we obtain that if $v \in X_{p,\lambda^*}^K$, then $\Upsilon(v) \in X_{p,\lambda^*}^K$.

Lemma 3.3 *Let* $\alpha = \left(\frac{T^{p-1}C^p}{\lambda^*}\right)^{\frac{1}{p}}$ *and assume that* λ^* *satisfies* $\lambda^* > T^{p-1}C^p$*. Then for any* $u, v \in X_{p,\lambda^*}^K$*, the following inequality is satisfied:*

$$
\rho_{p,\lambda^*}^K(\Upsilon u, \Upsilon v) \le \alpha \rho_{p,\lambda^*}^K(u, v). \tag{3}
$$

Proof. Consider that $v \in X_{p,\lambda^*}^K$, so $\Upsilon v \neq 0$. We can prove this by contradiction. Assume that $\Upsilon v = 0$, it follows that

$$
\mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t) f(v(s))\, ds = 0.
$$

If we set $t = 0$, this implies $v_0 = 0$, which contradicts the fact that $v_0 \neq 0$. Hence, $\Upsilon v \neq 0$. Subsequently, since S is a C_0-contractive semigroup and by applying Hölder's inequalities, for all $u, v \in X_{p,\lambda^*}^K$, we have

$$
\begin{aligned}
\rho_{p,\lambda}^K(\Upsilon u, \Upsilon v) &= \|\Upsilon u - \Upsilon v\|_{L_\lambda^p(0,T;H)} \\
&\le \left\| \int_0^t \mathfrak{R}(t)\left[f(u(s)) - f(v(s))\right] \, ds \right\|_{L_\lambda^p(0,T;H)} \\
&= \left(\int_0^T e^{-\lambda t}\left(\| \int_0^t \mathfrak{R}(t)\left[f(u(s)) - f(v(s))\right] \|_H \, ds\right)^p dt\right)^{1/p} \\
&\le \left(\int_0^T e^{-\lambda t}\left(\int_0^t \|f(u(s)) - f(v(s))\|_H \, ds\right)^p dt\right)^{1/p}
\end{aligned}
$$

$$\leq \left(\int_0^T e^{-\lambda t} \left(\int_0^t C \|u(s) - v(s)\|_H \, ds \right)^p dt \right)^{1/p}$$

$$\leq \left(C^p \int_0^T e^{-\lambda t} t^{p-1} \int_0^t \left(\|u(s) - v(s)\|_H \right)^p \, ds \, dt \right)^{1/p}$$

$$\leq \left(\frac{C^p T^{p-1}}{\lambda} \int_0^T e^{-\lambda s} \left(\|u(s) - v(s)\|_H \right)^p \, ds \right)^{1/p}.$$

Now, let $\lambda = \lambda^* > T^{p-1} C^p$, then we obtain

$$\rho^K_{p,\lambda^*}(\Upsilon u, \Upsilon v) \leq \alpha \rho^K_{p,\lambda^*}(u, v),$$

where

$$\alpha = \left(\frac{T^{p-1} C^p}{\lambda} \right)^{1/p} < 1.$$

Finally, we establish the existence and uniqueness of solution to the heat equation (1) through the following theorem.

Theorem 3.1 *For some λ^*, the heat equation (1) has a unique mild solution in the space $(X^K_{p,\lambda^*}, \rho^K_{p,\lambda^*})$.*

Proof. Let $G(x) = x$ and $\phi(x) = \alpha x = \left(\frac{T^{p-1} C^p}{\lambda^*} \right)^{1/p} x$, and

$$\lambda^* > \max \left\{ \frac{(2^{p-1} + 2^{3p-3} C^p T^p) \|v_0\|_H^p + 2^{2p-2} T^p \|f(v_0)\|_H^p}{K^p} + 2^{3p-3} C^p T^{p-1}, \right.$$
$$\left. T^{p-1} C^p \right\}.$$

Then, by Lemmas 3.2 and 3.3, and also by Definition 2.6, we have that $\Upsilon : X^K_{p,\lambda^*} \to X^K_{p,\lambda^*}$ is a forward ϕG-contraction. The necessary conditions for the quasi-metric space $(X^K_{p,\lambda^*}, \rho^K_{p,\lambda^*})$ in Theorem 2.1 are satisfied by Lemma 2.1 and Lemma 2.2. This implies that Υ is a forward Picard Operator, hence there exists a unique $u \in X^K_{p,\lambda^*} \subset L^p_{\lambda^*}(0, T; H)$ such that

$$u(t) = \Re(t) u_0 + \int_0^t \Re(t) f(u(s)) \, ds.$$

Therefore, $u(t)$ is the unique mild solution of the heat equation (1).

4 Conclusion

In this paper, we have demonstrated the existence and uniqueness of mild solutions to the heat equation (1) in the quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$. This result was obtained by using the fixed point theory of the ϕG-contraction, as stated in Theorem 2.1. We established that the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ satisfies the necessary conditions for Theorem 2.1 to hold. Moreover, we showed that the mapping $\Upsilon(u)(t) = u(t)$, where $u(t)$ is the mild solution of the heat equation, is a ϕG-contraction in this space. The final result, presented in Theorem 3.1, confirms that the mild solution to the heat equation exists and is unique within the given quasi-metric space framework. Future work may explore the extension of these results to more general classes of differential equations.

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