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NONLINEAR DYNAMICS AND SYSTEMS THEORY

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Resolvability of Boundary Value Problems with Integral Conditions and Solution's Approximation by Homotopy Perturbation Method

Necib Abdelhalim [1], Iqbal M. Batiha [2,3,*], Imad Rezzoug [1], Adel Ouannas [1], Osama Ogilat [4], Nidal Anakira [5] and Tala Sasa [6]

[1] *Department of Mathematics and Computer Science, University of Oum El Bouaghi, Oum El Bouaghi 04000, Algeria.*
[2] *Department of Mathematics, Al Zaytoonah University of Jordan, Amman 11733, Jordan.*
[3] *Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman 346, UAE.*
[4] *Department of Basic Sciences, Faculty of Arts and Science, Al-Ahliyya Amman University, Amman 19111, Jordan.*
[5] *Faculty of Education and Arts, Sohar University, Sohar 3111, Oman.*
[6] *Applied Science Research Center, Applied Science Private University, Amman, Jordan.*

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Abstract: The Telegraph problem is a linear time-partial differential equation that models the transmission of electrical impulses through a cable. It consists of two coupled partial differential equations that describe the voltage and current within the cable, capturing both wave-like propagation and diffusive effects. This equation integrates elements from the wave equation and the heat equation to account for resistive losses and signal propagation speed. The objective of this study is to investigate the existence and uniqueness of a strong solution to the Telegraph problem under purely integral conditions. The analysis is conducted using the operator density method derived from the problem framework and the energy inequality approach. To approximate the desired solution, a combination of the Laplace transform technique and the homotopy perturbation method is employed. This approach yields solutions in the form of rapidly convergent series, and the convergence of these series is rigorously established. The findings indicate that the proposed methodology is highly effective and applicable to a broad class of mathematical problems. To validate these results, several illustrative examples are provided, demonstrating the accuracy of the proposed method by comparing approximate solutions with exact solutions.

Keywords: *telegraph equation; purely integral conditions; a priori estimate; Laplace transform; homotopy perturbation method; Stehfest algorithm.*

Mathematics Subject Classification (2020): 30E25, 44A10, 70K99.

* Corresponding author: i.batiha@zuj.edu.jo

1 Introduction

Many problems in modern physics and engineering are effectively modeled using integral conditions, particularly when direct boundary measurements are not feasible [1–5]. These integral conditions often arise in scenarios where the data on the boundary cannot be explicitly determined. Several studies have explored such problems, as seen in [6–14]. The presence of integral terms in boundary conditions significantly complicates the application of standard numerical techniques such as the finite difference and finite element methods. Despite these challenges, the theoretical solvability of such equations has been extensively studied, focusing on the existence and uniqueness of their solutions. This study builds on previous theoretical findings and aims to further analyze the implications of integral conditions in solving partial differential equations, see [15–26].

The Homotopy Perturbation Method (HPM) is a semi-analytical technique used for solving both linear and nonlinear differential equations, including those with ordinary or partial derivatives. Additionally, this method can be applied to systems of linear and nonlinear differential equations. The HPM was first introduced by J. He [27] in 1998, and was later refined and improved in subsequent works [28, 29]. Traditional perturbation methods are often based on the assumption of small parameters, which can be a significant limitation in many practical applications. To address this issue, Liu [30] proposed the artificial parameter method, while Liao [31,32] contributed to the development of the Homotopy Analysis Method (HAM), which eliminates the dependence on small parameters. J. He [27] further developed the HPM technique, offering a more robust approach that does not rely on the assumption of small parameters. In recent years, the application of homotopy perturbation theory has gained widespread recognition among researchers [33–37]. Notably, S. Abbasbandy [33] successfully applied this method to functional integral equations, demonstrating its effectiveness as a powerful mathematical tool.

The contribution of this research lies in investigating the solvability of a problem primarily governed by the telegraph equation, where the classical boundary conditions are replaced with purely integral conditions. Furthermore, this study explores a novel approach that combines the Laplace transform with the homotopy perturbation method to approximate the solution using convergent series representations.

2 Existence and Uniqueness of the Solution

The well-posedness of partial differential equations is essential for ensuring the validity of their solutions. In this section, we analyze the existence and uniqueness of solutions for the telegraph equation with purely integral conditions using operator theory and energy inequalities.

2.1 Statement of the problem

The telegraph equation models signal transmission incorporating both diffusion and wave propagation effects. In this subsection, we formulate the problem with purely integral conditions, which introduce additional complexity compared to classical boundary-value problems. In the rectangular domain

$$Q = \{(x,t) : \ 0 < x < 1, \ 0 < t \leq T\},$$

we consider the telegraph equation

$$\pounds u = \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial t} = f(x,t) \tag{1}$$

subject to the initial conditions

$$\ell u = u(x,0) = \Phi(x), \ 0 < x < 1, \tag{2}$$
$$qu = u_t(x,0) = \Psi(x), \ 0 < x < 1, \tag{3}$$

and the purely integral conditions

$$\int_0^1 u(x,t)dx = a(t), \ 0 < t \le T, \tag{4}$$
$$\int_0^1 xu(x,t)dx = b(t), \ 0 < t \le T, \tag{5}$$

where f, Φ, Ψ, a, b are known functions, and α, β, T are strictly positive constants. Additionally, the functions Φ and Ψ satisfy the following compatibility conditions:

$$\int_0^1 \Phi dx = a(0), \int_0^1 x\Phi dx = b(0), \tag{6}$$
$$\int_0^1 \Psi dx = a'(0), \int_0^1 x\Psi dx = b'(0). \tag{7}$$

2.1.1 Reformulation of the problem

To facilitate the analysis, it is beneficial to transform the problem (1)-(5) into an equivalent formulation with homogeneous integral conditions. This transformation simplifies the mathematical treatment and enables a more structured approach to solving the problem. For this purpose, we introduce a new function $v(x,t)$ defined as

$$u(x,t) = v(x,t) + U(x,t), \tag{8}$$

where

$$U(x,t) = (12b(t) - 6a(t))\, x - (6b(t) - 4a(t)). \tag{9}$$

This substitution transforms the original problem (1)-(5) with inhomogeneous integral conditions (4)-(5) into the equivalent problem of determining $v(x,t)$, which satisfies

$$\pounds v = \frac{\partial^2 v}{\partial t^2} - \alpha \frac{\partial^2 v}{\partial x^2} + \beta \frac{\partial v}{\partial t} = g(x,t), \quad 0 < x < 1, \quad 0 < t \le T, \tag{10}$$

subject to the initial conditions

$$\ell v = v(x,0) = \varphi(x), \quad 0 < x < 1, \tag{11}$$
$$qv = v_t(x,0) = \psi(x), \quad 0 < x < 1, \tag{12}$$

and the homogeneous integral conditions

$$\int_0^1 v(x,t)\, dx = 0, \quad 0 < t \le T, \tag{13}$$
$$\int_0^1 xv(x,t)\, dx = 0, \quad 0 < t \le T. \tag{14}$$

The associated functions are defined as

$$\begin{aligned} g(x,t) &= f(x,t) - \pounds U, & (15)\\ \varphi(x) &= \Phi(x) - \ell U(x,t), & (16)\\ \psi(x) &= \Psi(x) - qU(x,t). & (17) \end{aligned}$$

Thus, instead of solving for u, we focus on finding v. The solution to the original problem (1)-(5) is then reconstructed using (8)-(9).

2.1.2 A priori estimates and their consequences

To establish the well-posedness of the problem (10)-(14), we derive a priori estimates, which play a crucial role in proving existence, uniqueness, and stability. These estimates ensure that the solution remains bounded within an appropriate function space. The solution of the problem (10)-(14) can be formulated in an operational form

$$Lv = \mathscr{F},$$

where $L = (\pounds, \ell, q)$ is considered as an operator from the Banach space B to the Hilbert space F. The space B is defined as the Banach space of functions $v \in L^2(Q)$, equipped with the norm

$$\|v\|_B = \left(\sup_{0\le\tau\le T} \int_0^1 \left(\left\| \Im_x \frac{\partial v}{\partial t}(x,t) \right\|^2 + \|\Im_x v(x,\tau)\|^2 \right) dx \right)^{\frac{1}{2}},$$

which is finite. Similarly, the space F is the Hilbert space consisting of all elements $\mathscr{F} = (g, \varphi, \psi)$, endowed with the norm

$$\|\mathscr{F}\|_F = \left(\int_{Q_\tau} \|g\|^2 \, dxdt + \int_0^1 \left(\|\psi(x)\|^2 + \|\varphi(x)\|^2 \right) dx \right)^{\frac{1}{2}},$$

which is finite. These function spaces provide the appropriate setting for analyzing the problem and deriving estimates that will aid in proving the existence and uniqueness of solutions.

The domain $D(L)$ of the operator L consists of all functions v such that

$$\frac{\partial v}{\partial t}, \quad \frac{\partial^2 v}{\partial t^2}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial^2 v}{\partial x^2} \in L^2(Q),$$

and v satisfies the integral conditions (13) and (14).

To analyze the solvability of the problem, we first establish an a priori estimate. This will immediately lead to results regarding the uniqueness and continuous dependence of the solution on the given data.

Theorem 2.1 *If $v(x,t)$ is a solution of the problem* (10)-(14) *and $g \in C(Q)$, then the following estimate holds:*

$$\|v\|_B \le C \|Lv\|_F, \tag{18}$$

where C is a positive constant independent of v, for all $v \in D(L)$, and is given by

$$C = \left(\frac{\max\left(\frac{1}{4}, \frac{\alpha}{2}, \frac{1}{8\beta}\right)}{\min\left(\frac{1}{2}, \alpha\right)} \right)^{\frac{1}{2}}.$$

Proof. We define the following integral operators:

$$\Im_x v = \int_0^x v(\xi,t)\,d\xi, \quad \Im_x^2 v = \int_0^x \int_0^\eta v(\xi,t)\,d\xi d\eta.$$

Multiplying equation (10) by the integro-differential operator

$$Mv = -\Im_x^2 \frac{\partial v}{\partial t},$$

and integrating over the subdomain

$$Q_\tau = (0,\tau) \times (0,1), \quad \text{where } 0 \le \tau \le T,$$

we obtain

$$-\int_{Q_\tau} \frac{\partial^2 v}{\partial t^2} \Im_x^2 \left(\frac{\partial v}{\partial t}\right) dxdt + \alpha \int_{Q_\tau} \frac{\partial^2 v}{\partial x^2} \Im_x^2 \left(\frac{\partial v}{\partial t}\right) dxdt - \beta \int_{Q_\tau} \frac{\partial v}{\partial t} \Im_x^2 \left(\frac{\partial v}{\partial t}\right) dxdt$$
$$= -\int_{Q_\tau} g \Im_x^2 \left(\frac{\partial v}{\partial t}\right) dxdt. \quad (19)$$

Applying integration by parts to each term on the left-hand side of (19), we obtain

$$-\int_{Q_\tau} \frac{\partial^2 v}{\partial t^2} \Im_x^2 \frac{\partial v}{\partial t} dxdt = \frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx - \frac{1}{2}\int_0^1 \|\Im_x \psi(x)\|^2 dx, \quad (20)$$

$$\alpha \int_{Q_\tau} \frac{\partial^2 v}{\partial x^2} \Im_x^2 \frac{\partial v}{\partial t} dxdt = \frac{\alpha}{2} \int_0^1 \|v(x,\tau)\|^2 dx - \frac{\alpha}{2} \int_0^1 \|\varphi(x)\|^2 dx, \quad (21)$$

$$-\beta \int_{Q_\tau} \frac{\partial v}{\partial t} \Im_x^2 \frac{\partial v}{\partial t} dxdt = \beta \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dxdt. \quad (22)$$

The substitution of (20), (21), and (22) into (19) yields

$$\frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx + \frac{\alpha}{2} \int_0^1 \|v(x,\tau)\|^2 dx + \beta \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dxdt$$
$$= \frac{1}{2} \int_0^1 \|\Im_x \psi(x)\|^2 dx + \frac{\alpha}{2} \int_0^1 \|\varphi(x)\|^2 dx - \int_{Q_\tau} g \Im_x^2 \frac{\partial v}{\partial t} dxdt. \quad (23)$$

When applying the Cauchy-Schwarz inequality with parameter ε, the right-hand side of (23) is bounded as follows:

$$\frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx + \frac{\alpha}{2} \int_0^1 \|v(x,\tau)\|^2 dx + \beta \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dxdt$$
$$\le \frac{1}{2} \int_0^1 \|\Im_x \psi(x)\|^2 dx + \frac{\alpha}{2} \int_0^1 \|\varphi(x)\|^2 dx + \frac{\varepsilon}{2} \int_{Q_\tau} \|g\|^2 dxdt + \frac{1}{2\varepsilon} \int_{Q_\tau} \left\| \Im_x^2 \frac{\partial v}{\partial t} \right\| dxdt. \quad (24)$$

Using the Poincaré-type inequalities

$$2\int_0^1 \left\|\Im_x v\left(x,\tau\right)\right\|^2 dx \leq \int_0^1 \left\|v\left(x,\tau\right)\right\|^2 dx,$$

$$\int_0^1 \left\|\Im_x^2 \frac{\partial v}{\partial t}\right\|^2 dx \leq \frac{1}{2}\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}\right\|^2 dx,$$

$$\int_0^1 \left\|\Im_x \psi\left(x\right)\right\|^2 dx \leq \frac{1}{2}\int_0^1 \left\|\psi\left(x\right)\right\|^2 dx,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}\left(x,\tau\right)\right\|^2 dx + \alpha\int_0^1 \left\|\Im_x v\left(x,\tau\right)\right\|^2 dx \\
&\quad + \left(\beta - \frac{1}{4\varepsilon}\right)\int_{Q_\tau} \left\|\Im_x \frac{\partial v}{\partial t}\right\|^2 dxdt \\
&\leq \frac{1}{4}\int_0^1 \left\|\psi\left(x\right)\right\|^2 dx + \frac{\alpha}{2}\int_0^1 \left\|\varphi\left(x\right)\right\|^2 dx + \frac{\varepsilon}{2}\int_{Q_\tau} \left\|g\right\|^2 dxdt.
\end{aligned} \tag{25}$$

For the choice $\varepsilon = \frac{1}{4\beta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}\left(x,\tau\right)\right\|^2 dx + \alpha\int_0^1 \left\|\Im_x v\left(x,\tau\right)\right\|^2 dx \\
&\leq \frac{1}{4}\int_0^1 \left\|\psi\left(x\right)\right\|^2 dx + \frac{\alpha}{2}\int_0^1 \left\|\varphi\left(x\right)\right\|^2 dx + \frac{1}{8\beta}\int_{Q_\tau} \left\|g\right\|^2 dxdt,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}\left(x,\tau\right)\right\|^2 dx + \int_0^1 \left\|\Im_x v\left(x,\tau\right)\right\|^2 dx \\
&\leq C^2\left(\int_{Q_\tau} \left\|g\right\|^2 dxdt + \int_0^1 \left\|\psi\left(x\right)\right\|^2 dx + \int_0^1 \left\|\varphi\left(x\right)\right\|^2 dx\right),
\end{aligned} \tag{26}$$

where the constant C is given by

$$C = \left(\frac{\max\left(\frac{1}{4}, \frac{\alpha}{2}, \frac{1}{8\beta}\right)}{\min\left(\frac{1}{2}, \alpha\right)}\right)^{\frac{1}{2}}.$$

Since the right-hand side of (26) is independent of τ, we take the supremum over τ in the interval $[0,T]$ on the left-hand side, yielding

$$\begin{aligned}
&\sup_{0\leq\tau\leq T}\int_0^1 \left(\left\|\Im_x \frac{\partial v}{\partial t}\left(x,\tau\right)\right\|^2 + \left\|\Im_x v\left(x,\tau\right)\right\|^2\right) dx \\
&\quad\leq C\left(\int_Q \left\|g\right\|^2 dxdt + \int_0^1 \left\|\psi\left(x\right)\right\|^2 dx + \int_0^1 \left\|\varphi\left(x\right)\right\|^2 dx\right).
\end{aligned}$$

Thus, we establish inequality (18).

Corollary 2.1 *If problem* (10)-(14) *admits a solution, then this solution is unique and depends continuously on the given data* $(g, \varphi, \psi) \in F$.

2.2 Existence of the solution

Establishing the existence of a strong solution is crucial for ensuring the well-posedness of the problem. The following theorem guarantees that under appropriate conditions, a unique solution exists.

Theorem 2.2 *If* $v\,(x,t)$ *satisfies the conditions stated in Theorem 2.1, then the problem* (10)-(14) *admits a unique strong solution given by*

$$v = \overline{L}^{-1}\,(g, \varphi, \psi) = \overline{L^{-1}}\,(g, \varphi, \psi)\,.$$

Proof. To establish the existence of a strong solution for the problem (10)-(14), we need to demonstrate that for any arbitrary $(f, \varphi, \psi) \in F$, the range of the operator L, denoted as $R(L)$, is dense in F. First, consider the case where L is reduced to $L_0 = (\pounds, \ell, q)$, with its domain given by

$$D(L_0) = \{v \in D(L) \mid \ell v = 0, \quad qv = 0\}.$$

To achieve this, we establish the following proposition.

Proposition 2.1 *Under the conditions of Theorem 2.1, for* $\omega \in L^2(Q)$ *and for all* $v \in D(L_0)$, *if*

$$\int_Q \pounds v \cdot \omega\, dx\, dt = 0, \tag{27}$$

then ω *is identically zero almost everywhere in* Q.

Proof. The equality (27) can be rewritten as

$$\int_{Q_\tau} \frac{\partial^2 v}{\partial t^2}\omega\, dx\, dt = \alpha \int_{Q_\tau} \frac{\partial^2 v}{\partial x^2}\omega\, dx\, dt - \beta \int_{Q_\tau} \frac{\partial v}{\partial t}\omega\, dx\, dt. \tag{28}$$

From equation (28), we express the function ω in terms of v as

$$\omega = -\Im_x^2 \frac{\partial v}{\partial t}. \tag{29}$$

Substituting ω from (29) into (28), we obtain

$$\int_{Q_\tau} \frac{\partial^2 v}{\partial t^2}\left(-\Im_x^2 \frac{\partial v}{\partial t}\right) dx\, dt = \alpha \int_{Q_\tau} \frac{\partial^2 v}{\partial x^2}\left(-\Im_x^2 \frac{\partial v}{\partial t}\right) dx\, dt - \beta \int_{Q_\tau} \frac{\partial v}{\partial t}\left(-\Im_x^2 \frac{\partial v}{\partial t}\right) dx\, dt. \tag{30}$$

Integrating by parts and considering conditions (13) and (14), we obtain

$$\begin{aligned}\int_{Q_\tau} \frac{\partial^2 v}{\partial t^2}\left(-\Im_x^2 \frac{\partial v}{\partial t}\right) dx\, dt &= \frac{1}{2}\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}(x,\tau)\right\|^2 dx - \frac{1}{2}\int_0^1 \left\|\Im_x qv\right\|^2 dx \\ &= \frac{1}{2}\int_0^1 \left\|\Im_x \frac{\partial v}{\partial t}(x,\tau)\right\|^2 dx, \end{aligned} \tag{31}$$

$$\alpha \int_{Q_\tau} \frac{\partial^2 v}{\partial x^2} \Im_x^2 \frac{\partial v}{\partial t} dx\, dt \quad = \quad \frac{\alpha}{2} \int_0^1 \|v(x,\tau)\|^2 dx - \frac{\alpha}{2} \int_0^1 \|lv\|^2 dx = \frac{\alpha}{2} \int_0^1 \|v(x,\tau)\|^2 dx,$$

and

$$-\beta \int_{Q_\tau} \frac{\partial v}{\partial t} \left(-\Im_x^2 \frac{\partial v}{\partial t} \right) dx\, dt = -\beta \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dx\, dt. \tag{32}$$

By substituting (31), (32), and (32) into (30), we obtain

$$\frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx + \beta \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dx\, dt + \frac{1}{2} \int_0^1 \|v(x,\tau)\|^2 dx \le 0. \tag{33}$$

Since norms are non-negative, the inequality simplifies to

$$\int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx + \int_{Q_\tau} \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dx\, dt + \int_0^1 \|v(x,\tau)\|^2 dx \le 0. \tag{34}$$

Since the right-hand side of (34) is independent of τ, we take the supremum over $\tau \in [0,T]$ on the left-hand side, yielding

$$\sup_{0\le\tau\le T} \int_0^1 \left\| \Im_x \frac{\partial v}{\partial t}(x,\tau) \right\|^2 dx + \int_Q \left\| \Im_x \frac{\partial v}{\partial t} \right\|^2 dxdt + \sup_{0\le\tau\le T} \int_0^1 \|v(x,\tau)\|^2 dx \le 0.$$

Since norms are non-negative, each integral in the above inequality must be zero. This implies that

$$\Im_x \frac{\partial v}{\partial t} = 0, \quad \Im_x v = 0, \quad \text{and} \quad v = 0 \quad \text{almost everywhere in } Q.$$

Thus, we conclude that $v \equiv 0$. Now, substituting $v = 0$ into (29), we obtain $\omega = 0$, which means $\omega \equiv 0$ in $L^2(Q)$.

3 Solution's Approximation via HPM

We assume that the function $u(x,t)$ is well-defined and of exponential order for $t \ge 0$, meaning there exist constants $A, \gamma > 0$, and $t_0 > 0$ such that

$$|u(x,t)| \le A \exp(\gamma t) \quad \text{for } t \ge t_0.$$

Furthermore, we assume that the Laplace transform of $u(x,t)$ exists and is given by

$$U(x,s) = \mathscr{L}\{u(x,t); t \to s\} = \int_0^\infty u(x,t) \exp(-st)\, dt,$$

where s is a positive parameter. Applying the Laplace transform to both sides of equation (1), we obtain

$$\alpha \frac{d^2 U}{dx^2}(x,s) = (s^2 + \beta s) U(x,s) - (F(x,s) + \psi(x) + (s+\beta)\varphi(x)), \tag{35}$$

where $F(x,s) = \mathscr{L}\{f(x,t); t \to s\}$. Similarly, the integral conditions in the Laplace transform form become

$$\int_0^1 U(\xi, s)\, d\xi = A(s), \tag{36}$$

$$\int_0^1 \xi U\left(\xi, s\right) d\xi = B(s), \tag{37}$$

where

$$A(s) = \mathscr{L}\left\{a(t); t \to s\right\}, \tag{38}$$
$$B(s) = \mathscr{L}\left\{b(t); t \to s\right\}. \tag{39}$$

We associate the problem (35)–(37) with the following mixed boundary value problem:

$$\frac{d^2V}{dx^2}(x,s) = k^2 V(x,s) - \frac{1}{\alpha}\left(F(x,s) + \psi(x) + (s+\beta)\varphi(x)\right), \tag{40}$$
$$V\left(0,s\right) = \lambda(s), \tag{41}$$
$$V_x\left(0,s\right) = \mu(s), \tag{42}$$

where

$$k^2 = \frac{s^2 + \beta s}{\alpha}. \tag{43}$$

Clearly, the function $V(x, s, \lambda, \mu)$ will be a solution to problem (35)–(37) if and only if the pair (λ, μ) satisfies the following system of equations:

$$\begin{cases} \displaystyle\int_0^1 V(\xi, s, \lambda, \mu) d\xi = A(s), \\ \displaystyle\int_0^1 \xi V(\xi, s, \lambda, \mu) d\xi = B(s). \end{cases} \tag{44}$$

It is evident that V can be expressed as the sum of two functions, $\overline{V}$ and $\widetilde{V}$, where these components satisfy the following problems:

- For $\widetilde{V}$, we have

$$\frac{d^2\widetilde{V}}{dx^2}(x,s) = k^2 \widetilde{V}(x,s) - \frac{1}{\alpha}\left(F(x,s) + \psi(x) + (s+\beta)\varphi(x)\right), \tag{45}$$
$$\widetilde{V}\left(0,s\right) = 0, \tag{46}$$
$$\widetilde{V}_x\left(0,s\right) = 0. \tag{47}$$

- For $\overline{V}$, we obtain

$$\frac{d^2\overline{V}}{dx^2}(x,s) = k^2 \overline{V}(x,s), \tag{48}$$
$$\overline{V}\left(0,s\right) = \lambda(s), \tag{49}$$
$$\overline{V}_x\left(0,s\right) = \mu(s). \tag{50}$$

One can easily verify that $\overline{V}$ is given by

$$\overline{V} = \lambda \cosh(kx) + \frac{\mu}{k}\sinh(kx). \tag{51}$$

Thus, we express U as

$$U = V = \widetilde{V} + \lambda \cosh(kx) + \frac{\mu}{k}\sinh(kx). \tag{52}$$

Substituting (52) into (44), we obtain the system

$$\begin{cases} \lambda\frac{\sinh k}{k} + \mu\frac{\cosh k - 1}{k^2} = A(s) - \int_0^1 \widetilde{V}(\xi, s)d\xi, \\ \lambda\frac{k\sinh k - \cosh k + 1}{k^2} + \mu\frac{k\cosh k - \sinh k}{k^3} = B(s) - \int_0^1 \xi\widetilde{V}(\xi, s)d\xi \end{cases}. \tag{53}$$

The determinant of this system is given by

$$D(k) = \frac{1}{k^4}\left(k\sinh k - 2\cosh k + 2\right) = \frac{1}{k^4}g(k). \tag{54}$$

Consequently, we have

$$\begin{aligned} g(k) &= k\sinh k - 2\cosh k + 2, \\ g'(k) &= \frac{d}{dk}g(k) = k\cosh k - \sinh k, \\ g''(k) &= \frac{d^2}{dk^2}g(k) = k\sinh k. \end{aligned}$$

Since $\forall k > 0 : g''(k) > 0$, it follows that

$$\forall k > 0 : g'(k) > g'(0) = 0 \Rightarrow g(k) > g(0) = 0.$$

Hence, for all positive values of k, the system (53) admits a unique solution given by

$$\begin{cases} \lambda(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} A(s) - \int_0^1 \widetilde{V}(\xi, s)d\xi & \frac{\cosh k - 1}{k^2} \\ B(s) - \int_0^1 \xi\widetilde{V}(\xi, s)d\xi & \frac{k\cosh k - \sinh k}{k^3} \end{vmatrix} \\ \mu(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} \frac{\sinh k}{k} & A(s) - \int_0^1 \widetilde{V}(\xi, s)d\xi \\ \frac{k\sinh k - \cosh k + 1}{k^2} & B(s) - \int_0^1 \xi\widetilde{V}(\xi, s)d\xi \end{vmatrix} \end{cases}. \tag{55}$$

To solve the problem (45)-(47), we construct the following homotopy:

$$\begin{aligned} \widetilde{V}(x, s, p) &: \quad Q \times [0, 1] \to \mathbb{R}, \\ \frac{d^2\widetilde{V}}{dx^2}(x, s) &= p\left[k^2\widetilde{V}(x, s) - \frac{1}{\alpha}\left(F(x, s) + \psi(x) + (s + \beta)\varphi(x)\right)\right]. \end{aligned} \tag{56}$$

We assume that the solution of (56) can be expressed as a power series

$$\widetilde{V} = \sum_{j=0}^{\infty} p^j\widetilde{V}_j. \tag{57}$$

Substituting (57) into (56), we obtain

$$\frac{d^2}{dx^2}\sum_{j=0}^{\infty} p^j\widetilde{V}_j = p\left[k^2\sum_{j=0}^{\infty} p^j\widetilde{V}_j - \frac{1}{\alpha}\left(F(x, s) + \psi(x) + (s + \beta)\varphi(x)\right)\right]. \tag{58}$$

Equating the coefficients of p with the same powers in (58) leads to

$$p^0 : \quad \frac{d^2\widetilde{V}_0}{dx^2} = 0, \quad \widetilde{V}_0(0) = 0, \quad \frac{d\widetilde{V}_0}{dx}(0) = 0.$$

$$\Rightarrow \quad \widetilde{V}_0(x,s) = 0.$$

For p^1, we have

$$\frac{d^2\widetilde{V}_1}{dx^2} = \left(k^2\widetilde{V}_0 - \frac{1}{\alpha}\left(F(x,s) + \psi(x) + (s+\beta)\varphi(x)\right)\right), \quad \widetilde{V}_1(0) = 0, \quad \frac{d\widetilde{V}_1}{dx}(0) = 0.$$

Solving the integral equation for $\widetilde{V}_1$ yields

$$\widetilde{V}_1(x,s) = -\frac{1}{\alpha}\int_0^x\int_0^\mu \left(F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)\right) d\xi d\mu.$$

$$= -\frac{1}{\alpha}\int_0^x (x-\xi)\left(F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)\right) d\xi.$$

For $p^{n+1}, \forall n \geq 1$, we have

$$\frac{d^2\widetilde{V}_{n+1}}{dx^2} = k^2\widetilde{V}_n, \quad \widetilde{V}_n(0) = 0, \quad \frac{d\widetilde{V}_n}{dx}(0) = 0.$$

Solving iteratively yields

$$\widetilde{V}_{n+1}(x,s) = k^2\int_0^x\int_0^\mu \widetilde{V}_n(\xi,s) d\xi d\mu = k^2\int_0^x (x-\xi)\widetilde{V}_n(\xi,s) d\xi.$$

Thus, we obtain the general expression

$$\begin{cases} \widetilde{V}_0(x,s) = 0, \\ \forall n \geq 1: \quad \widetilde{V}_n(x,s) = -\frac{1}{\alpha}\frac{k^{2n-2}}{(2n-1)!}\int_0^x (x-\xi)^{2n-1}\left(F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)\right) d\xi. \end{cases}$$

When $p \to 1$, equation (57) provides the approximate solution to the problem (45)-(47), given by

$$\widetilde{V}_{hpm}(x,s) = -\frac{1}{\alpha}\sum_{j=1}^{\infty}\frac{k^{2j-2}}{(2j-1)!}\int_0^x (x-\xi)^{2j-1}\left(F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)\right) d\xi. \quad (59)$$

Theorem 3.1 *If*

$$\sup_{\xi\in(0,1)} |F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)| = M < \infty,$$

then the series defined by equation (59) *is convergent.*

Proof. We define the function $G(\xi,s) = F(\xi,s) + \psi(\xi) + (s+\beta)\varphi(\xi)$. For all $n \geq 1$, we have

$$\begin{aligned} \left|\widetilde{V}_n\right| &= \left|-\frac{1}{\alpha}\frac{k^{2n-2}}{(2n-1)!}\int_0^x (x-\xi)^{2n-1} G(\xi,s)\, d\xi\right| \\ &\leq \frac{1}{\alpha}\frac{k^{2n-2}}{(2n-1)!}\int_0^x (x-\xi)^{2n-1} |G(\xi,s)|\, d\xi \\ &\leq \frac{M}{\alpha}\frac{k^{2n-2}}{(2n-1)!}\int_0^x (x-\xi)^{2n-1} d\xi \\ &\leq \frac{M}{\alpha}\frac{k^{2n-2}}{(2n)!}x^{2n}. \end{aligned}$$

Now, we define

$$W_n(x,s) = \frac{M}{\alpha}\frac{k^{2n-2}}{(2n)!}x^{2n}.$$

Taking the limit, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{W_{n+1}(x,s)}{W_n(x,s)} &= \lim_{n\to\infty}\frac{k^2x^2}{(2n+1)(2n+2)} \\ &= 0.\end{aligned}$$

Since the series $\sum W_n(x,s)$ is convergent, it follows that the series $\sum \widetilde{V}_n(x,s)$ is also convergent. The solution $U_{hpm}(x,t)$ can be approximately recovered from $U_{hpm}(x,s)$ either through an analytical approach or by using the Stehfest algorithm [38], where the inverse Laplace transform of $U_{hpm}(x,s)$ is computed as follows:

$$u_{hpm}(x,t) \simeq \frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n U_{hpm}\left(x, \frac{n\ln 2}{t}\right),$$

where the coefficients β_n are given by

$$\beta_n = (-1)^{n+m}\sum_{k=\left\lfloor\frac{n+1}{2}\right\rfloor}^{\min(n,m)}\frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}.$$

Here, m is an odd positive integer, and $\left\lfloor\frac{n+1}{2}\right\rfloor$ denotes the integer part of $\frac{n+1}{2}$. For $m=5$, the first ten coefficients β_n are given as

$$\begin{aligned}&\beta_1 = \frac{1}{12}, \quad \beta_2 = -\frac{385}{12}, \quad \beta_3 = 1279, \quad \beta_4 = -\frac{46871}{3}, \quad \beta_5 = \frac{505465}{6}, \\ &\beta_6 = -\frac{473915}{2}, \quad \beta_7 = \frac{1127735}{3}, \quad \beta_8 = -\frac{1020215}{3}, \quad \beta_9 = \frac{328125}{2}, \quad \beta_{10} = -\frac{65625}{2}.\end{aligned}$$

Thus, the approximate solution is given by

$$u_{hpm}(x,t) \approx \frac{\ln 2}{t}\sum_{n=1}^{10}\beta_n U_{hpm}\left(x, \frac{n\ln 2}{t}\right).$$

4 Numerical Example

In this section, we present a numerical example to illustrate the effectiveness and accuracy of the proposed method. The approximate solution obtained using the homotopy perturbation method (HPM) is compared with the exact solution to validate its convergence and reliability. The numerical results demonstrate the efficiency of the method in handling the Telegraph equation with integral boundary conditions.

Example 4.1 Consider the Telegraph equation with the following parameters:

$$\alpha = 1, \quad \beta = 1, \quad f(x,t) = e^t\left(2x^2 + 2t + 1\right),$$

$$a(t) = \frac{1}{3}e^t(3t+1), \quad b(t) = \frac{1}{4}e^t(2t+1),$$

$$\varphi(x) = x^2, \quad \psi(x) = x^2 + 1.$$

The exact solution to this problem is given by

$$u_{\mathrm{exa}}(x,t) = \left(t + x^2\right) e^t.$$

By substituting the given data into (43), (59), (55) and (52), we obtain

$$k = \sqrt{s^2 + s}$$

$$\begin{aligned}\widetilde{V}_{hpm}(x,s) &= -\sum_{j=1}^{\infty} k^{2j-2} \frac{1}{(2j-1)!} \int_0^x (x-\xi)^{2j-1} \left(\left(\frac{s^3 - s}{(s-1)^2} \right) \xi^2 + \frac{s^2 - s + 2}{(s-1)^2} \right) d\xi \\ &= -\frac{1}{(s-1)^2} \cosh kx + \frac{1}{(s-1)^2} \left(sx^2 - x^2 + 1 \right),\end{aligned}$$

$$\begin{cases} \lambda(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} \frac{\sinh k}{k(s-1)^2} & \frac{\cosh k - 1}{k^2} \\ \frac{1}{k^2(s-1)^2}\left(k\sinh k - \cosh k + 1\right) & \frac{k\cosh k - \sinh k}{k^3} \end{vmatrix} = \frac{1}{(s-1)^2} \\ \mu(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} \frac{\sinh k}{k} & \frac{\sinh k}{k(s-1)^2} \\ \frac{k\sinh k - \cosh k + 1}{k^2} & \frac{1}{k^2(s-1)^2}\left(k\sinh k - \cosh k + 1\right) \end{vmatrix} = 0 \end{cases},$$

and

$$U(x,s) = -\frac{1}{(s-1)^2} \cosh kx + \frac{1}{(s-1)^2} \left(sx^2 - x^2 + 1 \right) + \lambda \cosh kx + \frac{\mu}{k} \sinh kx.$$

Therefore, we obtain

$$U(x,s) = \frac{1}{(s-1)^2} \left(sx^2 - x^2 + 1 \right).$$

Taking the inverse Laplace transform yields

$$u(x,t) = \mathscr{L}^{-1} \left(\frac{1}{(s-1)^2} \left(sx^2 - x^2 + 1 \right) \right) = e^t x^2 + te^t = u_{exa}(x,t).$$

Example 4.2 The parameters for this example are given as

$$\alpha = 1, \quad \beta = 2, \quad f(x,t) = 2x\left(3t^2 + x^2\right),$$

$$a(t) = \frac{1}{2}t^3 + \frac{1}{4}t, \quad b(t) = \frac{1}{3}t^3 + \frac{1}{5}t, \quad \varphi(x) = 0, \quad \psi(x) = x^3.$$

The exact solution to this problem is given by

$$u_{exa}(x,t) = tx\left(t^2 + x^2\right).$$

By replacing the previous data in (43), (59), (55) and (52), we get

$$\begin{aligned} k &= \sqrt{s^2 + 2s}, \\ \widetilde{V}_{hpm}(x,s) &= -\sum_{j=1}^{\infty} \frac{k^{2j-2}}{(2j-1)!} \int_0^x (x-\xi)^{2j-1} \left(\left(\frac{2}{s} + 1 \right) \xi^3 + \frac{12}{s^3} \xi \right) d\xi, \\ &= \frac{1}{s^4} x \left(s^2 x^2 + 6 \right) - \frac{6}{ks^4} \sinh kx. \end{aligned}$$

Solving for $\lambda(s)$ and $\mu(s)$, we obtain

$$\begin{cases} \lambda(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} \frac{6}{k^2 s^4}(\cosh k - 1) & \frac{\cosh k - 1}{k^2} \\ -\frac{6}{k^3 s^4}(\sinh k - k\cosh k) & \frac{k\cosh k - \sinh k}{k^3} \end{vmatrix} = 0, \\ \mu(s) = \frac{k^4}{(k\sinh k - 2\cosh k + 2)} \begin{vmatrix} \frac{\sinh k}{k} & \frac{6}{k^2 s^4}(\cosh k - 1) \\ \frac{k\sinh k - \cosh k + 1}{k^2} & -\frac{6}{k^3 s^4}(\sinh k - k\cosh k) \end{vmatrix} = \frac{6}{s^4}. \end{cases}$$

Thus, the function $U(x,s)$ is given by

$$\begin{aligned} U(x,s) &= \frac{1}{s^4}x\left(s^2x^2+6\right) - \frac{6}{ks^4}\sinh kx + \lambda\cosh sx + \frac{\mu}{k}\sinh sx, \\ &= \frac{1}{s^4}x\left(s^2x^2+6\right). \end{aligned}$$

Taking the inverse Laplace transform yields

$$u(x,t) = \mathscr{L}^{-1}\left(\frac{1}{s^4}x\left(s^2x^2+6\right)\right) = tx\left(t^2+x^2\right) = u_{exa}(x,t).$$

5 Conclusion

In this paper, He's homotopy perturbation method (HPM) has been successfully applied to solve second-order differential equations with constant coefficients. The method has proven to be reliable and efficient, offering an analytical approximation that, in many cases, yields an exact solution in a rapidly convergent sequence with elegantly computed terms. The convergence of this series has been rigorously demonstrated in this study. Furthermore, by integrating the HPM with the Laplace transformation technique and the Stehfest algorithm, we effectively addressed the telegraph problem with integral boundary conditions. This combined approach not only provides a powerful framework for solving similar differential equations but also enhances the applicability of analytical methods in various scientific and engineering contexts.

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On the Positivity and Stability Analysis of Conformable Fractional COVID-19 Models

Saadia Benbernou Belmehdi [1], Djillali Bouagada [1], Boubakeur Benahmed [2] and Kamel Benyettou [3*]

[1] *Department of Mathematics and Computer Science, ACSY Team-Laboratory of Pure and Applied Mathematics, Abdelhamid Ibn Badis University Mostagenm(UMAB), P.O.Box 227/118 University of Mostaganem, 27000 Mostaganem, Algeria.*
[2] *National Higher School of Mathematics, Scientific and Technology Hub of Sidi Abdellah, P.O. Box 75, Algiers 16093, Algeria.*
[3] *National Higher School of Mathematics, Scientific and Technology Hub of Sidi Abdellah, ACSY Team-Laboratory of Pure and Applied Mathematics (UMAB), P.O. Box 75, Algiers 16093, Algeria.*

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Abstract: In this paper, we present a novel nonlinear fractional conformable SEIR compartmental model with a vaccination compartment. This SEIHRDV model is one of the next generation models that better explores nonlinear disease behaviour. The main aim of this study is to test when such a model is locally asymptotically stable, respectively, globally asymptotically stable around its equilibrium points: the disease-free state X_0 and the endemic state X_1. To achieve this, we introduce the Sumudu transform as an innovative integral approach to solve the differential equations, compute the basic reproduction number R_0, and construct a Lyapunov function to rigorously establish the stability conditions. Numerical solutions are then obtained using the Rang-Kutta 4th order method, and graphical simulations are performed in MATLAB (version 2023a) to further validate the theoretical findings and illustrate the applicability and the accuracy of the proposed approaches.

Keywords: *nonlinear model; conformable derivative; positivity analysis; local and global stability; Sumudu transform; COVID-19 models.*

Mathematics Subject Classification (2020): 70K75, 93-05, 93-08, 93-10, 93C10, 93D05, 93D20.

* Corresponding author: mailto:kamel.benyettou@nhsm.edu.dz

1 Introduction

Epidemiological nonlinear COVID-19 models are associated with a respiratory disease that emerged in December 2019 in Wuhan and other cities in China, which has become a global public health emergency declared by the World Health Organization (WHO) in the first quarter of 2020. The first transmission of the disease was from animal to human. Then, it spreaded rapidly among humans through direct and indirect contact. The most common clinical symptoms of SARS-CoV-2 infections are fever, cough, tiredness, loss of taste or smell, nausea, diarrhea, pneumonia, and respiratory issues.

Mathematical nonlinear models play a crucial role in predicting future trends of epidemics and developing effective strategies to control them based on data and microscopic hypotheses about the population. The Kermack and McKendrick model [6] presented in 1927, was the first SIR model capable of predicting epidemic infections. Since then, researchers have further studied and developed global and local stability of SEIR and SIR models. Recently, the development of mathematical models to predict the evolution of COVID-19 infections has attracted considerable attention. Researchers have explored various aspects of the disease, including its nature, transmission dynamics, basic reproduction number, and stability [3, 5, 9].

The stability of fractional nonlinear epidemiological models is crucial for understanding complex dynamical systems because it allows an accurate representation of natural phenomena by integrating long-term memories and non-local behaviors. Some researchers have examined the stability of fractional conformable pandemic models [7], in particular, simplifying calculations while preserving these advantages, thus facilitating the analysis of local and global stability. The particularity of this derivative is due to its simplicity of handling and interpreting and the use of the simplest initial conditions similar to those of classical derivatives. Some researchers have applied the Sumudu transform in fractional conformable models [2]. Similar to the Laplace and Fourier transforms, the Sumudu transform is an efficient method for solving differential and integral equations. It differs from the Laplace transform in that it also takes into account the units of the original function, making it more suitable in physics and engineering applications. Moreover, its connection with the Laplace transform allows a smooth transition between various analytical techniques, thus increasing its adaptability to different domains.

In this paper, by using the conformable derivative, we study Rabih Ghostine's COVID-19 model focusing on positivity analysis via the Sumudu transformation. Therefore, we apply this technique to understand in detail the local and global stability of this model using the concept of R_0. The results of our study reveal that the Sumudu transformation and the conformable derivative improve these analyses, providing more information about its behaviors as well as possible applications in epidemiological modeling or other related fields.

This paper is organized af follows. In Section 2, we present preliminaries and problem formulation. Section 3 is devoted to the study of the positivity and boundness of the solutions using the Sumudu transform. The choice of this transformation is due to its conservation of units, which facilitates the interpretation of results while simplifying calculations for certain differential and integral equations. Additionally, it provides a direct correspondence with power series and offers a simpler inversion in some cases. The Disease-Free and Endemic Equilibrium and the basic reproduction number are established in Section 4. Stability analysis is presented for the considered class of nonlinear models in Section 5. In Section 6, some numerical examples and simulations are presented to show the effectiveness and applicability of the proposed method.

2 Preliminaries and Problem Formulation

In this section, we present the mathematical formulation of the COVID-19 nonlinear model. We have chosen the conformable derivative for this formulation because it provides a simpler and more intuitive alternative for modelling, manipulating, and interpreting dynamic phenomena. Additionally, it uses simpler initial conditions similar to those of classical derivatives.

2.1 Preliminaries

This subsection presents essential theorems and definitions of conformable fractional operators and the Sumudu transform, which is crucial for the dynamic analysis of the results discussed in this paper.

Definition 2.1 [7] Let a function $x : [0, +\infty) \to \mathbb{R}$ be given. Then the conformable derivative of the function x of order α, with $\alpha \in (0, 1]$, is defined by

$$\mathbf{T}^{\alpha}(x)(t) = \lim_{\epsilon \to 0} \frac{x(t + \epsilon t^{1-\alpha}) - x(t)}{\epsilon}, \quad \forall t > 0.$$

If the conformable derivative of the function x of order α exists for all $t > 0$, then we simply say that x is α-differentiable.

Theorem 2.1 *[7] Let $\alpha \in (0, 1]$ and $x_1, x_2 : \mathbb{R}_+ \to \mathbb{R}$ be α-differentiable functions. Then, $\forall t > 0$,*

(a) $\mathbf{T}^{\alpha}(ax_1(t) + bx_2(t)) = a\mathbf{T}^{\alpha}(x_1)(t) + b\mathbf{T}^{\alpha}(x_2)(t)$ *for all* $a, b \in \mathbb{R}$*;*

(b) $\mathbf{T}^{\alpha}(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$*;*

(c) $\mathbf{T}^{\alpha}(\lambda) = 0$ *for all constant function* $x_1(t) = \lambda$*;*

(d) $\mathbf{T}^{\alpha}(x_1(t)x_2(t)) = x_1(t)\mathbf{T}^{\alpha}(x_2)(t) + x_2(t)\mathbf{T}^{\alpha}(x_1)(t)$*;*

(e) $\mathbf{T}^{\alpha}\left(\frac{x_1(t)}{x_2(t)}\right) = \frac{x_2(t)\mathbf{T}^{\alpha}(x_1)(t) + x_1(t)\mathbf{T}^{\alpha}(x_2)(t)}{x_2^2(t)}$*;*

(f) If x_1 is differentiable, then $\mathbf{T}^{\alpha}(x_1)(t) = t^{1-\alpha}\frac{dx_1(t)}{dt}$.

Definition 2.2 [7] We take into account functions with exponential order in the set $\mathcal{A}$ defined by

$$\mathcal{A} = \left\{x(t)\ \exists M, \tau_1, \tau_2 > 0, \mid x(t) \mid < Me^{-\frac{|t|}{\tau_j}},\ \text{if } t \in (-1)^j \times [0, \infty)\right\},$$

the Sumudu transform X of a continuous function x is represented by

$$\mathcal{S}[x(t)] = X(v) = \int_0^{\infty} x(vt)e^{-t}dt, \quad v \in (-\tau_1, \tau_2),$$

or a similar alternative

$$\mathcal{S}[x(t)] = X(v) = \frac{1}{v}\int_0^{\infty} x(t)e^{-\frac{t}{v}}dt, \quad v > 0.$$

Theorem 2.2 *[1] Let $x : [0,\infty) \to \mathbb{R}$ be a given function, $0 < \alpha \leq 1$. Then we have the following property:*

$$\mathcal{S}_\alpha[\mathbf{T}^\alpha x(t)](v) = \frac{1}{v}\left[\mathcal{S}_\alpha[x(t)](v) - x(0)\right], \quad \forall t > 0.$$

Theorem 2.3 *[1] Let c and $a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then*

1. $$\mathcal{S}_\alpha[e^{-a\frac{t^\alpha}{\alpha}}x(t)] = \frac{\mathcal{S}_\alpha[x(t)](\frac{1}{v}+a)}{v}, \quad v > 0;$$

2. $$\mathcal{S}_\alpha[c](v) = c;$$

3. $$\mathcal{S}_\alpha\left[\frac{t^{n\alpha}}{\alpha^n}\right](v) = \frac{\Gamma(1+n)^n}{v}, \quad v > 0;$$

4. $$\mathcal{S}_\alpha\left[e^{-\frac{at^\alpha}{\alpha}}\right](v) = \frac{1}{1+av}, \quad v > \frac{1}{a}.$$

2.2 Model formulation

Several models have been developed in the literature to describe and study the dynamics of COVID-19 disease. Rabih Ghostine's nonlinear model has been recognized for its comprehensive yet creative methods used to explain better and understand the outbreak and spread of COVID-19. Several factors contribute to its significance: Accuracy and Predictive Power, Involvement of Different Variables, Adjustability, Transdisciplinarity, Openness and Accessibility, and Influence in Reality. This work focuses on an extended and reformulated version of Ghostine's epidemiological nonlinear model [5], expressed in a fractional form using a conformable derivative, with a subsequent investigation of its global stability.

In our model, the human population denoted by N is composed of seven compartments (subpopulations) according to the status of the disease. The number of susceptible individuals $S(t)$, infected individuals in the incubation period $E(t)$, infectious individuals $I(t)$, hospitalized individuals $H(t)$, recovered $R(t)$, and vaccinated cases $V(t)$ are incorporated. The fractional-order SEIHRDV model, utilizing the conformable fractional derivative of order $0 < \alpha \leq 1$, is given by

$$
\begin{aligned}
\mathbf{T}^{\alpha} S(t) &= \Lambda^{\alpha} - \beta^{\alpha} S I - \eta^{\alpha} S - \mu^{\alpha} S, \\
\mathbf{T}^{\alpha} E(t) &= \beta^{\alpha} S I + (\sigma\beta)^{\alpha} V I - (\gamma^{\alpha} + \mu^{\alpha}) E, \\
\mathbf{T}^{\alpha} I(t) &= \gamma^{\alpha} E - (\delta^{\alpha} + \mu^{\alpha}) I, \\
\mathbf{T}^{\alpha} H(t) &= \delta^{\alpha} I - ((1-\kappa^{\alpha})\lambda^{\alpha} + (\kappa\rho)^{\alpha} + \mu^{\alpha}) H, \\
\mathbf{T}^{\alpha} R(t) &= (1-\kappa^{\alpha})\lambda^{\alpha} H - \mu^{\alpha} R, \\
\mathbf{T}^{\alpha} D(t) &= (\kappa\rho)^{\alpha} H, \\
\mathbf{T}^{\alpha} V(t) &= \eta^{\alpha} S - (\sigma\beta)^{\alpha} V I - \mu^{\alpha} V
\end{aligned} \tag{1}
$$

under the conditions

$S(0) = S_0 \geq 0, \quad E(0) = E_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad H(0) = H_0 \geq 0, \quad R(0) = R_0 \geq 0,$
$D(0) = D_0 \geq 0, \quad V(0) = V_0 \geq 0$

and $N(t) = S(t) + E(t) + I(t) + H(t) + R(t) + D(t) + V(t)$.

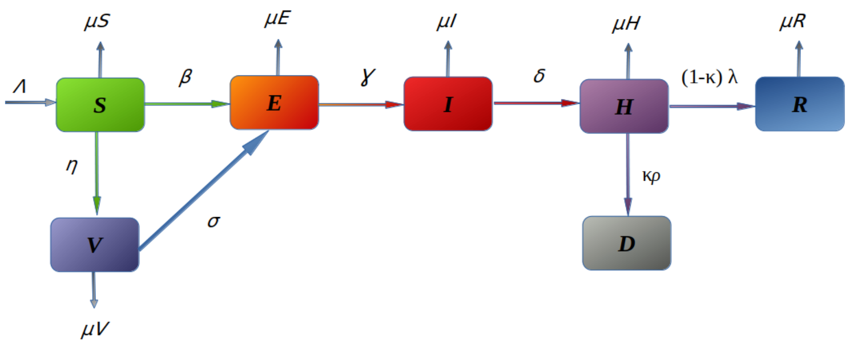


Figure 1: Compartmental diagram for the transmission dynamics of COVID-19.

In the following two sections, the analysis of the nonlinear epidemiological model is carried out under certain constraints on the parameters of the model to ensure the biological acceptability as well as its mathematical structure, the positive populations, plausible and sequential disease case rates. These characteristics are particularly important for describing the long-term development of an infectious disease process, including its retention or extinction. In terms of systems theory, the stability of these nonlinear models is of great importance in terms of the equilibrium point and its robustness against perturbations that revolve around the fundamental concepts of nonlinear dynamics.

3 Non-Negativity and Boundedness of Solutions

The objective of this section is to analyze and demonstrate the positivity and boundness of the system's solutions by introducing a new innovative integral technique named the Sumudu transform to solve differential systems.

Proposition 3.1 *The region* $\Omega = \{(S, E, I, H, R, D, V) \in \mathbb{R}^7 : 0 < N \leq \left(\frac{\Lambda}{\mu}\right)^\alpha\}$ *is non-negative invariant for the model (1) for* $t \geq 0$.

Proof. We have

$$\mathbf{T}^\alpha(S+E+I+H+R+D+V)(t) = \Lambda^\alpha - \mu^\alpha(S+E+I+H+R+D+V)(t).$$

$$\begin{aligned} \Longrightarrow \quad & \mathbf{T}^\alpha N(t) = \Lambda^\alpha - \mu^\alpha N(t) \\ \Longrightarrow \quad & \mathbf{T}^\alpha N(t) + \mu^\alpha N(t) = \Lambda^\alpha. \end{aligned}$$

Taking the Sumudu transform, we have

$$\begin{aligned} \frac{1}{v}\left[\mathcal{S}_\alpha[N(t)](v) - N(0)\right] + \mu^\alpha \mathcal{S}_\alpha[N(t)](v) &= \Lambda^\alpha \\ \Longrightarrow \quad \mathcal{S}_\alpha[N(t)](v) + v\mu^\alpha \mathcal{S}_\alpha[N(t)](v) &= N(0) + v\Lambda^\alpha \\ \Longrightarrow \quad \mathcal{S}_\alpha[N(t)](v) &= \frac{N(0) + v\Lambda^\alpha}{1 + v\mu^\alpha} \\ \Longrightarrow \quad \mathcal{S}_\alpha[N(t)](v) &= \frac{N(0)}{1 + v\mu^\alpha} - \frac{\Lambda^\alpha}{\mu^\alpha(1 + v\mu^\alpha)} + \frac{\Lambda^\alpha}{\mu^\alpha}. \end{aligned}$$

Applying the inverse Sumudu transform, we get

$$N(t) = \frac{\Lambda^\alpha}{\mu^\alpha} + \left(N(0) - \frac{\Lambda^\alpha}{\mu^\alpha}\right) e^{\frac{-\mu^\alpha t^\alpha}{\alpha}}.$$

Thus

$$\lim_{t \to \infty} Sup N(t) \leq \left(\frac{\Lambda}{\mu}\right)^\alpha.$$

As a result, the functions S, E, I, H, R, D and V are all non-negative.

4 Equilibrium Points and Basic Reproduction Number of the Model

The primary objective of this section is to determine two essential equilibrium states: the Disease-Free Equilibrium and the Endemic Equilibrium. Additionally, we aim to determine the Basic Reproduction Number $\mathcal{R}_0$.

Based on [12], Basic Reproduction Number $\mathcal{R}_0$ and the Disease-Free Equilibrium point (DFE) of the model (1) are given as follows:

$$\mathcal{R}_0 = \frac{(\beta\gamma\Lambda)^\alpha \epsilon_5}{\mu^\alpha \epsilon_1 \epsilon_2 \epsilon_3}, \quad X^0 = \left(\frac{\Lambda^\alpha}{\epsilon_1}, \quad 0, \quad 0, \quad 0, \quad 0, \quad \frac{(\eta\Lambda)^\alpha}{\mu^\alpha \epsilon_1}\right),$$

and the Endemic Equilibrium point (DEE) is given by

$$\begin{aligned} X^1 = \Bigg(& \frac{\Lambda^\alpha}{\beta^\alpha I^1 + \epsilon_1}, \quad \frac{\epsilon_3}{\gamma^\alpha} I^1, \quad \frac{\sqrt{M_1} + M_2}{M_3}, \quad \frac{\delta^\alpha}{\epsilon_4} I^1, \\ & \frac{(1-\kappa^\alpha)(\lambda\delta)^\alpha}{\mu^\alpha \epsilon_4} I^1, \quad \frac{(\eta\Lambda)^\alpha}{(\beta^\alpha I^1 + \epsilon_1)((\sigma\beta)^\alpha I^1 + \mu^\alpha)}, \Bigg), \end{aligned}$$

where

$$\epsilon_1 = \mu^\alpha + \eta^\alpha, \quad \epsilon_2 = \mu^\alpha + \gamma^\alpha, \quad \epsilon_3 = \mu^\alpha + \delta^\alpha,$$
$$\epsilon_4 = \mu^\alpha + \lambda^\alpha(1-\kappa^\alpha) + (\kappa\rho)^\alpha, \quad \epsilon_5 = \mu^\alpha + (\eta\sigma)^\alpha$$

and

$$\begin{aligned} M_1 =& (\epsilon_2\epsilon_3\mu^\alpha)^2 + (\Lambda\beta\gamma\sigma)^{2\alpha} + (\epsilon_1\epsilon_2\epsilon_3\sigma^\alpha)^2 - 2(\sigma\mu)^\alpha\epsilon_1\epsilon_2^2\epsilon_3^2 \\ &+ 4\mathcal{R}_0(\sigma\mu)^\alpha\epsilon_1\epsilon_2^2\epsilon_3^2 - 2(\Lambda\beta\sigma\gamma\mu)^\alpha\epsilon_2\epsilon_3 - 2(\beta\sigma^2\gamma\Lambda)^\alpha\epsilon_1, \\ M_2 =& (\beta\sigma\gamma\Lambda)^\alpha - \sigma^\alpha\epsilon_1\epsilon_2\epsilon_3 - \mu^\alpha\epsilon_2\epsilon_3, \quad M_3 = 2(\beta\sigma)^\alpha\epsilon_2\epsilon_3\epsilon_2\epsilon_3. \end{aligned} \tag{2}$$

5 Stability Analysis

In this section, our objective is to investigate the Local Asymptotic Stability (LAS) and to study also the Global Asymptotic Stability (GAS) of the Equilibrium points (DFE) X^0 and (DEE) X^1 for the model (1).

5.1 Local stability of the equilibrium points

By the use of the basic reproduction number $\mathcal{R}_0$ [12], we can present the theorem that establishes the local stability of the Disease-Free Equilibrium (DFE), which is defined as follows.

Theorem 5.1 *The disease-free equilibrium X^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix at the DFE point X^0 is given by

$$J(X^0) = \begin{pmatrix} -\epsilon_1 & 0 & -\dfrac{\mathcal{R}_0\mu^\alpha\epsilon_2\epsilon_3}{\epsilon_5\gamma^\alpha} & 0 & 0 & 0 \\ 0 & -\epsilon_2 & \dfrac{\epsilon_2\epsilon_3\mathcal{R}_0}{\gamma^\alpha} & 0 & 0 & 0 \\ 0 & \gamma^\alpha & -\epsilon_3 & 0 & 0 & 0 \\ 0 & 0 & \delta^\alpha & -\epsilon_4 & 0 & 0 \\ 0 & 0 & 0 & (1-\kappa^\alpha)\lambda^\alpha & -\mu^\alpha & 0 \\ \eta^\alpha & 0 & -\dfrac{\mathcal{R}_0(\sigma\eta)^\alpha\epsilon_2\epsilon_3}{\gamma^\alpha\epsilon_5} & 0 & 0 & -\mu^\alpha \end{pmatrix}.$$

The roots of the characteristic equation are defined as follows:

$$\begin{aligned} s_1 &= -\epsilon_1 < 0, \quad s_2 = -\epsilon_4 < 0, \quad s_3 = -\mu^\alpha < 0, \quad s_4 = -\mu^\alpha < 0, \\ s_5 &= -\frac{1}{2}\left(\epsilon_2 + \epsilon_3 + \sqrt{(\epsilon_2-\epsilon_3)^2 + 4\mathcal{R}_0\epsilon_2\epsilon_3}\right) < 0, \\ s_6 &= -\frac{1}{2}\left(\epsilon_2 + \epsilon_3 - \sqrt{(\epsilon_2-\epsilon_3)^2 + 4\mathcal{R}_0\epsilon_2\epsilon_3}\right) < 0 \quad \text{iff} \quad \mathcal{R}_0 < 1. \end{aligned}$$

The disease-free equilibrium X^0 is locally asymptotically stable or unstable according to $\mathcal{R}_0 < 1$ or $\mathcal{R}_0 > 1$.

5.2 Global stability of the equilibrium points

The theorems that establish the global stability of the Equilibrium points (DFE) and (DEE) are formulated as follows.

Theorem 5.2 *The disease-free equilibrium X^0 is global asymptotically stable in* $\Omega = \{(S,E,I,H,R,D,V) \in \mathbb{R}^7_+ / S+E+I+H+R+D+V \leq N, \quad S(0) > 0, E(0) > 0, I(0) > 0, H(0) > 0, R(0) > 0, D(0) > 0, V(0) > 0\}$ *if* $\mathcal{R}_0 \leq 1$*, otherwise unstable.*

Proof. Following [10], let us consider the Lyapunov function

$$\mathcal{V}(E,I) \quad = \quad E + \frac{\gamma+\mu}{\gamma} I.$$

The conformal derivative of the Lyapunov function, denoted by $\mathbf{T}^\alpha \mathcal{V}(E,I)$, is given by

$$\begin{aligned}
\mathbf{T}^\alpha \mathcal{V}(E,I) &= \mathbf{T}^\alpha E(t) + \mathbf{T}^\alpha I(t) \left(\frac{\epsilon_2}{\gamma^\alpha} \right), \\
&= \beta^\alpha S I + (\sigma\beta)^\alpha V I - \epsilon_2 E + \frac{\epsilon_2}{\gamma^\alpha} (\gamma^\alpha E - \epsilon_3 I), \\
&= \beta^\alpha (S + \sigma^\alpha V) - \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha}, \\
&= \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha} \left[\frac{\epsilon_1 \mu^\alpha \mathcal{R}_0 (S + \sigma^\alpha V)}{\Lambda^\alpha \epsilon_5} - 1 \right] I \quad \leq \quad \frac{\epsilon_2 \epsilon_3}{\gamma^\alpha} (\mathcal{R}_0 - 1) I.
\end{aligned}$$

Thus, $\mathbf{T}^\alpha \mathcal{V}(E,I) < 0$ if $\mathcal{R}_0 < 1$, and $\mathbf{T}^\alpha \mathcal{V}(E,I) = 0$ if $I(t) = 0$ and $\mathcal{R}_0 = 1$. Therefore, the largest invariant set contained in this set is

$$L = \{(S,E,I,H,R,D,V) \in \Omega / \mathbf{T}^\alpha \mathcal{V}(E,I) = 0\},$$

which is reduced to DFE. Hence, by LaSalle's Invariance Principle, it follows that the disease-free equilibrium point is Globally Asymptotically Stable in Ω whenever $\mathcal{R}_0 < 1$.

Theorem 5.3 *If $\mathcal{R}_0 > 1$, then the endemic equilibrium of the model (1) given by X^1 is globally asymptotically stable in Ω.*

Proof. The non-linear Lyapunov function of the Goh-Volterra form [8] is as follows:

$$\begin{aligned}
\mathcal{V}(S,E,I,H,V) &= \big(S - S^1 ln(S)\big) + \big(E - E^1 ln(E)\big) + \frac{\beta^\alpha S^1 I^1}{\gamma^\alpha E^1} \big(I - I^1 ln(I)\big) \\
&\quad + \frac{\beta^\alpha S^1}{\delta^\alpha} \big(H - H^1 ln(H)\big) + \big(V - V^1 ln(V)\big),
\end{aligned}$$

the conformable derivative of the Lyapunov function, $\mathbf{T}^\alpha \mathcal{V}(S,E,I,H,V)$, is expressed as

$$\begin{aligned}
\mathbf{T}^\alpha \mathcal{V}(S,E,I,H,V) = &\left(1 - \frac{S^1}{S}\right) \mathbf{T}^\alpha S(t) + \left(1 - \frac{E^1}{E}\right) \mathbf{T}^\alpha E(t) \\
&+ \frac{\beta^\alpha S^1 I^1}{\gamma^\alpha E^1} \left(1 - \frac{I^1}{I}\right) \mathbf{T}^\alpha I(t) \\
&+ \frac{\beta^\alpha S^1}{\delta^\alpha} \left(1 - \frac{H^1}{H}\right) \mathbf{T}^\alpha H(t) + \left(1 - \frac{V^1}{V}\right) \mathbf{T}^\alpha V(t).
\end{aligned}$$

Furthermore, we have

$$\mathbf{T}^\alpha S(t) \le \Lambda^\alpha - \beta^\alpha SI, \quad \mathbf{T}^\alpha E(t) \le \beta^\alpha SI + (\sigma\beta)^\alpha VI, \quad \mathbf{T}^\alpha I(t) \le \gamma^\alpha E$$

$$\mathbf{T}^\alpha H(t) \le \delta^\alpha I, \quad \mathbf{T}^\alpha V(t) \le \eta^\alpha S - (\sigma\beta)^\alpha VI.$$

Therefore,

$$\begin{aligned}\mathbf{T}^\alpha \mathcal{V}(S,E,I,H,V) < {} & \Lambda^\alpha \left(1 - \frac{S^1}{S}\right) + \eta^\alpha S^1 \left(\frac{S}{S^1} - \frac{V^1 S}{V S^1}\right) \\ & + (\sigma\beta)^\alpha V^1 I^1 \left(\frac{I}{I^1} - \frac{E^1 V I}{E V^1 I^1}\right) \\ & + \beta^\alpha S^1 I^1 \left(\frac{E}{E^1} + 2\frac{I}{I^1} - \frac{E^1 S I}{E S^1 I^1} - \frac{I^1 E}{I E^1} - \frac{H^1 I}{H I^1}\right) \\ < {} & 0\end{aligned}$$

so that the following inequalities hold:

$$1 - \frac{S^1}{S} \le 0, \quad \frac{S}{S^1} - \frac{V^1 S}{V S^1} \le 0, \quad \frac{I}{I^1} - \frac{E^1 V I}{E V^1 I^1} \le 0$$

and

$$\frac{E}{E^1} + 2\frac{I}{I^1} - \frac{E^1 S I}{E S^1 I^1} - \frac{I^1 E}{I E^1} - \frac{H^1 I}{H I^1} \le 0.$$

Thus $\mathbf{T}^\alpha \mathcal{V}(t) \le 0$ for $\mathcal{R}_0 > 1$. The point X^1 is globally asymptotically stable if $\mathcal{R}_0 > 1$.

6 Numerical Results and Discussion

In this section, we present a numerical study that simulates the SEIHRDV model with a conformable fractional derivative applied to the spread of COVID-19 in Algeria. The parameters are determined using real-time data on COVID-19 cases in Algeria provided by the World Health Organization (WHO) and the Algerian Ministry of Health. The primary objective of this simulation is to assess the impact of the vaccination campaign on disease transmission. We employ the Runge-Kutta 4th order (RK4) method in MATLAB for this study.

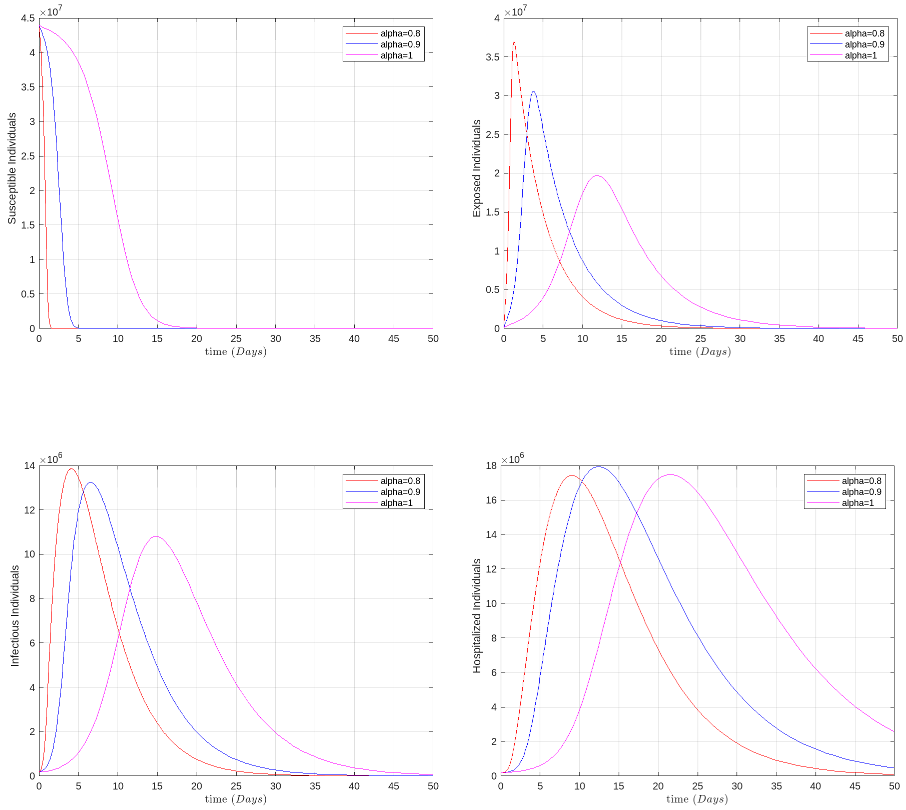
The baseline parameters and the initial values for our proposed model are shown in Tables 1 and 2.

Parameter	Interpretation	Initial Value	Reference
μ	Natural birth and death rate	3×10^{-5} person/day	Estimated
Λ	New births	1350 person/day	[14]
γ^{-1}	Incubation period	5.5 days	[4]
δ^{-1}	Infection time	3.8 days	[4]
λ^{-1}	Recovery time	10 days	[4]
ρ^{-1}	Time until death	15 days	[4]
η	Vaccination rate (rate of people who are vaccinated)	4×10^{-4} day^{-1}	Estimated
σ	Vaccine inefficacy	0.05 day^{-1}	[11]
β	Transmission rate divided by N	5.7×10^{-8} day^{-1}	Estimated
κ	Case fatality rate	0.028	[15]

Table 1: Initial model parameters.

Variable	N	I_0	E_0	H_0	R_0	D_0	V_0	S_0
Initial Value	44,600,000	202,122	150,000	180,000	138,362	5,739	0	44,457,156
Reference	[13]	[15]	Estimated	Estimated	[15]	[15]	[15]	Estimated

Table 2: The initial values for the model variables.



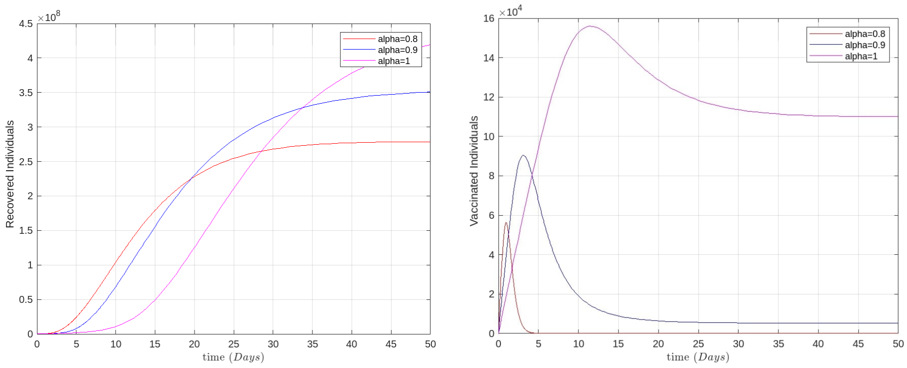


Figure 2: Plots of S(t), E(t), I(t), H(t), R(t) and V(t) for different values of $\alpha = 0.6, 0.8, 1.0$ with respect to time (days).

The graphs highlight the dynamics of disease spread and the impact of the α parameter. Over time, the number of susceptible individuals decreases as more people become exposed and infected, with lower α values resulting in fewer susceptibles. Exposed individuals peak and decline more quickly with lower α values, showing a faster transition to the infectious stage. Infectious individuals exhibit a distinct peak and subsequent decline, with lower α values leading to an earlier and higher peak, indicating a more rapid and intense spread. Hospitalized individuals peak and then decline, with lower α values causing a later and lower peak, suggesting a reduced burden on the healthcare system. The number of recovered individuals increases over time, with the lowest recovery rates at $\alpha = 1$. Vaccinated individuals rise rapidly at first and then stabilize, with lower α values leading to faster vaccination rates but reaching this stabilization sooner.

7 Conclusion

In this paper, we analyze fractional order derivatives utilizing the confomable derivatives with an order of $0 < \alpha \leq 1$ within the SEIHRDV nonlinear epidemiological models with vaccination. To this end, COVID-19 case data from Algeria at the beginning of vaccine implementation (September, 2021) are used. To prove the positivity of the system's solutions, we introduce the Sumudu transform (an integral technique for solving differential systems) as a new innovative approach. After that, we establish the local stability of the equilibrium points by calculating the Jacobian matrix and proving that its eigenvalues are negative. To demonstrate the global stability of the equilibrium points, we construct a Lyapunov function. Our study shows that a lower value of α is associated with a more effective response to the disease. However, it is important to note that α is not the only factor that determines the effectiveness of a response. Other factors such as the effectiveness of the vaccine and the availability of healthcare resources, also play a role in the analysis of Covid-19 epidemiological models.

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μ-S^p-Pseudo Almost Automorphic Solutions for Multidimensional Systems of Nonlinear Delay Integral Equations

A. El Haddouchi [1] and A. Sadrati [2*]

[1] *LMPA Laboratory, M2A Group, Department of Mathematics, FST, Errachidia, University Moulay Ismaïl of Meknes, BP 509, Boutalamine, 52000, Errachidia, Morocco.*
[2] *LMPA Laboratory, MASD Group, Department of Mathematics, FST, Errachidia, University Moulay Ismaïl of Meknes, BP 509, Boutalamine, 52000, Errachidia, Morocco.*

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Abstract: In the present work, we use a topological method to establish the existence of positive μ-S^p-pseudo almost automorphic solutions for some systems of nonlinear delay integral equations which, from a biological point of view, model the evolution in time of interacting species. Specifically, we use the contraction mapping principle, Leray–Schauder alternative and Krasnoselskii's theorem to obtain our results.

Keywords: *dynamical systems; delay integral equations; μ-S^p-pseudo almost automorphy; compact operators; fixed point.*

Mathematics Subject Classification (2020): 45G15, 92D15, 92D25, 92D30.

1 Introduction

The notion of almost automorphy is a natural generalization of the classical almost periodicity [2, 3]. It was first introduced by Bochner in the mid-1960s; one can find more details about this topic in [7, 11, 13] and the references therein. Since then, this notion has been generalized in different directions. In [12], N'Guérékata and Pankov introduced the concept of Stepanov-like almost automorphy and applied this concept to investigate the existence and uniqueness of an almost automorphic solution to the autonomous semilinear equation. Then, new generalizations of the Stepanov-like automorphy have been discovered. Among the most important of these generalizations, we have the notion of Stepanov-like weighted pseudo almost automorphic function presented by Xia and Fan

* Corresponding author: mailto:abdo2sadrati@gmail.com

in [17] and that of weighted pseudo almost automorphic functions in the light of the measure theory introduced by Blot et al. in [1].

Recently, the concept of μ-Stepanov-like pseudo almost automorphic functions was introduced and applied to study the existence of solutions to some evolution equations in Banach spaces (see, for instance, [5, 6]). Note that with the development of the theory of almost automorphy, its applications have attracted a great deal of attention of many mathematicians due to their significance and applications in physics, mathematical biology, control theory, and so on.

However, since the concept of μ-Stepanov-like pseudo almost automorphy is more general than weighted pseudo almost automorphy, our goal in this paper is to investigate the existence of Stepanov-like μ-pseudo almost automorphic solutions to the following multidimensional systems of nonlinear delay integral equations:

$$x(s) = \mathcal{F}(s, x(s-\ell)) + \int_0^{\Gamma(s)} f(s, \sigma, x(s-\sigma-\ell))d\sigma, \tag{1}$$

where $x = (x_1, ..., x_n) : \mathbb{R} \longrightarrow \mathbb{R}_+^n$, $\mathcal{F} = (\mathcal{F}_1, ..., \mathcal{F}_n) : \mathbb{R} \longrightarrow \mathbb{R}_+^n$, $\Gamma = (\Gamma_1, ..., \Gamma_n) : \mathbb{R} \longrightarrow \mathbb{R}_+^n$ and $f = (f_1, ..., f_n) : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$ are appropriate functions specified later. Hence, system (1) means that, for each $i \in \{1, 2, ..., n\}$,

$$x_i(s) = \mathcal{F}_i(s, x_1(s-\ell), ..., x_n(s-\ell)) + \int_0^{\Gamma_i(s)} f_i(s, \sigma, x_1(s-\sigma-\ell), ..., x_n(s-\sigma-\ell))d\sigma.$$

System (1) with positive solution describes the evolution in time of n species with interaction. In fact, for $i = 1, 2, ..., n$, $x_i(t)$ is the number of individuals present in the population x_i at time t, which live to the age $\Gamma_i(t)$. The function f_i is the number of new births per time unit in x_i and $\mathcal{F}_i$ is a nonlinear perturbation. Another explanation can be given to (1) by considering it as an epidemic model combined with population ecology. In this context, $x_i(t)$ is the population at time t of infectious individuals in the species x_i, f_i represents the instantaneous rate of infection in the species x and $\Gamma_i(t)$ is the duration of infectivity in x_i.

Note that, in the case $n = 2$, system (1) generalizes the one studied in [15] if one does the change of variable $s - \sigma = u$ and $\ell = 0$,

$$\begin{aligned} x(t) &= \gamma_1(t)x(t-\beta_1) + \int_{t-\sigma_1(t)}^{t} f(s, x(s), y(s))ds, \\ y(t) &= \gamma_2(t)y(t-\beta_2) + \int_{t-\sigma_2(t)}^{t} g(s, x(s), y(s))ds. \end{aligned} \tag{2}$$

In [14], the authors show the existence of positive Stepanov-like almost automorphic solutions for multidimensional systems of the type

$$x(s) = \int_0^{\tau(s)} f(s, \sigma, x(s-\sigma-l))d\sigma.$$

Therefore, our system (1), in the present work, generalizes the previous and other systems.

The paper is organized as follows. Section 2 is devoted to basic definitions and results that are known in the literature. In Section 3, we present some results which are essential in the proof of our results, namely, composition theorems. Finaly, in Section 4, we prove the existence of solutions.

2 Preliminaries

Troughout this paper, we will use the following notations. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{N}$ is the set of nonnegative integers and for every element $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $\|x\| = \sum_{i=1}^{n} |x_i|$. Also, $C(\mathbb{R}, \mathbb{R}^n)$ denotes the space of all continuous functions from $\mathbb{R}$ into $\mathbb{R}^n$ and $BC(\mathbb{R}, \mathbb{R}^n)$ consists of the bounded ones in $C(\mathbb{R}, \mathbb{R}^n)$. Equipped with the norm sup $\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|$, $BC(\mathbb{R}, \mathbb{R}^n)$ is a Banach space. $L^p_{Loc}(\mathbb{R}, \mathbb{R}^n)$ denotes the space of all equivalence classes of measurable functions $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, \mathbb{R}^n)$.

Let us begin by giving some basic definitions and results on almost automorphic and μ-pseudo almost automorphic functions.

Definition 2.1 [2]

(i) A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is called almost automorphic if for every sequence of real numbers $\{s'_n\}$, there exists a subsequence $\{s_n\}$ such that

$$f^*(t) = \lim_{n\to+\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to+\infty} f^*(t - s_n), \ \forall t \in \mathbb{R}.$$

The collection of all such functions will be denoted by $AA(\mathbb{R}, \mathbb{R}^n)$.

(ii) A continuous function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$ is said to be almost automorphic if $f(s, \sigma, u)$ is almost automorphic in $s \in \mathbb{R}$ uniformly for all $(\sigma, u) \in K$, where K is any bounded subset of $\mathbb{R}_+ \times \mathbb{R}^n_+$. The collection of all such functions will be denoted by $AA(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n)$.

Example 2.1 The function

$$t \longrightarrow \cos \frac{1}{2 - \sin t - \sin \pi t}$$

is almost automorphic but not almost periodic since it is not uniformly continuous.

Throughout this work, we denote by $\mathfrak{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathfrak{M}$ the set of all positive measures μ on $\mathfrak{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < +\infty$ for all $a, b \in \mathbb{R}$ with $a < b$.

Definition 2.2 [1] Let $\mu \in \mathfrak{M}$.

(i) A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be μ-ergodic if

$$\lim_{r\to+\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \|f(s)\| d\mu(s) = 0.$$

We denote the space of all such functions by $\varepsilon(\mathbb{R}, \mathbb{R}^n, \mu)$ (or $\varepsilon(\mathbb{R}^n, \mu)$ for short).

(ii) A bounded continuous function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \longrightarrow \mathbb{R}^n$ is said to be μ-ergodic if $f(., \sigma, x)$ is bounded for each $(\sigma, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ and

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])} \int_{[-r,r]} \|f(s,\sigma,x)\| d\mu(s) = 0.$$

We denote the space of all such functions by $\varepsilon\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n, \mu\right)$ (or $\varepsilon\left(\mathbb{R}_+^{n+1}, \mathbb{R}^n, \mu\right)$ for short).

Lemma 2.1 *[1] Let $\mu \in \mathfrak{M}$. Then $(\varepsilon(\mathbb{R}^n, \mu), \|.\|_\infty)$ is a Banach space.*

Definition 2.3 [1] Let $\mu \in \mathfrak{M}$. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be μ-pseudo almost automorphic if f can be expressed as

$$f = g + h,$$

where $g \in AA(\mathbb{R}, \mathbb{R}^n)$ and $h \in \varepsilon(\mathbb{R}^n, \mu)$. We denote the space of all such functions by $PAA(\mathbb{R}, \mathbb{R}^n, \mu)$.

For $\mu \in \mathfrak{M}$, denote by μ_τ the positive measure on $(\mathbb{R}, \mathfrak{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}), for A \in \mathfrak{B}.$$

In the sequel, we assume the following hypothesis.

(H_0) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu_\tau(A) \leq \beta\mu(A),$$

where any $A \in \mathfrak{B}$ satisfies $A \cap I = \phi$.

Lemma 2.2 *[1] Let $\mu \in \mathfrak{M}$ satisfy (H_0). Then $\varepsilon(\mathbb{R}^n, \mu)$ is translation invariant, and therefore, $PAA(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant.*

Lemma 2.3 *[1] Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant. Then the decomposition of μ-pseudo almost automorphic function in the form $f = g + h$, where $g \in AA(\mathbb{R}, \mathbb{R}^n)$ and $h \in \varepsilon(\mathbb{R}^n, \mu)$, is unique.*

Lemma 2.4 *[1] Let $\mu \in \mathfrak{M}$. Assume that $PAA(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant. Then $(PAA(\mathbb{R}, \mathbb{R}^n, \mu), \|.\|_\infty)$ is a Banach space.*

Now, we give some preliminaries on μ-Stepanov-like pseudo almost automorphic functions.

Definition 2.4 [8,12] The Bochner transform $f^b(t, s), t \in \mathbb{R}, s \in [0, 1]$, of a function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is defined by

$$f^b(t, s) := f(t + s).$$

Remark 2.1 [8] Note that a function $\varphi(t, s), t \in \mathbb{R}, s \in [0, 1]$, is the Bochner transform of a certain function $f(t)$,

$$\varphi(t, s) = f^b(t, s)$$

if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}, s \in [0, 1]$ and $\tau \in [s - 1, s]$.

Definition 2.5 [8] The Bochner transform $f^b(t,s,\sigma,u), t \in \mathbb{R}, s \in [0,1], (\sigma,u) \in \mathbb{R} \times \mathbb{R}^n$, of a function $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by

$$f^b(t,s,\sigma,u) := f(t+s,\sigma,u).$$

Definition 2.6 [12] Let $p \in [1,+\infty)$.

i) The space $BS^p(\mathbb{R},\mathbb{R}^n)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{R}^n$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1],\mathbb{R}^n))$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p ds \right)^{\frac{1}{p}}.$$

ii) The space $BS^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n\right)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}^n$ such that

$$f^b(.,.,\sigma,u) \in L^\infty(\mathbb{R}, L^p([0,1],\mathbb{R}^n)), t \to f^b(t,.,\sigma,u) \in L^p([0,1],\mathbb{R}^n),$$

for each $t \in \mathbb{R}$ and each $(\sigma,u) \in \mathbb{R}_+ \times \mathbb{R}^n_+$.

Definition 2.7 [12] The space $AS^p(\mathbb{R},\mathbb{R}^n)$ of Stepanov-like almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(\mathbb{R},\mathbb{R}^n)$ such that $f^b \in AA(\mathbb{R}, L^p([0,1],\mathbb{R}^n))$.

In other words, a function $f \in L^p_{\text{loc}}(\mathbb{R},\mathbb{R}^n)$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p([0,1],\mathbb{R}^n)$ is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}$, there exist a subsequence $\{s_n\}$ and a function $f^* \in L^p_{\text{loc}}(\mathbb{R},\mathbb{R}^n)$ such that

$$\begin{aligned} &\Big(\int_t^{t+1} \Big\| f(s+s_n) - f^*(s) \Big\|^p ds \Big)^{1/p} \to 0, \\ &\Big(\int_t^{t+1} \Big\| f^*(s-s_n) - f(s) \Big\|^p ds \Big)^{1/p} \to 0 \end{aligned} \tag{3}$$

as $n \to +\infty$ pointwise on $\mathbb{R}$.

Example 2.2 [12] Let $\{f_n\} \subset \mathbb{R}$ be an almost automorphic sequence, and $\varepsilon \in (0, \frac{1}{2})$. Let

$$f(t) = \begin{cases} f_n \text{ if } t \in (n-\varepsilon, n+\varepsilon); \\ 0 \text{ otherwise.} \end{cases}$$

Then $f \in AS^p(\mathbb{R},\mathbb{R})$ for all $p \in [1,+\infty)$ but $f \notin AA(\mathbb{R},\mathbb{R})$.

Lemma 2.5 *[12]*

(i) $(AS^p(\mathbb{R},\mathbb{R}^n), \|.\|_{S^p})$ *is a Banach space.*

(ii) $AA(\mathbb{R},\mathbb{R}^n)$ *is continuously embeded in* $AS^p(\mathbb{R},\mathbb{R}^n)$.

Remark 2.2 (1) The operator $J : AS^p(\mathbb{R},\mathbb{R}^n) \longrightarrow AS^p(\mathbb{R},\mathbb{R}^n)$ such that $(Jx)(s) := x(-s)$ is well defined and linear. Moreover, it is an isometry and $J^2 = I$.

(2) the operator τ_β defined by $(\tau_\beta x)(s) := x(s-\beta)$ for a fixed $\beta \in \mathbb{R}$ leaves $AS^p(\mathbb{R}, \mathbb{R}^n)$ invariant.

Definition 2.8 [12] A function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \longrightarrow \mathbb{R}^n$, $(s, \sigma, u) \longrightarrow f(s, \sigma, u)$ with $f(., \sigma, u) \in L^p_{Loc}(\mathbb{R}, \mathbb{R}^n)$ for each $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is said to be Stepanov-like almost automorphic in $s \in \mathbb{R}$ uniformly for $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ if $s \longrightarrow f(s, \sigma, u)$ is Stepanov-like almost automorphic for each $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$. That is, for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ and a function $f^* : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \longrightarrow \mathbb{R}^n$ with $f^*(., \sigma, u) \in L^p_{Loc}(\mathbb{R}, \mathbb{R}^n)$ such that

$$\left(\int_t^{t+1} \|f(s + s_n, \sigma, u) - f^*(s, \sigma, u)\|^p ds\right)^{\frac{1}{p}} \longrightarrow 0,$$
$$\left(\int_t^{t+1} \|f^*(s - s_n, \sigma, u) - f(s, \sigma, u)\|^p ds\right)^{\frac{1}{p}} \longrightarrow 0$$

as $n \longrightarrow +\infty$ for all $t \in \mathbb{R}$ and $(\sigma, u) \in \mathbb{R}_+ \times \mathbb{R}_+^n$.

Denote by $AS^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n\right)$ the set of all such functions.

Definition 2.9 [6] Let $\mu \in \mathfrak{M}$. A function $f \in BS^p(\mathbb{R}, \mathbb{R}^n)$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p(\mathbb{R}, \mathbb{R}^n)$ and $h^b \in \varepsilon\left(L^p([0,1], \mathbb{R}^n), \mu\right)$. In other words, a function $f \in L^p_{loc}(\mathbb{R}, \mathbb{R})$ is said to be μ-Stepanov-like pseudo almost automorphic relatively to the measure μ if its Bochner transform $f^b : \mathbb{R} \longrightarrow L^p([0,1], \mathbb{R}^n)$ is μ-pseudo almost automorphic in the sense that there exist two functions $g, h : \mathbb{R} \longrightarrow \mathbb{R}^n$ such that $f = g + h$, where $g \in AS^p(\mathbb{R}, \mathbb{R}^n)$ and $h^b \in \varepsilon\left(L^p([0,1], \mathbb{R}^n), \mu\right)$, that is, $h^b \in BC\left(\mathbb{R}, L^p([0,1], \mathbb{R}^n)\right)$ and

$$\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r,r]} \left[\int_t^{t+1} \|h(s)\|^p ds\right]^{\frac{1}{p}} d\mu(t) = 0.$$

The set of all such functions will be denoted by $PAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$.

Definition 2.10 [6] Let $\mu \in \mathfrak{M}$. A function $f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n, (s, \sigma, x) \to f(s, \sigma, x)$ with $f(., \sigma, x) \in L^p_{Loc}(\mathbb{R}, \mathbb{R}^n)$ for each $(\sigma, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is said to be μ-Stepanov-like pseudo almost automorphic (or μ-S^p-pseudo almost automorphic) if it can be expressed as $f = g + h$, where $g \in AS^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n\right)$ and $h^b \in \varepsilon\left(\mathbb{R}_+^{n+1}, L^p([0,1], \mathbb{R}^n), \mu\right)$. We denote by $PAA^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n, \mu\right)$ the set of all such functions.

Theorem 2.1 *[6] Let $\mu \in \mathfrak{M}$ and I be a bounded interval (eventually $I = \varnothing$). Assume that $f(.) \in BS^p(\mathbb{R}, \mathbb{R}^n)$. Then the following assertions are equivalent:*

(i) $f^b(.) \in \varepsilon\left(L^p([0,1], \mathbb{R}^n), \mu\right)$.

(ii) $\lim_{r \to +\infty} \dfrac{1}{\mu([-r, r] \setminus I)} \int_{[-r,r]\setminus I} \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} d\mu(t) = 0.$

(iii) For any $\varepsilon > 0$, $\lim_{r \to +\infty} \dfrac{\mu\left(\{t \in [-r, r] \setminus I : \left(\int_t^{t+1} \|f(s)\|^p ds\right)^{\frac{1}{p}} > \varepsilon\}\right)}{\mu([-r, r] \setminus I)} = 0.$

Theorem 2.2 *[6] Let $\mu \in \mathfrak{M}$ satisfy (H_0). Then $\varepsilon\left(L^p([0,1],\mathbb{R}^n),\mu\right)$ is translation invariant, therefore $PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ is also translation invariant.*

Theorem 2.3 *[6] Let $\mu \in \mathfrak{M}$ satisfy (H_0) and $f \in PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ be such that $f = g + h$, where $g \in AS^p\left(\mathbb{R},\mathbb{R}^n\right)$ and $h^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n),\mu\right)$. If $PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ is translation invariant, then*

$$\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}} \quad (\textit{the closure of range } f).$$

Theorem 2.4 *[6] Let $\mu \in \mathfrak{M}$. Assume that $PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ is translation invariant. Then the decomposition of the μ-S^p-pseudo almost automorphic function in the form $f = g + h$, where $g \in AS^p(\mathbb{R},\mathbb{R}^n)$ and $h^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n),\mu\right)$, is unique.*

Theorem 2.5 *[6] Let $\mu \in \mathfrak{M}$. Assume that $PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ is translation invariant. Then $\left(PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right), \|.\|_{S^p}\right)$ is a Banach space.*

Recall the following theorem, whose proof and more details can be found in [6].

Theorem 2.6 *[6] Let $\mu \in \mathfrak{M}$. Suppose that $f = g + h \in PAA^p\left(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n,\mu\right)$ with $g \in AS^p\left(\mathbb{R}\times\mathbb{R}^n_+,\mathbb{R}^n\right)$, $h^b \in \varepsilon\left(\mathbb{R}^n_+, L^p([0,1],\mathbb{R}^n),\mu\right)$ and the following hypothesis holds:*

- *There exists a constant $L > 0$ such that for all $u, v \in \mathbb{R}^n_+$ and $s \in \mathbb{R}$,*

$$\|f(s,u) - f(s,v)\| \leq L\|u - v\|.$$

If $x = \alpha + \psi \in PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$ with $\alpha \in AS^p\left(\mathbb{R},\mathbb{R}^n\right)$, $\psi^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n),\mu\right)$ and $\overline{\{\alpha(s), s \in \mathbb{R}\}}$ is compact, then $f(.,x(.)) \in PAA^p\left(\mathbb{R},\mathbb{R}^n,\mu\right)$.

To establish the existence of solutions for system (1), we will apply the fixed point theorem of Krasnoselskii-Schaefer type, where the Schauder-type condition is substituted by the Schaefer-type condition.

Theorem 2.7 (Krasnoselskii-Schauder [16]) *Let Ω be a nonempty bounded closed convex subset of a Banach space $(X,\|.\|)$. Suppose C, $T : \Omega \longrightarrow X$ are two mappings satsfying*

(i) $Cx + Ty \in \Omega,\ \forall x, y \in \Omega$,

(ii) C is a contraction and

(iii) T is completely continuous.

Then the mapping $A = C + T$ has a fixed point $x \in \Omega$, that is, $Ax = Cx + Tx = x$.

Proposition 2.1 *[4] If $(X,\|.\|)$ is a normed space, if $0 < \lambda < 1$, and if $C : X \to X$ is a contraction mapping with the contraction constant δ, then*

$$\lambda C\frac{1}{\lambda} : X \to X$$

is also a contraction mapping with the contraction constant δ, independent of λ; in particular,

$$\|\lambda C(x/\lambda)\| \leq \delta\|x\| + \|C(0)\|.$$

Theorem 2.8 (Krasnoselskii-Schaefer [4]) *Let $(X,\|.\|)$ be a Banach space and let $C,T : X \to X$ be such that C is a contraction with the contraction constant $\delta < 1$ and T being completely continuous. Then either*

(a) *$x = \lambda C(x/\lambda) + \lambda Tx$ has a solution in X for $\lambda = 1$, or*

(b) *The set $\{x \in X : x = \lambda C(x/\lambda) + \lambda Tx, \lambda \in (0,1)\}$ is unbounded.*

3 Composition Theorems

In this section, we will prove two composition theorems, which will allow us to study, in Section 4, the existence of μ-Stepanov-like pseudo almost automorphic solutions for systems of type (1), as well as for the scalar case of system (2). Let us begin by introducing some notations.

Let $\mu \in \mathfrak{M}$. Assume that $PAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant. Consider the set of all bounded functions $BAS^p(\mathbb{R}, \mathbb{R}^n) \subset AS^p(\mathbb{R}, \mathbb{R}^n)$ (resp. $BPAA^p(\mathbb{R}, \mathbb{R}^n, \mu) \subset PAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$), that is, for each $x \in BAS^p(\mathbb{R}, \mathbb{R}^n)$ (resp. $x \in BPAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$), we have $\|x\|_\infty = \sup_{s\in\mathbb{R}} \|x(s)\| < \infty$. It is clear that $(BAS^p(\mathbb{R}, \mathbb{R}^n), \|.\|_{S^p})$ (resp. $(BPAA^p(\mathbb{R}, \mathbb{R}^n, \mu), \|.\|_{S^p})$) is a Banach space. Also, if $f = g + h \in BPAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$, then both g and h are bounded.

Consider $L^{p,1}_{Loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ being the space of all equivalence classes of measurable functions $\varphi : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n$, $(s, \sigma) \longrightarrow \varphi(s, \sigma)$ such that the restriction of φ to every bounded subset of $\mathbb{R} \times \mathbb{R}_+$ is in $L^{p,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n) = L^p\left(\mathbb{R}, L^1(\mathbb{R}_+, \mathbb{R}^n)\right)$. Then, let $AS^{p,1}\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n\right)$ be the subset of $AS^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n\right)$ consisting of all functions f such that $f(., ., u) \in L^{p,1}_{Loc}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ for all $u \in \mathbb{R}^n_+$ and let $\varepsilon\left(\mathbb{R}^{n+1}_+, L^{p,1}([0, 1] \times \mathbb{R}_+, \mathbb{R}^n), \mu\right)$ be the subset of $\varepsilon\left(\mathbb{R}^{n+1}_+, L^p([0, 1], \mathbb{R}^n), \mu\right)$ consisting of all function h satisfying $h(., ., u) \in L^{p,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^n)$ for all $u \in \mathbb{R}^n_+$.

Denote by $PAA^{p,1}\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n, \mu\right)$ the subset of $PAA^p\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n, \mu\right)$ consisting of all functions $f = g + h$ such that $g \in AS^{p,1}\left(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n\right)$ and $h^b \in \varepsilon\left(\mathbb{R}^{n+1}_+, L^{p,1}([0, 1] \times \mathbb{R}_+, \mathbb{R}^n), \mu\right)$.

Next, as general assumptions, we make the following ones for functions $\Phi : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$:

(H_1) For each compact subset $K \subset \mathbb{R}^n_+$, there exist constants $L_K(\Phi), M_K(\Phi) > 0$ such that

(i)
$$\|\Phi(s, \sigma_1, u) - \Phi(s, \sigma_2, v)\| \leq L_K(\Phi) \left(|\sigma_1 - \sigma_2| + \|u - v\|\right)$$
for all $u, v \in K$, all $\sigma_1, \sigma_2 \in \mathbb{R}_+$ and all $s \in \mathbb{R}$.

(ii)
$$\|\Phi(s, \sigma, u)\| \leq M_K(\Phi)\|u\|,$$
for all $(s, \sigma) \in \mathbb{R} \times \mathbb{R}_+$ and all $u \in K$.

In the sequel, we say that a function f satisfies (H_1) means that (H_1) holds for f instead of Φ. Then we prove the following composition theorem relevant to our system (1).

Theorem 3.1 *Let $\mu \in \mathfrak{M}$ be such that $BPAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$ is translation invariant. Assume that*

(i) $f = g + h \in PAA^{p,1}\left(\mathbb{R} \times \mathbb{R}^{n+1}_+, \mathbb{R}^n_+, \mu\right)$ *with* $g \in AS^{p,1}\left(\mathbb{R} \times \mathbb{R}^{n+1}_+, \mathbb{R}^n_+\right)$ *and* $h^b \in \varepsilon\left(\mathbb{R}^{n+1}_+, L^{p,1}([0, 1] \times \mathbb{R}_+, \mathbb{R}^n_+), \mu\right)$ *such that f and g satisfy (H_1).*

(ii) $\Gamma = \tau + \phi$, $x = \alpha + \psi \in BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$ *with* $\tau, \alpha \in BAS^p\left(\mathbb{R}, \mathbb{R}^n_+\right)$ *and* $\phi^b, \psi^b \in \varepsilon\left(L^p([0, 1], \mathbb{R}^n_+), \mu\right)$.

Then the function $Tx : \mathbb{R} \longrightarrow \mathbb{R}^n$ defined by

$$Tx(s) = \int_0^{\Gamma(s)} f(s, \sigma, x(s - \sigma - \ell)d\sigma$$

belongs to $BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$.

Proof. First, since $x, \Gamma \in BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$ and f satisfies $(H_1)(ii)$, we get that $Tx(.)$ is a bouned function. Then, taking into account all assumptions of the theorem, it is easily verified that

$$\begin{aligned}
Tx(s) \quad &= \quad \int_0^{\Gamma(s)} f(s, \sigma, x(s - \sigma - \ell)d\sigma \\
&= \quad \int_0^{\tau(s)} g(s, \sigma, \alpha(s - \sigma - \ell)d\sigma \\
&\quad + \int_0^{\tau(s)} [f(s, \sigma, \alpha(s - \sigma - \ell) - g(s, \sigma, \alpha(s - \sigma - \ell)]d\sigma \\
&\quad + \int_0^{\tau(s)} [f(s, \sigma, x(s - \sigma - \ell) - f(s, \sigma, \alpha(s - \sigma - \ell)]d\sigma \\
&\quad + \int_{\tau(s)}^{\Gamma(s)} f(s, \sigma, x(s - \sigma - \ell)d\sigma \\
&= \quad G(s) + H(s) + I(s) + J(s),
\end{aligned}$$

where

$$\begin{aligned}
G(s) &= \int_0^{\tau(s)} g(s, \sigma, \alpha(s - \sigma - \ell)d\sigma, \\
H(s) &= \int_0^{\tau(s)} [f(s, \sigma, \alpha(s - \sigma - \ell) - g(s, \sigma, \alpha(s - \sigma - \ell)]d\sigma \\
&= \int_0^{\tau(s)} h(s, \sigma, \alpha(s - \sigma - \ell)d\sigma, \\
I(s) &= \int_0^{\tau(s)} [f(s, \sigma, x(s - \sigma - \ell) - f(s, \sigma, \alpha(s - \sigma - \ell)]d\sigma, \\
J(s) &= \int_{\tau(s)}^{\Gamma(s)} f(s, \sigma, x(s - \sigma - \ell)d\sigma.
\end{aligned}$$

In view of [14, Theorem 3.1], $G \in BAS^p\left(\mathbb{R}, \mathbb{R}^n_+\right)$. So, it is enough to show that $H^b, I^b, J^b \in \varepsilon\left(L^p([0,1], \mathbb{R}^n_+), \mu\right)$.

Let us prove that $H^b \in \varepsilon\left(L^p([0,1], \mathbb{R}^n_+), \mu\right)$. We first point out that since f and g satisfy (H_1), the function $h = f - g$ also satisfies (H_1). In addition, since $\alpha(s)$ is bounded, $K = \overline{\{\alpha(s) : s \in \mathbb{R}\}}$ is compact. So, for $\epsilon > 0$ (small enough), there exist a finite number of points $\alpha_k \in K$, $k = 1, 2, ..., m$, such that for any $u \in K$, we have $\|u - \alpha_k\| < \frac{\varepsilon}{8\|\tau\|_\infty L_K(h)} = \varepsilon_0$ for some $1 \le k \le m$. Let

$$\vartheta_k = \{s \in \mathbb{R} : \|\alpha(s) - \alpha_k\| < \varepsilon_0\}, \; k = 1, 2, ..., m.$$

It is clear that $\mathbb{R} = \cup_{k=1}^{m} \vartheta_k$. Let

$$E_1 = \vartheta_1,\ E_k = \vartheta_k \setminus \Big(\cup_{i=1}^{k-1} \vartheta_i \Big),\ k = 2, 3, ..., m.$$

Then we have

$$\mathbb{R} = \bigcup_{k=1}^{m} E_k \text{ and } E_i \cap E_j = \emptyset,\ i \neq j,\ i, j \in \{1, ..., m\}. \tag{4}$$

Define the step function $\overline{\alpha} : \mathbb{R} \to \mathbb{R}^n$ by $\overline{\alpha}(s) = \alpha_k,\ s \in E_k,\ k = 1, ..., m$. Then $\|\alpha(s) - \overline{\alpha}(s)\| < \varepsilon_0$. Hence, from $(H_1)(i)$ and (4), one can easily get

$$\begin{aligned}
&\Big(\sum_{k=1}^{m} \int_{E_k \cap [t,t+1]} \|h(s,\sigma,\alpha(s-\sigma-\ell)) - h(s,\sigma,\alpha_k)\|^p ds\Big)^{\frac{1}{p}} \\
&= \Big(\int_t^{t+1} \|h(s,\sigma,\alpha(s-\sigma-\ell)) - h(s,\sigma,\overline{\alpha}(s-\sigma-\ell))\|^p ds\Big)^{\frac{1}{p}} \\
&< \varepsilon_0 L_K(h) = \frac{\varepsilon}{8\|\tau\|_\infty}.
\end{aligned}$$

On the other hand, since $h^b \in \varepsilon\left(\mathbb{R}_+^{n+1}, L^{p,1}([0,1] \times \mathbb{R}_+, \mathbb{R}_+^n), \mu\right)$, there exists $r_0 > 0$ such that

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \Big[\int_t^{t+1} \Big\| \int_0^{\|\tau\|_\infty} h(s,\sigma,\alpha_k) d\sigma \Big\|^p ds \Big]^{\frac{1}{p}} d\mu(t) < \frac{\varepsilon}{8m} \tag{5}$$

for all $r > r_0$ and $1 \leq k \leq m$.

Now, using (4)-(5), we have for all $r > r_0$,

$$\begin{aligned}
&\Big[\int_t^{t+1} \Big\| \int_0^{\tau(s)} h(s,\sigma,\alpha(s-\sigma-\ell)) d\sigma \Big\|^p ds \Big]^{\frac{1}{p}} \\
&\leq \|\tau\|_\infty^{\frac{p-1}{p}} \Big[\int_t^{t+1} \int_0^{\|\tau\|_\infty} \|h(s,\sigma,\alpha(s-\sigma-\ell))\|^p d\sigma ds \Big]^{\frac{1}{p}} \\
&= \|\tau\|_\infty^{\frac{p-1}{p}} \Big[\sum_{k=1}^{m} \int_{E_k \cap [t,t+1]} \Big(\int_0^{\|\tau\|_\infty} \|h(s,\sigma,\alpha(s-\sigma-\ell))\|^p d\sigma \Big) ds \Big]^{\frac{1}{p}} \\
&\leq 2^{1+\frac{1}{p}} \|\tau\|_\infty^{\frac{p-1}{p}} \Big[\int_0^{\|\tau\|_\infty} \Big(\sum_{k=1}^{m} \int_{E_k \cap [t,t+1]} \|h(s,\sigma,\alpha(s-\sigma-\ell)) - h(s,\sigma,\alpha_k)\|^p ds \Big) d\sigma \Big]^{\frac{1}{p}} \\
&\quad + 2^{1+\frac{1}{p}} \|\tau\|_\infty^{\frac{p-1}{p}} \Big[\sum_{k=1}^{m} \int_{E_k \cap [t,t+1]} \Big(\int_0^{\|\tau\|_\infty} \|h(s,\sigma,\alpha_k)\|^p d\sigma \Big) ds \Big]^{\frac{1}{p}},
\end{aligned}$$

which proves that for all $r > r_0$,

$$\frac{1}{\mu([-r,r])} \int_{[-r,r]} \Big[\int_t^{t+1} \Big\| \int_0^{\tau(s)} h(s,\sigma,\alpha(s-\sigma-\ell)) d\sigma \Big\|^p ds \Big]^{\frac{1}{p}} d\mu(t) < 2^{1+\frac{1}{p}} \Big(\frac{\varepsilon}{8} + \frac{m^{\frac{1}{p}} \varepsilon}{8m} \Big) < \varepsilon.$$

This ensures the desired result.

Next, we prove that $I^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n_+),\mu\right)$. Set $\Lambda = \overline{\{x(s) : s \in \mathbb{R}\}}$, then

$$\begin{aligned}
&\Big(\int_t^{t+1} \|I(s)\|^p ds\Big)^{\frac{1}{p}} \\
&= \Big(\int_t^{t+1} \|\int_0^{\tau(s)} [f(s,\sigma,x(s-\sigma-\ell)) - f(s,\sigma,\alpha(s-\sigma-\ell))]d\sigma\|^p ds\Big)^{\frac{1}{p}} \\
&\le L_\Lambda(f)\|\tau\|_\infty^{\frac{p-1}{p}}\Big(\int_t^{t+1}\int_0^{\|\tau\|_\infty} \|x(s-\sigma-\ell)-\alpha(s-\sigma-\ell)\|^p d\sigma ds\Big)^{\frac{1}{p}} \\
&= L_\Lambda(f)\|\tau\|_\infty^{\frac{p-1}{p}}\Big(\int_0^{\|\tau\|_\infty}\int_t^{t+1} \|\psi(s-\sigma-\ell)\|^p ds d\sigma\Big)^{\frac{1}{p}}.
\end{aligned}$$

Since $\psi^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n),\mu\right)$, using Theorem 2.1 and the fact that $\varepsilon\left(L^p([0,1],\mathbb{R}^n_+),\mu\right)$ is translation invariant, we get

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \Big(\int_t^{t+1} \|I(s)\|^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0.$$

Finally, we show that $J^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n_+),\mu\right)$.

$$\begin{aligned}
\Big(\int_t^{t+1} \|J(s)\|^p ds\Big)^{\frac{1}{p}} &= \Big(\int_t^{t+1} \|\int_{\tau(s)}^{\Gamma(s)} [f(s,\sigma,x(s-\sigma-\ell))d\sigma\|^p ds\Big)^{\frac{1}{p}} \\
&\le M_\Lambda(f)\|x\|_\infty\Big(\int_t^{t+1} \|\Gamma(s)-\tau(s)\|^p ds\Big)^{\frac{1}{p}} \\
&= M_\Lambda(f)\|x\|_\infty\Big(\int_t^{t+1} \|\phi(s)\|^p ds\Big)^{\frac{1}{p}}.
\end{aligned}$$

Also, since $\phi^b \in \varepsilon\left(L^p([0,1],\mathbb{R}^n_+),\mu\right)$, we have

$$\lim_{r\to+\infty} \frac{1}{\mu([-r,r])}\int_{[-r,r]} \Big(\int_t^{t+1} \|J(s)\|^p ds\Big)^{\frac{1}{p}} d\mu(t) = 0.$$

This completes the proof.

Proposition 3.1 *Let $\mu \in \mathfrak{M}$ and $f \in C_u(\mathbb{R},\mathbb{R})$. Then the following statements hold:*

(i) $f \in AS^p(\mathbb{R},\mathbb{R})$ implies that $f \in AA(\mathbb{R},\mathbb{R})$;

(ii) $f^b \in \varepsilon(\mathbb{R}, L^p(0,1;\mathbb{R}),\mu)$ implies that $f \in \varepsilon(\mathbb{R},\mathbb{R},\mu)$;

(iii) $f \in PAA^p(\mathbb{R},\mathbb{R},\mu)$ implies that $f \in PAA(\mathbb{R},\mathbb{R},\mu)$.

Proof. The proof of the above proposition is analogous to that from [10, Proposition 3.3]. We omit all details here.

Now, we prove the composition theorem which corresponds to the system (2) in its scalar case.

Theorem 3.2 *Let $\mu \in \mathfrak{M}$ be such that $BPAA^p(\mathbb{R},\mathbb{R},\mu)$ is translation invariant.*

(1) *Assume that* $f, \Gamma \in BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$. *Then the function* F *defined by*

$$F(s) = \int_{s-\Gamma(s)}^{s} f(\theta)d\theta$$

is in $BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$.

(2) *If* $f, g \in BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$, *then the product* $f.g \in BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$.

Proof. **(1)** Since $f, \Gamma \in PAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$, we have by definition that $f = h + \varphi$ and $\Gamma = \tau + \xi$, where $h, \tau \in BAS^p(\mathbb{R}, \mathbb{R}_+)$ and $\varphi^b, \xi^b \in \varepsilon(\mathbf{R}, L^p(0,1; \mathbb{R}_+), \mu)$. Hence,

$$\begin{aligned} F(s) = \int_{s-\Gamma(s)}^{s} f(\theta)d\theta &= \int_{s-\Gamma(s)}^{s-\tau(s)} f(\theta)d\theta + \int_{s-\tau(s)}^{s} h(\theta)d\theta + \int_{s-\tau(s)}^{s} \varphi(\theta)d\theta \\ &= H(s) + \Phi(s), \end{aligned}$$

where $H(s) = \int_{s-\tau(s)}^{s} h(\theta)d\theta$ and $\Phi(s) = \int_{s-\Gamma(s)}^{s-\tau(s)} f(\theta)d\theta + \int_{s-\tau(s)}^{s} \varphi(\theta)d\theta$.

We prove that $H \in BAS^p(\mathbb{R}, \mathbb{R}_+)$. Since $h, \tau \in BAS^p(\mathbb{R}, \mathbb{R}_+)$, it is easy to see that $h, \tau \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R}_+)$. Moreover, for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ and functions $h^*, \tau^* \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R}_+)$ such that (3) holds. Let

$$H^*(s) = \int_{s-\tau^*(s)}^{s} h^*(\theta)d\theta.$$

Then we have

$$\begin{aligned} &\Big[\int_t^{t+1} \Big|H(s+s_n) - H^*(s)\Big|^p ds\Big]^{\frac{1}{p}} \\ &= \Big[\int_t^{t+1} \Big|\int_{s+s_n-\tau(s+s_n)}^{s+s_n} h(\theta)d\theta - \int_{s-\tau^*(s)}^{s} h^*(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} \\ &= \Big[\int_t^{t+1} \Big|\int_{s-\tau(s+s_n)}^{s} h(\theta+s_n)d\theta - \int_{s-\tau^*(s)}^{s} h^*(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} \\ &\leq \Big[\int_t^{t+1} \Big|\int_{s-\tau(s+s_n)}^{s-\tau^*(s)} h(\theta+s_n)d\theta\Big|^p ds\Big]^{\frac{1}{p}} \\ &+ \Big[\int_t^{t+1} \Big|\int_{s-\tau^*(s)}^{s} [h(\theta+s_n) - h^*(\theta)]d\theta\Big|^p ds\Big]^{\frac{1}{p}} \\ &\leq \|h\|_\infty \Big[\int_t^{t+1} \Big|\tau(s+s_n) - \tau^*(s)\Big|^p ds\Big]^{\frac{1}{p}} \\ &+ \|\tau^*\|_\infty^{\frac{p-1}{p}} \Big[\int_t^{t+1} \int_{s-\|\tau^*\|}^{s} \Big|h(\theta+s_n) - h^*(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}. \end{aligned}$$

Since $\tau \in BAS^p(\mathbb{R}, \mathbb{R}_+)$, we have

$$\lim_{n\to+\infty} \Big[\int_t^{t+1} \Big|\tau(s+s_n) - \tau^*(s)\Big|^p ds\Big]^{\frac{1}{p}} = 0.$$

Moreover,

$$\begin{aligned}
&\Big[\int_t^{t+1}\int_{s-\|\tau^*\|}^{s}\Big|h(\theta+s_n)-h^*(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}\\
&\leq \Big[\int_t^{t+1}\int_{-\infty}^{s}\Big|h(\theta+s_n)-h^*(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}\\
&= \Big[\int_0^{+\infty}\int_t^{t+1}\Big|h(s-\theta+s_n)-h^*(s-\theta)\Big|^p ds d\theta\Big]^{\frac{1}{p}}\\
&= \Big[\int_0^{+\infty}K_t(\theta,s_n)d\theta\Big]^{\frac{1}{p}},
\end{aligned}$$

where $K_t(\theta,s_n)=\int_t^{t+1}\Big|h(s-\theta+s_n)-h^*(s-\theta)\Big|^p ds$.

It is clear that K_t is bounded, $K_t\geq 0$ and $\lim_{n\to\infty}K_t(\theta,s_n)=0$. Hence, by the Lebegue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\Big[\int_t^{t+1}\int_{s-\|\tau^*\|}^{s}\Big|h(\theta+s_n)-h^*(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}=0.$$

Thus,

$$\lim_{n\to\infty}\Big[\int_t^{t+1}\Big|H(s+s_n)-H^*(s)\Big|^p ds\Big]^{\frac{1}{p}}=0.$$

Analogously, we prove that $\lim_{n\to\infty}\Big[\int_t^{t+1}\Big|H^*(s-s_n)-H(s)\Big|^p ds\Big]^{\frac{1}{p}}=0$.

Thus, $H\in BAS^p(\mathbb{R},\mathbb{R}_+)$.

On the other hand,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{s-\tau(s)}^{s}\varphi(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}}d\mu(t)\\
&\leq \frac{\|\tau\|_\infty^{\frac{p-1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\int_{s-\|\tau\|_\infty}^{s}\Big|\varphi(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}d\mu(t)\\
&\leq \frac{\|\tau\|_\infty^{\frac{p-1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\int_{-\infty}^{s}\Big|\varphi(\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}d\mu(t)\\
&= \frac{\|\tau\|_\infty^{\frac{p-1}{p}}}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\int_0^{+\infty}\Big|\varphi(s-\theta)\Big|^p d\theta ds\Big]^{\frac{1}{p}}d\mu(t)\\
&\leq \frac{\|\tau\|_\infty^{\frac{p-1}{p}}}{(\mu([-r,r]))^{\frac{1}{p}}}\Big[\int_{[-r,r]}\int_t^{t+1}\int_0^{+\infty}\Big|\varphi(s-\theta)\Big|^p d\theta ds d\mu(t)\Big]^{\frac{1}{p}}\\
&= \|\tau\|_\infty^{\frac{p-1}{p}}\Big[\int_0^{+\infty}\Big(\frac{1}{\mu([-r,r])}\int_{[-r,r]}\int_t^{t+1}\Big|\varphi(s-\theta)\Big|^p ds d\mu(t)\Big)d\theta\Big]^{\frac{1}{p}}.
\end{aligned}$$

Since $\varepsilon\,(\mathbb{R},L^p(0,1;\mathbb{R}_+),\mu)$ is translation invariant, and by the Lebegue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{s-\tau(s)}^{s}\varphi(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}}d\mu(t)=0.$$

Also,

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{s-(\tau(s)+\xi(s))}^{s-\tau(s)} f(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{s-(\tau(s)+\xi(s))}^{s-\tau(s)} h(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{-\infty}^{s} \varphi(\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&\le \frac{\|h\|_\infty}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\xi(s\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&+\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{0}^{+\infty} \varphi(s-\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t).
\end{aligned}$$

Then we get

$$\lim_{n\to+\infty}\frac{\|h\|_\infty}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\xi(s\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)=0$$

and

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{0}^{+\infty} \varphi(s-\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{(\mu([-r,r]))^{\frac{1}{p}}}\Big[\int_{[-r,r]}\int_t^{t+1}\int_0^{+\infty}\Big|\varphi(s-\theta)\Big|^p d\theta ds d\mu(t)\Big]^{\frac{1}{p}}\\
&=\Big[\int_0^{+\infty}\Big(\frac{1}{\mu([-r,r])}\int_{[-r,r]}\int_t^{t+1}\Big|\varphi(s-\theta)\Big|^p ds d\mu(t)\Big)d\theta\Big]^{\frac{1}{p}}.
\end{aligned}$$

Again, since $\varepsilon\left(\mathbb{R}, L^p(0,1;\mathbb{R}_+),\mu\right)$ is translation invariant, by the Lebegue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|\int_{0}^{+\infty} \varphi(s-\theta)d\theta\Big|^p ds\Big]^{\frac{1}{p}} d\mu(t)=0.$$

Consequently, $\Phi^b\in\varepsilon\left(\mathbb{R}, L^p(0,1;\mathbb{R}_+),\mu\right)$. This completes the proof of the assertion **(1)**.

(2) By definition, there exist $h,k\in BAS^p\left(\mathbb{R},\mathbb{R}_+\right)$ and $\varphi,\psi\in\varepsilon\left(\mathbb{R},\mathbb{R}_+,\mu\right)$ such that $f=h+\varphi$ and $g=k+\psi$.

Obviously,

$$f.g=h.k+h.\psi+\varphi.k+\varphi.\psi.$$

First, we prove that $h.k\in BAS^p\left(\mathbb{R},\mathbb{R}\right)$. By Definition 2.7, for every sequence of real numbers $(s'_n)_n$, there exist a subsequence $(s_n)_n$ and functions $h^*,k^*\in L^p_{\text{loc}}(\mathbb{R},\mathbb{R})$ such that (3) holds.

It is easy to see that $h^*.k^* \in L^p_{\text{loc}}(\mathbb{R},\mathbb{R}_+)$. In addition,

$$\begin{aligned}
&\Big[\int_t^{t+1}\Big|h(\theta+s_n)k(\theta+s_n)-h^*(\theta)k^*(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&\le \Big[\int_t^{t+1}\Big|h(\theta+s_n)k(\theta+s_n)-h(\theta+s_n)k^*(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&+\Big[\int_t^{t+1}\Big|h(\theta+s_n)k^*(\theta)-h^*(\theta)k^*(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&\le \|h\|_\infty\Big[\int_t^{t+1}\Big|k(\theta+s_n)-k^*(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&+\|k^*\|_\infty\Big[\int_t^{t+1}\Big|h(\theta+s_n)-h^*(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}.
\end{aligned}$$

This assures the assertion.

On the other hand,

$$\begin{aligned}
&\Big[\int_t^{t+1}\Big|h(\theta)\psi(\theta)+\varphi(\theta)k(\theta)+\varphi(\theta)\psi(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&\le 3^{1+\frac{1}{p}}\|h\|_\infty\Big[\int_t^{t+1}\Big|\psi(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}+3^{1+\frac{1}{p}}\|k\|_\infty\Big[\int_t^{t+1}\Big|\varphi(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}\\
&+3^{1+\frac{1}{p}}\Big[\int_t^{t+1}\Big|\varphi(\theta)\psi(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}},
\end{aligned}$$

which gives

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}\Big|h(\theta)\psi(\theta)+\varphi(\theta)k(\theta)+\varphi(\theta)\psi(\theta)\Big|^p d\theta\Big]^{\frac{1}{p}}d\mu(t)=0.$$

4 Existence of μ-S^p-Pseudo Almost Automorphic Solutions

We will give sufficient conditions for system (1) to admit a solution in the space $PAA^p\left(\mathbb{R},\mathbb{R}^n_+,\mu\right)$. In this case, we apply Theorem 2.8. As a corollary, we treat the scalar case of system (2). We begin first by the following lemma.

Lemma 4.1 *Let $\mu\in\mathfrak{M}$ be such that $PAA^p(\mathbb{R},\mathbb{R}^n,\mu)$ is translation invariant. Let $\{x_k\}$ be a sequence of μ-S^p-pseudo almost automorphic functions (that is, $\{x_k\}\subset PAA^p(\mathbb{R},\mathbb{R}^n,\mu)$) such that*

$$\lim_{k\to\infty}\int_t^{t+1}\|x_k(s)-x(s)\|^p ds=0 \tag{6}$$

for each $t\in\mathbb{R}$, then $x\in PAA^p(\mathbb{R},\mathbb{R}^n,\mu)$.

Proof. To prove the lemma, we refer to [9, Lemma 2.7]. From (6), one can easily deduce that $\{x_k\}$ is a Cauchy sequence with respect to $\|.\|_{S^p}$, and by definition, we can write $x_k=\alpha_k+\phi_k$, where $\{\alpha_k\}\subset AS^p(\mathbb{R},\mathbb{R}^n)$ and $\{\phi^b_k\}\subset\varepsilon(\mathbb{R},L^p([0,1],\mathbb{R}^n),\mu)$. From Theorem 2.3, we have

$$\{\alpha_k(t):t\in\mathbb{R}\}\subset\overline{\{x_k(t):t\in\mathbb{R}\}}.$$

It follows that $\{\alpha_k\}$ is also a Cauchy sequence for the norm $\|.\|_{S^p}$. Thus, there exists a function $\alpha \in AS^p(\mathbb{R}, \mathbb{R}^n)$ such that $\lim_{k\to\infty} \|\alpha_k - \alpha\|_{S^p} = 0$. Hence, $\phi_k = x_k - \alpha_k$ is a Cauchy sequence for the norm $\|.\|_{S^p}$. So, there exists a function $\phi \in BS^p(\mathbb{R}, \mathbb{R}^n)$ such that $\lim_{k\to\infty} \|\phi_k - \phi\|_{S^p} = 0$. Let us prove that $\phi \in \varepsilon(\mathbb{R}, L^p([0,1], \mathbb{R}^n), \mu)$. For $r > 0$, we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}|\phi(s)|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&\le \frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}|\phi_k(s)-\phi(s)|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&+ \frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}|\phi_k(s)|^p ds\Big]^{\frac{1}{p}} d\mu(t)\\
&\le \|\phi_k - \phi\|_{S^p} + \frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}|\phi_k(s)|^p ds\Big]^{\frac{1}{p}} d\mu(t).
\end{aligned}$$

Since $\phi_k^b \in \varepsilon(\mathbb{R}, L^p([0,1], \mathbb{R}^n), \mu)$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} \|\phi_k - \phi\|_{S^p} = 0$, we deduce that

$$\lim_{r\to+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\Big[\int_t^{t+1}|\phi(s)|^p ds\Big]^{\frac{1}{p}} d\mu(t) = 0.$$

Consequently, by the uniqueness of the limit, we obtain $x = \alpha + \phi \in PAA^p(\mathbb{R}, \mathbb{R}^n, \mu)$.

Theorem 4.1 *Suppose that all assumptions of Theorem 3.1 are verified. In addition, the function $\mathcal{F} \in PAA^p\left(\mathbb{R}\times\mathbb{R}^n_+, \mathbb{R}^n_+, \mu\right)$ and satisfies the following conditions:*

(H_2) *There exists a constant $\delta \in [0,1)$ such that*

$$\|\mathcal{F}(s,x) - \mathcal{F}(s,y)\| \le \delta\|x-y\|,\ \forall x,y \in \mathbb{R}^n_+, \forall s \in \mathbb{R}.$$

(H_3) *There exists a constant $M > 0$ such that for any compact subset $K \subset \mathbb{R}^n_+$,*

$$\|\mathcal{F}(s,x)\| \le M\|x\|,\ \forall s \in \mathbb{R},\ \forall x \in K,$$

and

$$\delta + M\|\Gamma\|_\infty < 1.$$

Then system (1) has a positive solution x in $BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$.

Proof. Define the operators $C, T : BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right) \to BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$ by

$$Cx(s) = \mathcal{F}(s, x(s-\ell)) \text{ and } Tx(s) = \int_0^{\Gamma(s)} f(s,\sigma,x(s-\sigma-\ell)d\sigma.$$

Then, by Theorem 2.6 and (H_2), the operator C is well defined. Moreover, it is easy to see that

$$\|Cx - Cy\|_{S^p} \le \delta\|x-y\|_{S^p},\ \forall x,y \in BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right),$$

which implies that C is a contraction operator.

Let us prove that $T : BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right) \to BPAA^p\left(\mathbb{R}, \mathbb{R}^n_+, \mu\right)$ is a completly continuous operator, that is, T is compact and continuous. Firstly, we prove that

T is a compact operator, which is equivalent to showing that for every bounded sequence $\{x_k\} \in BPAA^p\left(\mathbb{R}, \mathbb{R}_+^n, \mu\right)$, the sequence $\{Tx_k\}$ has a convergent subsequence in $BPAA^p\left(\mathbb{R}, \mathbb{R}_+^n, \mu\right)$. Indeed, the proof is similar to step 2 in the proof from [14], the only difference is that we use Lemma 4.1 instead of Lemma 2.7 from [9]. The continuity of T is also similar to that in step 2 from [14, Theorem 3.5].

Finally, if $\lambda \in (0,1)$ and if $x \in BPAA^p\left(\mathbb{R}, \mathbb{R}_+^n, \mu\right)$ satisfying

$$x(s) = \lambda \mathcal{F}\left(s, \frac{x(s-\ell)}{\lambda}\right) + \lambda \int_0^{\Gamma(s)} f(s, \sigma, x(s-\sigma-\ell))d\sigma,$$

then, from Proposition 2.1, we have

$$\left\| \lambda \mathcal{F}\left(s, \frac{x(s-\ell)}{\lambda}\right) \right\| \leq \delta \|x(s-\ell)\| + R$$

for some $R > 0$. Moreover, according to (H_3), we get

$$\|x(s)\| \leq \delta \|x(s-\ell)\| + R + \int_0^{\|\Gamma\|_\infty} M \|x(s-\sigma-\ell)\| d\sigma.$$

It follows that

$$\begin{aligned} \|x\|_{S^p} &\leq \delta \|x\|_{S^p} + R + \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \Big\| \int_0^{\|\Gamma\|_\infty} M \|x(s-\sigma-\ell)\| d\sigma \Big\|^p ds \Big)^{\frac{1}{p}} \\ &\leq \delta \|x\|_{S^p} + R + \|\Gamma\|_\infty^{\frac{p-1}{p}} M \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \int_0^{\|\Gamma\|_\infty} \|x(s-\sigma-\ell)\|^p d\sigma ds \Big\|^p ds \Big)^{\frac{1}{p}}, \end{aligned}$$

which implies that

$$\|x\|_{S^p} \leq \delta \|x\|_{S^p} + M \|\Gamma\|_\infty \|x\|_{S^p} + R.$$

Consequently,

$$\|x\|_{S^p} \leq \frac{R}{1 - \delta - M\|\Gamma\|_\infty}.$$

The proof is complete.

As a special case, we prove the existence of Stepanov-like μ-pseudo almost automorphic solutions to the scalar case of system (2) in $BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$. Let us consider

$$x(s) = \gamma(s)x(s-\beta) + \int_{s-\Gamma(s)}^{s} f(\sigma, x(\sigma))d\sigma, \tag{7}$$

where $\gamma, \Gamma : \mathbb{R} \times \mathbb{R}_+$ and $f : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbf{R}_+$ are positive functions.

Corollary 4.1 *Suppose that* $\gamma, \Gamma \in BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$ *and* $f \in PAA^{p,1}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}_+, \mu)$. *In addition,*

(H_4) *For all compact subset* $K \subset \mathbb{R}_+$, *there exist constant* $L, M > 0$ *such that*

(iii)
$$|f(s,u) - f(s,v)| \leq L|u-v|, \ \forall u, v \in K, \ \forall s \in \mathbb{R};$$

(iv)
$$|f(s,u)| \leq M|u|, \forall u \in K, \ \forall s \in \mathbb{R}.$$

(H_5) *There exists a constant* $\delta \in (0,1)$ *such that*

$$\|\gamma\|_\infty \leq \delta.$$

If

$$\delta + M\|\Gamma\|_\infty < 1,$$

then equation (7) has a positive solution $x \in BPAA^p(\mathbb{R}, \mathbb{R}_+, \mu)$.

Proof. We will apply Theorem 2.8 once again. Indeed, it is enough to verify that assertion (a) holds.

$$\|x(s)\| \leq \delta\|x(s-\beta)\| + M\int_{s-\|\Gamma\|_\infty}^{s} \|x(\sigma)\|d\sigma.$$

Hence,

$$\begin{aligned}
\|x\|_{S^p} &\leq \delta\|x\|_{S^p} + M\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\Big\|\int_{s-\|\Gamma\|_\infty}^{s}\|x(\sigma)\|d\sigma\Big\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \delta\|x\|_{S^p} + M\|\Gamma\|_\infty^{\frac{p-1}{p}}\sup_{t\in\mathbb{R}}\Big(\int_t^{t+1}\int_0^{\|\Gamma\|_\infty}\|x(s-\sigma)\|^p d\sigma ds\Big\|^p ds\Big)^{\frac{1}{p}} \\
&\leq \delta\|x\|_{S^p} + M\|\Gamma\|_\infty\|x\|_{S^p} + 1.
\end{aligned}$$

Thus,

$$\|x\|_{S^p} \leq \frac{1}{1-\delta-M\|\Gamma\|_\infty}.$$

5 Conclusion

In conclusion, the model studied in this work is a generalization of the one treated in [14] by adding a nonlinear perturbation in a more general Banach space. The topological method used allowed us to get rid of any form of monotony considered in many works of this type.

Indeed, when we assume the monotony of the functions with respect to each variable x_i, $i = 1, ..., n$, we predetermine the nature of the interaction between the populations, for example, cooperation, competition, prey-predator, etc. In our case, the topological method allowed us to study the model for a broader class of functions.

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Double-Phase System with Neumann Boundary Condition

M. Knifda [1], A. Aberqi [2] and A. Ouaziz [1*]

[1] *Departement of Mathematics, Sidi Mohamed ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Fez, Morocco.*

[2] *Departement of Mathematics, Sidi Mohamed ben Abdellah University, National School of Applied Sciences, Fez, Morocco.*

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Abstract: This paper investigates the existence of multiple solutions for double-phase systems subject to Neumann conditions. The study is conducted within the framework of Sobolev spaces featuring variable exponents. Assuming appropriate conditions on the given data, we establish the existence of at least two weak solutions, each characterized by distinct energy signs. We employ the Nehari manifold and variational method as the foundation for our approach.

Keywords: *Neumann boundary; two-phase operator; weak solutions; convex-concave source.*

Mathematics Subject Classification (2020): 35J62, 70K05, 35J70, 93A10.

1 Introduction

Let a bounded domain $\mathcal{U} \subseteq \mathbb{R}^N$, $N \geq 2$, with the Lipschitz boundary $\partial\mathcal{U}$ be given. Consider the following double-phase system:

$$\begin{cases} \mathcal{L}^{\mu_1(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = \lambda_1|\mathrm{u}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{u}(\mathrm{y}) + \dfrac{2r(\mathrm{y})}{r(\mathrm{y})+s(\mathrm{y})}|\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})-2}\mathrm{u}(\mathrm{y})|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} & \text{in} \quad \mathcal{U}, \\ \mathcal{L}^{\mu_2(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{v} = \lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{v}(\mathrm{y}) + \frac{2s(\mathrm{y})}{r(\mathrm{y})+s(\mathrm{y})}|\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})-2}\mathrm{v}(\mathrm{y}) & \text{in} \quad \mathcal{U}, \\ \left(\mathrm{Du}(\mathrm{y})\,|^{p(\mathrm{y})-2}\mathrm{Du} + \mu_1(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q(\mathrm{y})-2}\,\mathrm{Du}\right).\nu = h_1(\mathrm{y},\mathrm{u}(\mathrm{y})) & \text{on} \quad \partial\mathcal{U}, \\ \left(|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})-2}\mathrm{Dv} + \mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})-2}\mathrm{Dv}\right).\nu = h_2(\mathrm{y},\mathrm{v}(\mathrm{y})) & \text{on} \quad \partial\mathcal{U}, \end{cases} \tag{1}$$

* Corresponding author: mailto:abdesslam.ouaziz1994@gmail.com

where ν the outer unit normal parameters λ_1, λ_1 are positive, $\mu_1, \mu_2 : \overline{\mathcal{U}} \to (0, \infty)$ are Lipschitz continuous, and s, r, q, $p : \mathcal{U} \to \mathbb{R}$ are continuous functions that satisfy the following conditions:

$$1 < q^- \leq q^+ < p^- \leq p^+ < s^- + r^- = \min_{y \in \mathcal{U}} s(y)) + \min_{y \in \mathcal{U}} r(y)) \leq s^+ + r^+ < \infty, \tag{2}$$

$$\frac{p^-}{s^+ + r^+} < \left(\frac{p^- - q^+}{s^+ + r^+ - q^+} \right) \left(\frac{s^- + r^- - q^+}{p^+ - q^-} \right), \tag{3}$$

$$\mathcal{L}^{\mu_i(y)}_{p(y),q(y)} u = -\operatorname{div} \Big(\,|\, Du(y) \,|^{p(y)-2} Du + \mu_i(y) \,|\, Du(y) \,|^{q(y)-2} \, Du \Big) \quad \text{for all } i = 1, 2. \tag{4}$$

We assume $h_i : \partial\mathcal{U} \times \mathbb{R} \to \mathbb{R}$ are Carathéodry functions satisfying the following conditions:

($\mathcal{H}_1$) There exists $\beta > r^+ + s^+$ for some B such that for each $|\theta| > B$, we get

$$0 < \int_{\partial\mathcal{U}} H_i(y, \theta) d\nu \leq \int_{\partial\mathcal{U}} h_i(y, \theta) \frac{\theta}{\beta} d\nu \quad a.e\, x \in \partial\mathcal{U} \quad \text{with } H_i(y, \theta) = \int_0^\theta h_i(y, t) dt,$$

($\mathcal{H}_2$) $h_i(y, 0) = 0$,

($\mathcal{H}_3$) $\lim\limits_{\varepsilon \to 0} \dfrac{h_i(y, \varepsilon)}{|\varepsilon|^{q(y)-2}\varepsilon} = 0$ and $\lim\limits_{\varepsilon \to \pm\infty} \dfrac{h_i(y, \varepsilon)}{|\varepsilon|^{q(y)-2}\varepsilon} = +\infty$ uniformly a.e.y $\in \partial\mathcal{U}$.

Due to the presence of two different elliptic growths $p(y)$ and $q(y)$, the problem (1) is said to be the double-phase type system. Recently, great attention has been devoted to treating the functional

$$v \mapsto \int_{\mathcal{U}} (|Dv|^p + a(y)|Dv|^q) \, dy \quad \text{with } 1 < p < q \quad \text{and } a(y) \geq 0, \quad \text{for all } y \in \mathcal{U} \subset \mathbb{R}^N. \tag{5}$$

This kind of functional was introduced by Zhikov [19–21]. The functional (5) has been used in various scientific fields. Zhikov [20, 21] discusses its numerous applications in the study of duality theory. In recent works, several researchers have focused on studying equations or systems involving the double-phase operator. The reader may consult the work of Liu and Dai [11] who studied the following equations:

$$\begin{cases} \mathcal{L}^{\mu(y)}_{p,q} u &= f(y, u) \quad \text{in } \mathcal{U}, \\ u \,=\, 0 & \qquad\qquad \text{on } \partial\mathcal{U}, \end{cases}$$

where $\mathcal{L}^{\mu(y)}_{p,q} u = -\operatorname{div}(\,|\, Du(y) \,|^p Du + \mu(y) \,|\, Du(y) \,|^q \, Du)$. For the performed comparable processing via Nehari's manifold method, see Gasiński et al. in [10], see also Arora et al. in [3]. Along the same lines, refer to [1, 4, 14–18]. In the same context, Gasinski and Winkert [9] studied the double-phase equation with a non-linear boundary condition of the following form:

$$\begin{cases} \mathcal{L}^{\mu(y)}_{p(y),q(y)} u = f(x, u) - |\, u(y) \,|^{p-2} \, u(y) - \mu(y) |\, u(y) \,|^{q-2} \, u(y) & \text{in } \mathcal{U}, \\ (\,|\, Du(y) \,|^{p-2} Du + \mu_2(y) \,|\, Du(y) \,|^{q-2} \, Dv).\nu = h(x, u(y)) & \text{on } \partial\mathcal{U}. \end{cases}$$

Choudhuri, Repovš and Saudi [6] proved the existence of solutions to a double-phase problem with a nonlinear boundary condition that is specified and nonlinear. Concerning the analysis of nonlinear systems with Dirichlet-type boundaries, Aberqi et al. [2] demonstrated the existence of at least two nonnegative nontrivial solutions to a $p(z)$-Laplacian system with critical nonlinearity. Furthermore, Winkert et al. [13] investigated a system involving the double-phase operator with sources that depend on Neumann boundary conditions, and the gradient structured as follows:

$$\begin{cases} \mathcal{L}^{\mu_1(\mathrm{y})}_{p(\mathrm{y}),q(\mathrm{y})}\mathrm{u} = f_1(x,\mathrm{u},\mathrm{v},\mathrm{Du},\mathrm{Dv}) & \text{in } \mathcal{U}, \\ \mathcal{L}^{\mu_2(\mathrm{y})}_{p_1(x),q_1(x)}(v) = f_2(x,\mathrm{u},\mathrm{v},\mathrm{Du},\mathrm{Dv}) & \text{in } \mathcal{U}, \\ (\,|\,\mathrm{Du}(\mathrm{y})\,|^{p_1(x)-2}\mathrm{Du} + \mu_1(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q_1(x)-2}\,Du).\nu = h_1(x,\mathrm{u},\mathrm{v}) & \text{on } \partial\mathcal{U}, \\ (\,|\,\mathrm{Du}(\mathrm{y})\,|^{p_2(x)-2}\mathrm{Du} + \mu_2(\mathrm{y})\,|\,\mathrm{Du}(\mathrm{y})\,|^{q_2(x)-2}\,\mathrm{Du}).\nu = h_2(x,\mathrm{u},\mathrm{v}) & \text{on } \partial\mathcal{U}. \end{cases}$$

More recently, Marino and Winkert [13] showed the existence of at least one weak solution of such systems. Some research related to our contribution are: Marano, Marino and Moussaoui [12], Chen and Wu [5].

Remark 1.1
- *Neumann boundary conditions may cause non-uniqueness unless the solution is normalized.*
- *Neumann conditions often stabilize the system compared to Dirichlet conditions.*
- *They allow global attractors or stationary solutions to emerge depending on the nonlinearities.*

Motivated by the aforementioned works and the remarks above, in this study, we will employ a variational approach to investigate the existence of at least two positive non-trivial solutions to the system (1) under conditions $(\mathcal{H}_1) - (\mathcal{H}_3)$, (2)-(3).
The theorem presented below constitutes the primary result of this paper.

Theorem 1.1 *If hypotheses* $(\mathcal{H}_1)$ *to* $(\mathcal{H}_3)$ *are satisfied, and there exists a positive constant* $K > 0$ *such that* $\lambda_1 + \lambda_2 \in (0, K)$, *then the system* (1) *possesses at least two non-negative, non-trivial solutions.*

The paper is structured as follows. In Section 2, we review established properties of the variable exponent spaces $L^{p(.)}(\mathcal{U})$ and Musielak-Orlicz Sobolev space $W^{1,p(.)}(\mathcal{U})$ that are compatible with the variable exponent double-phase operator, along with other technical tools that will be employed later. We present our main results in Section 3.

2 Background Results

This section presents the essential definitions and properties of the Sobolev-Orlicz space featuring variable exponents. To delve deeper into the theory of Sobolev-Orlicz spaces, refer to [2, 7, 8].

Consider the following sets: $P(\mathcal{U}) = \{w : \mathcal{U} \to \mathbb{R}, w \text{ m}\textit{easurable}\}$ and $C_+(\bar{\mathcal{U}}) = \{p : \mathcal{U} \to (1,\infty)) \text{ continuous } : p_- > 1\}$.

Definition 2.1 (see [7]) Let $q \in C_+(\bar{\mathcal{U}})$. We denote by $L^{q(.)}(\mathcal{U})$ the Lebesgue space with variable exponent, that is,

$$L^{q(.)}(\mathcal{U}) = \left\{ w \in P(\mathcal{U}) : \int_{\mathcal{U}} |w|^{q(\mathrm{y})}\mathrm{dy} < \infty \right\},$$

whose modular is given by $\varrho_{q(.)}(w) = \int_{\mathcal{U}} |w|^{q(\mathrm{y})} \mathrm{dy}$ and which is endowed with its corresponding Luxemburg norm

$$\|w\|_{q(.)} = \inf\left\{\lambda > 0 : \int_{\mathcal{U}} \left(\frac{|w|}{\lambda}\right)^{q(y)} \mathrm{dy} \leqslant 1\right\}.$$

Definition 2.2 (see [7]) Let $w : \mathcal{U} \to \mathbb{R}$, we define the Sobolev space by

$$W^{1,q(\mathrm{y})}(\mathcal{U}) = \left\{w \in L^{q(\mathrm{y})}(\mathcal{U}) : |\nabla w| \in L^{q(\mathrm{y})}(\mathcal{U})\right\},$$

it is equipped with the norm $\|w\|_{1,q(.)} = \|w\|\|_{q(.)} + \|\nabla w\|_{q(.)}$, and let $W_0^{1,q(\mathrm{y})}(\mathcal{U}) = \overline{C^\infty(\mathcal{U})}^{W^{1,q(\mathrm{y})}(\mathcal{U})}$.

Proposition 2.1 *(see [7]) Let $q \in C_+(\bar{\mathcal{U}})$ be such that there exists a vector $X \in \mathbb{R}^N \setminus \{0\}$ with the property that for all $x \in \mathcal{U}$, the function*

$$g_x(t) = q(x + tX) \ \ with \ \ K_x = \{t \in \mathbb{R} : x + tX \in \mathcal{U}\}$$

is monotone. Then there exsits a constant $C > 0$ such that

$$\varrho_{q(.)}(w) \leqslant C.\varrho_{q(.)}(\nabla w) \quad for\ all \ \ w \in W^{1,q(.)}(\mathcal{U}),\ where\, \varrho_{q(.)}(\nabla w) = \varrho_{q(.)}(|\nabla w|).$$

Proposition 2.2 *(See [8]) Let $q \in C_+(\bar{\mathcal{U}})$, then there exists $C_0 > 0$ such that*

$$\|w\|_{q(.)} \leqslant C_0 \|\nabla w\|_{q(.)},$$

thus, we can define the equivalent norm on $W_0^{1,q(.)}(\mathcal{U})$, $\|u\|_{1,q(.),0} = \|\nabla u\|_{q(.)}$.

Proposition 2.3 *(See [8]) Let $u \in L^{r(.)}(\mathcal{U})$, $v \in L^{r'(.)}(\mathcal{U})$. Then we have*

$$\int_{\mathcal{U}} |u(\mathrm{y})\mathrm{v}(\mathrm{y})| d\mathrm{y} \leq \left(\frac{1}{r^-} + \frac{1}{r'^-}\right) \|u\|_{L^{r(.)}(\mathcal{U})}.\|v\|_{L^{r'(.)}(\mathcal{U})},$$

where $\frac{1}{r'(\mathrm{y})} + \frac{1}{r(\mathrm{y})} = 1$.

Proposition 2.4 *(See [2]) Let $w \in L^{r(.)}(\mathcal{U})$, $\{w_j\} \in L^{r(.)}(\mathcal{U})$, $j \in \mathbb{N}$. Then we have*
(i) $\|w\|_{r(x)} < 1\,(resp. = 1, > 1) \Longleftrightarrow \rho_{r(.)} < 1\,(resp. = 1, > 1)$,
(ii) $\|w\|_{r(x)} < 1 \Rightarrow \|w\|_{r(x)}^{r^-} \leq \rho_{r(.)} \leq \|w\|_{r(x)}^{r^+}$,
(iii) $\lim\limits_{j\to\infty} \|w_j - w\|_{r(x)} = 0 \Longleftrightarrow \lim\limits_{j\to\infty} \rho_{r(x)}(w_j - w) = 0.$

$$min\left\{\rho_{r(x)}(w)^{\frac{1}{r^-}}; \rho_{r(x)}(w)^{\frac{1}{r^+}}\right\} \leq \|w\|_{r(x)} \leq max\left\{\rho_{r(x)}(w)^{\frac{1}{r^-}}; \rho_{r(x)}(w)^{\frac{1}{r^+}}\right\}.$$

Theorem 2.1 *(See [2]) Let $q(\mathrm{y}), p(\mathrm{y}), r(\mathrm{y}) + s(\mathrm{y}) \in L^\infty(\mathcal{U}) \cap C(\bar{\mathcal{U}})$. If*

$$q(\mathrm{y}) < r(\mathrm{y}), \quad p(\mathrm{y}) < r(\mathrm{y}) + s(\mathrm{y}) < q^* = \frac{Nq(\mathrm{y})}{N - q(\mathrm{y})},$$

then

$$W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow\hookrightarrow L^{p(\mathrm{y})}(\mathcal{U}) and\ W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow\hookrightarrow L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}).$$

In addition, we consider the $(N-1)$-dimensional Hausdorff (surface) measure σ on $\partial\mathcal{U}$. Introduce the space $(L^{r(.)}(\partial\mathcal{U}), \|.\|_{r(.),\partial\mathcal{U}})$ and the continuous trace map $\gamma : W^{1,r(.)}(\mathcal{U}) \to L^{\bar{r}}(\partial\mathcal{U})$ with $\bar{r} < r_*$, such that

$$\gamma(u) = u|_{\partial\mathcal{U}} \quad for\ all\ u \in W^{1,p} \cap C^0(\bar{\mathcal{U}}),$$

and

$$r_* = \frac{(N-1)r}{N-1}\ if\, r < N;\ r* = \quad \text{any}\, l \in (r, +\infty)\ if\ r \geq N.$$

According to the trace embedding theorem, γ is compact for any $\bar{r} < r_*$. So, we understand all restrictions of Sobolev functions to $\partial\mathcal{U}$ in the sense of traces.

Lemma 2.1 *Let* $(\mathrm{u}, \mathrm{v}) \in W^{1.q(\mathrm{y})}(\mathcal{U}) \times W^{1.q(\mathrm{y})}(\mathcal{U})$, *then we have*

$$\begin{aligned}
&\int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\right) d\mathrm{y} \\
&\leq c_2(\lambda_1+\lambda_2) \max\left(\|u\|^{q^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{q^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|u\|^{q^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{q^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}\right),
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\int_{\mathcal{U}} |u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} d\mathrm{y} \\
&\leq c_3 \max\left(\|u\|^{r^-+s^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{r^-+s^-}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|u\|^{r^++s^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}, \|v\|^{r^++s^+}_{W_0^{1,q(\mathrm{y})}(\mathcal{U})}\right).
\end{aligned} \tag{7}$$

Proof. The proof is similar to the proof in [2], we will omit it.
The weighted variable exponent Lebesgue space $L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$ is defined as follows:

$$L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U}) = \left\{ w : \mathcal{U} \to \mathbb{R}\ is\ measurable\ ; \int_{\mathcal{U}} \mu(\mathrm{y})|w|^{q(\mathrm{y})} d\mathrm{y} < \infty \right\}$$

and endowed with

$$\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} = \inf\left\{ \eta > 0 : \int_{\mathcal{U}} |\frac{w}{\eta}|^{q(\mathrm{y})} \mu(\mathrm{y}) d\mathrm{y} \leq 1 \right\}.$$

Moreover, the weighted modular on $L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$ is $\rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w) = \displaystyle\int_{\mathcal{U}} \mu(\mathrm{y})|w(x)|^{q(\mathrm{y})} dv_g(\mathrm{y})$.

Proposition 2.5 *(See [2]) Let* w *and* $\{w_n\} \subset L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U})$, *then we get*
(i) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} < 1\, (resp.\ = 1, > 1) \Longleftrightarrow \rho_{q(\mathrm{y}),\mu(\mathrm{y})} < 1\, (resp.\ = 1, > 1)$,
(ii) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} < 1 \Rightarrow \|w\|^{q^-}_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \rho_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \|w\|^{q^+}_{q(\mathrm{y}),\mu(\mathrm{y})}$,
(iii) $\|w\|_{q(\mathrm{y}),\mu(\mathrm{y})} > 1 \Rightarrow \|w\|^{q^+}_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \rho_{q(\mathrm{y}),\mu(\mathrm{y})} \leq \|w\|^{q^-}_{q(\mathrm{y}),\mu(\mathrm{y})}$,
(iv) $\lim\limits_{n\to\infty} \|w_n\|_{q(\mathrm{y}),\mu(\mathrm{y})} = 0 \Longleftrightarrow \lim\limits_{n\to\infty} \rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w_n) = 0$,
(v) $\lim\limits_{n\to\infty} \|w_n\|_{q(\mathrm{y}),\mu(\mathrm{y})} = \infty \Longleftrightarrow \lim\limits_{n\to\infty} \rho_{q(\mathrm{y}),\mu(\mathrm{y})}(w_n) = \infty$.
It should be noted that the non-negative weighted function $\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}^+_*$ *satisfies the following condition:*
$\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}^+_*$ *such that* $\mu(.) \in L^{\varsigma(x)}(\mathcal{U})$ *with*

$$\frac{Np(\mathrm{y})}{Np(\mathrm{y}) - q(\mathrm{y})(N - p(\mathrm{y}))} < \varsigma(x) < \frac{p(\mathrm{y})}{p(\mathrm{y}) - q(\mathrm{y})}. \tag{8}$$

In fact, because $\mu(.) : \bar{\mathcal{U}} \to \mathbb{R}^+_*$, then there exists $\mu_0 > 0$, and for all $x \in \mathcal{U}$, we get that $\mu(\mathrm{y}) > \mu_0$.

Theorem 2.2 *(See [2]) Assume that $q(\mathrm{y}) \in C(\bar{\mathcal{U}}) \cap L^\infty(\mathcal{U})$ and $\mathcal{U}$ is a bounded domain with smooth boundaries. Suppose that the assumption* (8) *is verified. Then we have the compact embedding*

$$W^{1,q(\mathrm{y})}(\mathcal{U}) \hookrightarrow L^{q(\mathrm{y})}_{\mu(\mathrm{y})}(\mathcal{U}).$$

3 Proof of Theorem 1.1

Let us denote $W = W^{1.q(\mathrm{y})}(\mathcal{U}) \times W^{1.q(\mathrm{y})}(\mathcal{U})$, endowed with norm $\|(\mathrm{u}, \mathrm{v})\| = \|\mathrm{u}\| + \|\mathrm{v}\|$, and $D(\mathcal{U})$ is the space of C^∞_c functions with compact support in $\mathcal{U}$.

3.1 Nehari analysis for system (1)

First, we define the weak solution of the system (1) as follows.

Definition 3.1 We say that $(\mathrm{u}, \mathrm{v}) \in W$ is a weak solution of the system (1) if

$$\begin{aligned}
&\int_{\mathcal{U}} \Big(|Du(\mathrm{y})|^{p(\mathrm{y})-2} g(Du(\mathrm{y}), D\phi(\mathrm{y})) + |Dv(\mathrm{y})|^{p(\mathrm{y})-2} g(Dv(\mathrm{y}), D\psi(\mathrm{y}))\Big)\,\mathrm{dy} \\
&+ \int_{\mathcal{U}} \Big(\mu_1(\mathrm{y})|Du(\mathrm{y})|^{q(\mathrm{y})-2} g(Du(\mathrm{y}), D\phi(\mathrm{y})) + \mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})-2} g(Dv(\mathrm{y}) D\psi(\mathrm{y}))\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (H_1(\mathrm{y}, u(\mathrm{y}))\phi(x) + h_2(\mathrm{y}, \mathrm{v}(\mathrm{y}))\psi(x))\,\mathrm{d}\nu \\
&= \int_{\mathcal{U}} \Big(\lambda_1 |\, u(\mathrm{y})|^{q(\mathrm{y})-2} u(\mathrm{y})\phi(\mathrm{y}) + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})-2} v(\mathrm{y})\psi(\mathrm{y})\Big)\,\mathrm{dy} \\
&+ \int_{\mathcal{U}} \Big(\frac{2r(\mathrm{y})}{r(\mathrm{y}) + s(\mathrm{y})} |u(\mathrm{y})|^{n(\mathrm{y})-2} u(\mathrm{y})\phi(\mathrm{y}) + \frac{2s(\mathrm{y})}{r(\mathrm{y}) + s(\mathrm{y})} |v(\mathrm{y})|^{m(\mathrm{y})-2} v(\mathrm{y})\psi(\mathrm{y})\Big)\,\mathrm{dy}
\end{aligned}$$

for all $(\phi, \psi) \in D(\mathcal{U}) \times D(\mathcal{U})$.

Let $J_{\lambda_1,\lambda_2} : W \to \mathbb{R}$ be the energy functional defined by

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u}, \mathrm{v}) = &\int_{\mathcal{U}} \frac{1}{p(\mathrm{y})} \Big(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} \Big(\mu_1 |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + |\mu_2 Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (H_1(\mathrm{y}, u(\mathrm{y})) + H_2(\mathrm{y}, \mathrm{v}(\mathrm{y})))\mathrm{d}\nu \\
&- \int_{\mathcal{U}} \frac{1}{q(\mathrm{y})} \Big(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} - \int_{\mathcal{U}} \frac{2}{s(\mathrm{y}) + r(\mathrm{y})} |u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}$$

By a direct calculation, we have $J_{\lambda_1,\lambda_2} \in C^1(W, \mathbb{R})$. Consider the Nehari manifold $N_{\lambda_1,\lambda_2} = \Big\{(\mathrm{u}, \mathrm{v}) \in W \backslash \{(0, 0)\} : \langle J'_{\lambda_1,\lambda_2}(\mathrm{u}, \mathrm{v}), (\mathrm{u}, \mathrm{v})\rangle = 0\Big\}$. Then, $(\mathrm{u}, \mathrm{v}) \in N_{\lambda_1,\lambda_2}$ equivalent:

$$\begin{aligned}
&\int_{\mathcal{U}} \Big(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |Dv(\mathrm{y})|^{p(\mathrm{y})}\Big)\,\mathrm{dy} + \int_{\mathcal{U}} \Big(\mu_1 |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2 |Dv(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\partial\mathcal{U}} (h_1(\mathrm{y}, u(\mathrm{y})).u(\mathrm{y}) + h_2(\mathrm{y}, \mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}))\,\mathrm{d}\nu - \int_{\mathcal{U}} \Big(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})}\Big)\,\mathrm{dy} \\
&- \int_{\mathcal{U}} 2|u(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy} = 0.
\end{aligned}$$

Lemma 3.1 *Suppose that the hypotheses $(\mathcal{H}_1)-(\mathcal{H}_3)$ are satisfied, then the energy functionals J_{λ_1,λ_2} are bounded and coercive on W.*

Proof. For $(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}$, according to (3), (4) and Proposition 2.2

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) &= \int_{\mathcal{U}}\left(\frac{1}{p(\mathrm{y})}-\frac{1}{r(\mathrm{y})+s(\mathrm{y})}\right)\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
&+\int_{\mathcal{U}}\left(\frac{1}{q(\mathrm{y})}-\frac{1}{r(\mathrm{y})+s(\mathrm{y})}\right)\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})}h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})-H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})}h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})-H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+\int_{\mathcal{U}}\left(\frac{1}{r(\mathrm{y})+s(\mathrm{y})}-\frac{1}{q(\mathrm{y})}\right)\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|v(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&\geq c_0\left(\frac{1}{p^+}-\frac{1}{r^-+s^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}+\frac{\mu_0}{k^{p+}(c+1)^{p^+}q^+}\left(\frac{1}{q^+}-\frac{1}{r^-+s^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}\\
&+c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^++s^+}-\frac{1}{q^-}\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}.
\end{aligned}$$

As $\frac{1}{p^+}>\frac{1}{r^++s^+}>\frac{1}{\beta}$ and $p^->q^+$, then $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})\to\infty$ as $\|(\mathrm{u},\mathrm{v})\|\to\infty$. As a result J_{λ_1,λ_2} is coercive and bounded below on N_{λ_1,λ_2}. Then we look at the energy $\xi_{\lambda_1,\lambda_2}:N_{\lambda_1,\lambda_2}\to\mathbb{R}$ given by

$$\xi_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})=\langle J'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \text{ for all } (\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}.$$

Therefore, we divide N_{λ_1,λ_2} into

$$\begin{aligned}
N^+_{\lambda_1,\lambda_2}&=\left\{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}:\langle\xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle>0\right\},\\
N^0_{\lambda_1,\lambda_2}&=\left\{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}:\langle\xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle=0\right\},\\
N^-_{\lambda_1,\lambda_2}&=\left\{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}:\langle\xi'_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle<0\right\}.
\end{aligned}$$

Lemma 3.2 *For each $(\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus\{(0,0)\}$, there exists a constant $K_1>0$ such that for all $0<\lambda_1+\lambda_2<K_1$, we have $N^0_{\lambda_1,\lambda_2}=\emptyset$.*

Proof. Suppose $N^0_{\lambda_1,\lambda_2}\neq\emptyset$ for all $(\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. Take $(\mathrm{u},\mathrm{v})\in N^0_{\lambda_1,\lambda_2}$ such that $\|(\mathrm{u},\mathrm{v})\|>1$, then the definition of $N^0_{\lambda_1,\lambda_2}$, $(\mathcal{H}_1)-(\mathcal{H}_3)$ and (4), lead to

$$\begin{aligned}
0&=\langle\xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle\\
&\geq\left(p^--q^+\right)\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+\left(q^--q^+\right)\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&+\int_{\partial\mathcal{U}}\left(q^+h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})-H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu+\int_{\partial\mathcal{U}}\left(q^+h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})-H_1(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+2\left(q^+-(r^++s^+)\right)\int_{\mathcal{U}}\|u(\mathrm{y})\|^{r(\mathrm{y})}\|\mathrm{v}(\mathrm{y})\|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}$$

Since $q^+ > \frac{1}{q^+} > \frac{1}{\beta}$, by $(\mathcal{H}_1)$, we obtain

$$\begin{aligned}
0 &= \langle \xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \\
&\geq \left(p^- - q^+\right)\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+\left(q^- - q^+\right)\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy} \\
&+2\left(q^+ - (r^+ + s^+)\right)\int_{\mathcal{U}}\|u(\mathrm{y})\|^{r(\mathrm{y})}\|\mathrm{v}(\mathrm{y})\|^{s(\mathrm{y})}\mathrm{dy}.
\end{aligned}$$

Therefore, from Theorem 2.2, Poincare's inequality, and Lemma 2.1, we have

$$0 \geq \frac{1}{c}\left(p^- - q^+\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}+c_1\left(q^- - q^+\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}+2\left(q^+ - (r^+ + s^+)\right)c_3\|(\mathrm{u},\mathrm{v})\|^{r^+ + s^+}.$$

As $p^- > q^+$, then

$$0 \geq \left(\frac{1}{c}\left(p^- - q^+\right)+c_1\left(q^- - q^+\right)\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}+2\left(q^+ - (r^+ + s^+)\right)c_3\|(\mathrm{u},\mathrm{v})\|^{r^+ + s^+}.$$

$$\text{Then, } \|(\mathrm{u},\mathrm{v})\| \geq \left(\frac{\frac{1}{c}(p^- - q^+)+c_1(q^- - q^+)}{2\left(q^+ - (r^+ + s^+)\right)c_3}\right)^{\frac{1}{r^+ + s^+ - q^+}}. \tag{9}$$

Analogously,

$$\begin{aligned}
0 &= \langle \xi''_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}),(\mathrm{u},\mathrm{v})\rangle \\
&\leq \left(p^+ - r^- - s^-\right)\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+\int_{\partial\mathcal{U}}\left((r^- + s^-)h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})-H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu \\
&+\left(r^- + s^- - q^-\right)\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}+\int_{\partial\mathcal{U}}\left((r^- + s^-)h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})-H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu \\
&+\left(q^+ - r^- - s^-\right)\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|Dv(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}.
\end{aligned}$$

According to Theorem 2.1, (3) and $(\mathcal{H}_1)$, we have

$$\begin{aligned}
0 \leq &(p^+ - r^- - s^-)\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+(r^- - s^- - q^+)\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy} \\
&+(r^- + s^-)\int_{\partial\mathcal{U}}\left(h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})+h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})\right)\mathrm{d}\nu \\
&\leq \frac{1}{c}(p^+ - r^- - s^-)\|(\mathrm{u},\mathrm{v})\|^{p^-}+c_2(\lambda_1+\lambda_2)(r^- - s^- - q^+)\|(\mathrm{u},\mathrm{v})\|^{q^-}+(r^- + s^-)\|(\mathrm{u},\mathrm{v})\|^{q^-}.
\end{aligned}$$

$$\text{Then } \|(\mathrm{u},\mathrm{v})\| \leq \left(\frac{c(c_2(\lambda_1+\lambda_2)+1)(r^- + s^-)}{r^- + s^- - p^+}\right)^{\frac{1}{p^- - q^-}}. \tag{10}$$

According to (9) and (10), we deduce that

$$\left(\frac{\frac{1}{c}(p^- - q^+)+c_1(q^- - q^+)}{2\left(q^+ - (r^+ + s^+)\right)c_3}\right)^{\frac{1}{r^+ + s^+ - q^+}} \leq \left(\frac{c(c_2(\lambda_1+\lambda_2)+1)(r^- + s^-)}{r^- + s^- - p^+}\right)^{\frac{1}{p^- - q^-}}.$$

Then $\lambda_1+\lambda_2 > K_1$, which is a contradiction, hence we can conclude that for any $0<\lambda_1+\lambda_2<K_1$, we have $N^0_{\lambda_1,\lambda_2}=\emptyset$ for all $(\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus\{(0,0)\}$.

Remark 3.1 As a conclusion of Lemma 3.1, we can write $N_{\lambda_1,\lambda_2}=N^+_{\lambda_1,\lambda_2}\cup N^-_{\lambda_1,\lambda_2}$ and we define

$$\gamma^+_{\lambda_1,\lambda_2}=\inf_{(\mathrm{u},\mathrm{v})\in r^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \text{ and } \gamma^-_{\lambda_1,\lambda_2}=\inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}).$$

Lemma 3.3 *Suppose that* $(\mathcal{H}_1)-(\mathcal{H}_3)$ *are satisfied. If* $0<\lambda_1+\lambda_2<K_2$ *then for all* $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})<0$.

Proof. Suppose $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, then the definition of J_{λ_1,λ_2} leads to

$$\begin{aligned}
&J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \leq \frac{1}{p^-}\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|Dv(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
&+\frac{1}{q^-}\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\int_{\partial\mathcal{U}}\left(H_1(\mathrm{y},u(\mathrm{y}))+H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu-\frac{1}{q^-}\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\frac{2}{r^++s^+}\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}\\
&\leq\left(\frac{1}{p^-}-\frac{1}{q^-}\right)\int_{\mathcal{U}}\left[|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}+\mu_1|u(\mathrm{y})|^{q(\mathrm{y})}+\mu_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right]\mathrm{dy}\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{q^+}h_1(\mathrm{y},u(\mathrm{y})).u(\mathrm{y})-H_1(\mathrm{y},u(\mathrm{y}))\right)\mathrm{d}\nu\\
&+\int_{\partial\mathcal{U}}\left(\frac{1}{q^+}h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y})-H_1(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu\\
&+2\left(\frac{1}{q^+}-\frac{1}{r^++s^+}\right)\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}\\
&\leq\frac{1}{c}\left(\frac{1}{p^-}-\frac{1}{q^-}\right)\|(\mathrm{u},\mathrm{v})\|^{p^-}+\left(c_1(\frac{1}{p^-}-\frac{1}{q^+})+\frac{1}{q^+}\right)\|(\mathrm{u},\mathrm{v})\|^{q^+}\\
&+2c_3\left(\frac{1}{q^+}-\frac{1}{r^++s^+}\right)\|(\mathrm{u},\mathrm{v})\|^{r^-+s^-},
\end{aligned}\tag{11}$$

with (c_1,c_3) being the embedding constant of (6), (7). As $(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}$, we get

$$\begin{aligned}
&p^+\int_{\mathcal{U}}\left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+q^+\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}+\mu_2(\mathrm{y})|\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-\int_{\partial\mathcal{U}}\left(H_1(\mathrm{y},u(\mathrm{y}))+H_2(\mathrm{y},\mathrm{v}(\mathrm{y}))\right)\mathrm{d}\nu-q^-\int_{\mathcal{U}}\left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})}+\lambda_2|\mathrm{v}(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
&-2(r^-+s^-)\int_{\mathcal{U}}|u(\mathrm{y})|^{r(\mathrm{y})}|\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})}\mathrm{dy}>0.
\end{aligned}\tag{12}$$

Multiplying (3.1) by $-(r^- + s^-)$, by a direct calculation

$$\begin{aligned}
&-(r^- + s^-)\int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} \\
&-(r^- + s^-)\left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+(r^- + s^-)\int_{\partial\mathcal{U}} \left(h_1(\mathrm{y}, u(\mathrm{y})).u(\mathrm{y}) + h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)}\right) \mathrm{d}\nu \\
&+(r^- + s^-)\int_{\mathcal{U}} \left(\lambda_1|u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2|\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+2(r^- + s^-)\int_{\mathcal{U}} |\mathrm{u(y)}|^{r(\mathrm{y})}|\mathrm{v(y)}|^{s(\mathrm{y})}\mathrm{dy} = 0.
\end{aligned} \tag{13}$$

Combining (12) with (13), we get that

$$\begin{aligned}
&\left(p^+ - (r^- + s^-)\right)\int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} + \left(q^+ - (r^- + s^-)\right) \\
&\times \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+\int_{\partial\mathcal{U}} \left((r^- + s^-)h_1(\mathrm{y}, \mathrm{u(y)}).\mathrm{u(y)} - H_1(\mathrm{y}, \mathrm{u(y)})\right) \mathrm{d}\nu \\
&+\int_{\partial\mathcal{U}} \left((r^- + s^-)h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)} - H_2(\mathrm{y}, \mathrm{v(y)})\right) \mathrm{d}\nu \\
&(r^- + s^- - q^+)\int_{\mathcal{U}} \left(\lambda_1|\mathrm{u(y)}|^{q(\mathrm{y})} + \lambda_2|\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} > 0.
\end{aligned}$$

Combining $(\mathcal{H}_1)$ with Poincare's inequality and Lemma 2.1, we have

$$\begin{aligned}
&(r^- + s^- - q^+)\int_{\mathcal{U}} \left(\lambda_1|\mathrm{u(y)}|^{q(\mathrm{y})} + \lambda_2|\mathrm{v(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&+(r^- + s^-)\int_{\partial\mathcal{U}} \left(h_1(\mathrm{y}, \mathrm{u(y)}).\mathrm{u(y)} + h_2(\mathrm{y}, \mathrm{v(y)}).\mathrm{v(y)}\right) \mathrm{d}\nu \\
&> \left((r^- + s^-) - p^+\right)\int_{\mathcal{U}} \left(|\mathrm{Du(y)}|^{p(\mathrm{y})} + |\mathrm{Dv(y)}|^{p(\mathrm{y})}\right) \mathrm{dy} + \left((r^- + s^-) - q^+\right) \\
&\times \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du(y)}|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv(y)}|^{q(\mathrm{y})}\right) \mathrm{dy} \\
&> \frac{1}{c}\left((r^- + s^-) - p^+\right)\|(\mathrm{u,v})\|^{p^-} + \frac{\mu_0\left((r^- + s^-) - q^+\right)}{D^{p^+}(c+1)^{p^+}}\|(\mathrm{u,v})\|^{p^-},
\end{aligned}$$

then

$$\begin{aligned}
&\left(c_2(r^- + s^- - q^+) - q^+(\lambda_1 + \lambda_2) + (r^- + s^-)\right)\|(\mathrm{u,v})\|^{q^+} \\
&> \left((r^- + s^-) - p^+\right)\left[\frac{1}{c} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\right]\|(\mathrm{u,v})\|^{p^-}
\end{aligned}$$

and $\|(\mathrm{u,v})\|^{p^-} < \dfrac{[c_2(r^- + s^- - q^+)(\lambda_1 + \lambda_2) + (r^- + s^-)]}{((r^- + s^-) - p^+)\left[\frac{1}{c} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\right]}\|(\mathrm{u,v})\|^{q^+}.$

As $(r^- + s^-) > q^+$, using (11), we get

$$\begin{aligned} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) < & \left[-(\frac{p^- - q^+}{p^- q^+}) \times \frac{[c_2(r^- + s^- - q^+)(\lambda_1 + \lambda_2) + (r^- + s^-)]}{c\left((r^- + s^-) - p^+\right)\left[\frac{1}{c} + \frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\right]} \right] \|(\mathrm{u},\mathrm{v})\|^{q^+} \\ & + \left[\left(c_1(\frac{1}{p^-} - \frac{1}{q^+}) + \frac{1}{q^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^+} \\ & + \left[2c_3 \left(\frac{1}{q^+} - \frac{1}{r^+ + s^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^+}. \end{aligned}$$

Finally, for $\lambda_1 + \lambda_2$ sufficiently large, we get $\gamma^+_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v}) \in N^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) < 0$.

Lemma 3.4 *Under assumptions* $(\mathcal{H}_1) - (\mathcal{H}_3)$*, if* $0 < \lambda_1 + \lambda_2 < K_3$*, then for all* $(\mathrm{u},\mathrm{v}) \in N^-_{\lambda_1,\lambda_2}$*,* $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$.

Proof. Let $(\mathrm{u},\mathrm{v}) \in N^-_{\lambda_1,\lambda_2}$. By definition of J_{λ_1,λ_2}, (4), $(\mathcal{H}_1)$, and (3.1), we get

$$\begin{aligned} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq & \frac{1}{p^+} \int_{\mathcal{U}} \left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\ & + \frac{1}{q^+} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y}) |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y}) |\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\ & - \int_{\partial\mathcal{U}} (H_1(\mathrm{y},\mathrm{u}(\mathrm{y})) + H_2(\mathrm{y},\mathrm{v}(\mathrm{y})))\, \mathrm{d}\nu - \frac{1}{q^-} \int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\ & - \frac{2}{r^- + s^-} \int_{\mathcal{U}} |\mathrm{u}(\mathrm{y})|^{r(\mathrm{y})} |\mathrm{v}(\mathrm{y})|^{s(\mathrm{y})} \mathrm{dy} \\ \geq & \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) \int_{\mathcal{U}} \left(|\mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\ & + \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \int_{\mathcal{U}} \left(\mu_1(\mathrm{y}) |\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y}) |\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\ & - \int_{\partial\mathcal{U}} \left(\frac{1}{r^- + s^-} h_1(\mathrm{y},\mathrm{u}(\mathrm{y})).\mathrm{u}(\mathrm{y}) - H_1(\mathrm{y},\mathrm{u}(\mathrm{y})) \right) \mathrm{d}\nu \\ & - \int_{\partial\mathcal{U}} \left(\frac{1}{r^- + s^-} h_2(\mathrm{y},\mathrm{v}(\mathrm{y})).\mathrm{v}(\mathrm{y}) - H_2(\mathrm{y},\mathrm{v}(\mathrm{y})) \right) \mathrm{d}\nu \\ & + \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \int_{\mathcal{U}} \left(\lambda_1 |u(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2 |v(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy}. \end{aligned}$$

Since $\beta > r^- + s^-$, we get $\frac{1}{r^- + s^-} > \frac{1}{\beta}$, and by $(\mathcal{H}_1)$, we have

$$\begin{aligned} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq & \frac{1}{c} \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) \|(\mathrm{u},\mathrm{v})\|^{p^-} + \mu_0 \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \|(\mathrm{u},\mathrm{v})\|^{q^-} \\ & + c_2(\lambda_1 + \lambda_2) \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \|(\mathrm{u},\mathrm{v})\|^{q^+}. \end{aligned}$$

Since $q^- \leq q^+ < p^-$, we have

$$\begin{aligned} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) \geq & \left[\frac{1}{c} \left(\frac{1}{p^+} - \frac{1}{r^- + s^-} \right) + \mu_0 \left(\frac{1}{q^+} - \frac{1}{r^- + s^-} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^-} \\ & + \left[\frac{1}{c} c_2(\lambda_1 + \lambda_2) \left(\frac{1}{r^- + s^-} - \frac{1}{q^+} \right) \right] \|(\mathrm{u},\mathrm{v})\|^{q^-}. \end{aligned}$$

So, if we take $\lambda_1 + \lambda_2 \leq \left[\frac{1}{cc_2}(\frac{1}{p^+} - \frac{1}{r^- + s^-}) + \frac{\mu_0}{c_2}(\frac{1}{q^+} - \frac{1}{r^- + s^-})\right]\left[\frac{q^+(s^+ + r^+)}{s^- + r^- - q^+}\right] = K_3$, we obtain that $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$, this implies that $\gamma^-_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$. Hence, $N_{\lambda_1,\lambda_2} = N^-_{\lambda_1,\lambda_2} \cup N^+_{\lambda_1,\lambda_2}$, $N^-_{\lambda_1,\lambda_2} \cap N^+_{\lambda_1,\lambda_2} = \emptyset$, by the above lemma, we must have $(\mathrm{u},\mathrm{v}) \in N^-_{\lambda_1,\lambda_2}$.

4 Minimizer on $N^+_{\lambda_1,\lambda_2}$ and $N^-_{\lambda_1,\lambda_2}$.

We will show that there are two nonnegative solutions to the system.

Theorem 4.1 *Under assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$, there exists a minimizer $(\mathrm{u}^+_0, \mathrm{v}^+_0)$ of $J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})$ on $N^+_{\lambda_1,\lambda_2}$, for every $\lambda_1 + \lambda_2 < K = \min(K_1, K_2)$, such that $J_{\lambda_1,\lambda_2}(\mathrm{u}^+_0, \mathrm{v}^+_0) = \gamma^+_{\lambda_1,\lambda_2}$.*

Proof. Lemma 3.1 implies J_{λ_1,λ_2} is bounded below on N_{λ_1,λ_2}, so it is bounded below in $N^+_{\lambda_1,\lambda_2}$, then there exists a minimizing sequence $\{\mathrm{u}^+_n, \mathrm{v}^+_n\} \in r^+_{\lambda_1,\lambda_2}$ such that

$$\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}^+_n, \mathrm{v}^+_n) = \inf_{(\mathrm{u},\mathrm{v})\in r^+_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma^+_{\lambda_1,\lambda_2} < 0. \tag{14}$$

Note that J_{λ_1,λ_2} is bounded on W. Hence, without loss of generality, we suppose $(\mathrm{u}^+_n, \mathrm{v}^+_n) \to (\mathrm{u}^+_0, \mathrm{v}^+_0)$ on W; and by the compact embedding, we get

$$\begin{aligned}
&\mathrm{u}^+_n \to \mathrm{u}^+_0 \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{\alpha(x)}(\mathcal{U}) \text{ and } L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty,\\
&\mathrm{v}^+_n \to \mathrm{v}^+_0 \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{\alpha(x)}(\mathcal{U}) \text{ and } L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty,\\
&\mathrm{u}^+_n \to u^+_0 \text{ and } \mathrm{v}^+_n \to \mathrm{v}^+_0 \text{ a.e in } \mathcal{U} \text{ as } n \to \infty.
\end{aligned} \tag{15}$$

Next, we will prove that $\mathrm{u}^+_n \to \mathrm{u}^+_0$ and $\mathrm{v}^+_n \to \mathrm{v}^+_0$ on $W^{1,p(\mathrm{y})}(\mathcal{U})$ as $n \to \infty$. Otherwise, let $\mathrm{u}^+_n \to u^+_0$ and $\mathrm{v}^+_n \to \mathrm{v}^+_0$ on $W^{1,p(\mathrm{y})}(\mathcal{U})$ as $n \to \infty$, then we have

$$\rho_{q(\mathrm{y})}(\mathrm{u}^+_0) \leq \lim_{n\to\infty} \inf \rho_{q(\mathrm{y})}(\mathrm{u}^+_n), \text{ and } \rho_{q(\mathrm{y})}(\mathrm{v}^+_0) \leq \lim_{n\to\infty} \inf \rho_{q(\mathrm{y})}(\mathrm{v}^+_n).$$

Since $\langle J_{\lambda_1,\lambda_2}(\mathrm{u}^+_n, \mathrm{v}^+_n), (\mathrm{u}^+_n, \mathrm{v}^+_n)\rangle = 0$, we get

$$\begin{aligned}
J_{\lambda_1,\lambda_2}(\mathrm{u}^+_n, \mathrm{v}^+_n) \geq{}& \frac{1}{c}\left(\frac{1}{p^+} - \frac{1}{r^- + s^-}\right)\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{p^-} + \mu_0\left(\frac{1}{q^+} - \frac{1}{r^- + s^-}\right)\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{q^-}\\
&+ c_1(\lambda_1 + \lambda_2)\left(\frac{1}{r^+ + s^+} - \frac{1}{q^+}\right)\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{q^+}.
\end{aligned}$$

That is,

$$\begin{aligned}
\lim_{n\to\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}^+_n, \mathrm{v}^+_n) \geq{}& \frac{1}{c}\left(\frac{1}{p^+} - \frac{1}{r^- + s^-}\right)\lim_{n\to\infty}\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{p^-}\\
&+ \mu_0\left(\frac{1}{q^+} - \frac{1}{r^- + s^-}\right)\lim_{n\to\infty}\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{q^-}\\
&+ c_1(\lambda_1 + \lambda_2)\left(\frac{1}{r^+ + s^+} - \frac{1}{q^+}\right)\lim_{n\to\infty}\|(\mathrm{u}^+_n, \mathrm{v}^+_n)\|^{q^+}.
\end{aligned}$$

By (15) and (4), we have

$$\sigma^{+}_{\lambda_1,\lambda_2} > \frac{1}{c}\left(\frac{1}{p^{+}} - \frac{1}{r^{-}+s^{-}}\right)\|(u_0^{+},\mathrm{v}_0^{+})\|^{p^{-}} + \mu_0\left(\frac{1}{q^{+}} - \frac{1}{r^{-}+s^{-}}\right)\|(\mathrm{u}_0^{+},\mathrm{v}_0^{+})\|^{q^{-}}$$
$$+ c_1(\lambda_1+\lambda_2)\left(\frac{1}{r^{+}+s^{+}} - \frac{1}{q^{+}}\right)\|(\mathrm{u}_0^{+},\mathrm{v}_0^{+})\|^{q^{+}}.$$

Since $r^{-} + s^{-} > p^{-} > q^{+}$ for $\|(\mathrm{u},\mathrm{v})\| > 1$, we deduce that $\sigma^{+}_{\lambda_1,\lambda_2} = \inf_{(\mathrm{u},\mathrm{v})\in r^{+}_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) > 0$, which is in contradiction with Lemma 3.3, hence

$$\mathrm{u}_n^{+} \to \mathrm{u}_0^{+} \text{ and } \mathrm{v}_n^{+} \to \mathrm{v}_0^{+} \text{ on } W_0^{1,p(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty.$$

Consequently, $(\mathrm{u}_0^{+},\mathrm{v}_0^{+})$ is a minimizer of J_{λ_1,λ_2} on $N^{+}_{\lambda_1,\lambda_2}$.

Theorem 4.2 *Suppose that conditions* $(\mathcal{H}_1)$- $(\mathcal{H}_3)$ *are true. Then there exists a minimizer* $(\mathrm{u}_0^{-},\mathrm{v}_0^{-})$ *of* J_{λ_1,λ_2} *on* $N^{-}_{\lambda_1,\lambda_2}$ *for all* $0 < \lambda_1 + \lambda_2 < K = \min\{K_1, K_2\}$ *such that* $J_{\lambda_1,\lambda_2}(u_0^{-},\mathrm{v}_0^{-}) = \sigma^{-}_{\lambda_1,\lambda_2}$.

Proof. J_{λ_1,λ_2} is bounded below on N_{λ_1,λ_2}, and so on $r^{-}_{\lambda_1,\lambda_2}$, then there exists a minimizing sequence $\{\mathrm{u}_n^{-},\mathrm{v}_n^{-}\} \in N^{-}_{\lambda_1,\lambda_2}$ such that

$$\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^{-},\mathrm{v}_n^{-}) = \inf_{(\mathrm{u},\mathrm{v})\in N^{-}_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma^{-}_{\lambda_1,\lambda_2} > 0.$$

So, the sequence$\{\mathrm{u}_n^{-},\mathrm{v}_n^{-}\}_{n\in\mathbb{N}}$ is bounded in W. There exists $(u_0^{-},\mathrm{v}_0^{-}) \in W$ such that up to a subsequence $(\mathrm{u}_n^{-},\mathrm{v}_n^{-}) \rightharpoonup (u_0^{+},\mathrm{v}_0^{+})$ in W. Thanks to Theorem 2.1, we obtain

$$\begin{cases} \mathrm{u}_n^{-} \to u_0^{-} \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty, \\ \mathrm{v}_n^{-} \to \mathrm{v}_0^{-} \text{ strongly in } L^{p(\mathrm{y})}(\mathcal{U}), L^{r(\mathrm{y})+s(\mathrm{y})}(\mathcal{U}) \text{ as } n \to \infty, \\ \mathrm{u}_n^{-} \to u_0^{-} \text{ and } \mathrm{v}_n^{-} \to \mathrm{v}_0^{-} \text{ a.e in } \mathcal{U} \text{ as } n \to \infty. \end{cases} \tag{16}$$

Hence, $(\mathrm{u}_0^{-},\mathrm{v}_0^{-}) \in N^{-}_{\lambda_1,\lambda_2}$, $\exists t > 0$, such that $(t\mathrm{u}_0^{-},t\mathrm{v}_0^{-}) \in N^{-}_{\lambda_1,\lambda_2}$ and $J_{\lambda_1,\lambda_2}(\mathrm{u}_0^{-},\mathrm{v}_0^{-}) \geq J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^{-},t\mathrm{v}_0^{-})$. According to $(\mathcal{H}_1)$ and the definition of $\xi'_{\lambda_1,\lambda_2}$, we have

$$
\begin{aligned}
\left\langle \xi^{'}_{\lambda_1,\lambda_2}(tu_0^-, t\mathrm{v}_0^-), (tu_0^-, t\mathrm{v}_0^-) \right\rangle &= \int_{\mathcal{U}} p(\mathrm{y}) \left(|Dtu_0^-(\mathrm{y})|^{p(\mathrm{y})} + |Dt\mathrm{v}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\
&- 2\int_{\mathcal{U}} (r(\mathrm{y}) + s(\mathrm{y})|tu_0^-(x)|^{r(\mathrm{y})} |t\mathrm{v}_0^-(x)|^{s(\mathrm{y})} \mathrm{dy} \\
&+ \int_{\mathcal{U}} q(\mathrm{y}) \left(\mu_1(\mathrm{y})|Dtu_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|Dt\mathrm{v}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- \int_{\partial\mathcal{U}} \left(H_1(x, t\mathrm{u}_0^-(x)) + H_2(x, t\mathrm{v}_0^-(x)) \right) \mathrm{d}\nu \\
&- \int_{\mathcal{U}} q(\mathrm{y}) \left(\lambda_1 |t\mathrm{u}_0^-(x)|^{q(\mathrm{y})} + \lambda_2 |t\mathrm{v}_0^-(x)|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- 2\int_{\mathcal{U}} (r(\mathrm{y}) + s(\mathrm{y})|tu_0^-(x)|^{r(\mathrm{y})} |t\mathrm{v}_0^-(x)|^{s(\mathrm{y})} \mathrm{dy} \\
&\le p^+ t^{p^+} \int_{\mathcal{U}} \left(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \\
&+ q^+ t^{q^-} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- q^- t^{q^-} \int_{\mathcal{U}} \left(\lambda_1 |\mathrm{u}_0^-(x)|^{q(\mathrm{y})} + \lambda_2 |\mathrm{v}_0^-(x)|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&- 2(r^- + s^-) t^{r^- + s^-} \int_{\mathcal{U}} |\mathrm{u}_0^-(x)|^{r(\mathrm{y})} |\mathrm{v}_0^-(x)|^{s(\mathrm{y})} \mathrm{dy}.
\end{aligned}
$$

Due to $q^- \le q^+ < p^+ < s^- + r^-$, and by Propositions 2.2, 2.3, it follows that $\langle \xi^{'}_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-), (t\mathrm{u}_0^-, t\mathrm{v}_0^-) \rangle < 0$. Hence, by definition of $N^-_{\lambda_1,\lambda_2}$, $(t\mathrm{u}_0^-, t\mathrm{v}_0^-) \in N^-_{\lambda_1,\lambda_2}$. Next, we show that $(\mathrm{u}_n^-, \mathrm{v}_n^-) \to (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W(\mathcal{U})$. Assume that $(\mathrm{u}_n^-, \mathrm{v}_n^-) \nrightarrow (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W$, by Fatou's Lemma, we have

$$
\begin{aligned}
&\int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy} \\
&\le \lim_{n\to+\infty} \int_{\mathcal{U}} \left(\mu_1(\mathrm{y})|\mathrm{Du}_n^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_n^-(\mathrm{y})|^{q(\mathrm{y})} \right) \mathrm{dy},
\end{aligned}
$$

by (16), we get

$$
\int_{\mathcal{U}} \left(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy} \le \lim_{n\to+\infty} \int_{\mathcal{U}} \left(|\mathrm{Du}_n^-(\mathrm{y})|^{p(\mathrm{y})} |\mathrm{Dv}_n^-(\mathrm{y})|^{p(\mathrm{y})} \right) \mathrm{dy}.
$$

Then, by $(\mathcal{H}_1)$, we have

$$
\begin{aligned}
J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-) \leq\ & \frac{t^{p^+}}{p^-}\int_{\mathcal{U}}\left(|\mathrm{Du}_0^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_0^-(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
& + \frac{t^{q^+}}{q^-}\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_0^-(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
& - \int_{\partial\mathcal{U}}\left(H_1(x, t\mathrm{u}_0^-(x)) + H_2(x, t\mathrm{v}_0^-(\mathrm{y}))\right)\mathrm{d}\nu\\
& - \frac{t^{q^-}}{q^+}\int_{\mathcal{U}}\left(\lambda_1|\mathrm{u}_0^-(\mathrm{y})|^{q(\mathrm{y})} + \lambda_2|\mathrm{v}_0^-(x)|^{q(\mathrm{y})}\right)\mathrm{dy}\\
& - 2\frac{t^{r^-+s^-}}{r^+ + s^+}\int_{\mathcal{U}}|\mathrm{u}_0^-(x)|^{r(\mathrm{y})}|\mathrm{v}_0^-(x)|^{s(\mathrm{y})}\mathrm{dy}\\
& \leq \lim_{n\to+\infty}\frac{t^{p^+}}{p^-}\int_{\mathcal{U}}\left(|\mathrm{Du}_n^-(\mathrm{y})|^{p(\mathrm{y})} + |\mathrm{Dv}_n^-(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}\\
& + \lim_{n\to+\infty}\frac{t^{q^+}}{q^-}\int_{\mathcal{U}}\left(\mu_1(\mathrm{y})|\mathrm{Du}_n^-(\mathrm{y})|^{q(\mathrm{y})} + \mu_2(\mathrm{y})|\mathrm{Dv}_n^-(\mathrm{y})|^{q(\mathrm{y})}\right)\mathrm{dy}\\
& - \lim_{n\to+\infty}\int_{\partial\mathcal{U}}\left(H_1(x, t\mathrm{u}_n^-(x)) + H_2(x, t\mathrm{v}_n^-(x))\right)\mathrm{d}\nu\\
& - \lim_{n\to+\infty}\frac{t^{q^-}}{q^+}\int_{\mathcal{U}}\left(\lambda_1|\mathrm{u}_n^-(x)|^{q(\mathrm{y})} + \lambda_2|\mathrm{v}_n^-(x)|^{q(\mathrm{y})}\right)\mathrm{dy}\\
& - \lim_{n\to+\infty}\frac{2t^{r^-+s^-}}{r^+ + s^+}\int_{\mathcal{U}}|\mathrm{u}_n^-(x)|^{r(\mathrm{y})}|\mathrm{v}_n^-(x)|^{s(\mathrm{y})}\mathrm{dy}\\
& \leq \lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(t\mathrm{u}_n^-, t\mathrm{v}_n^-)\\
& < \lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^-, \mathrm{v}_n^-) = \inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma^-_{\lambda_1,\lambda_2}.
\end{aligned}
$$

Hence

$$
J_{\lambda_1,\lambda_2}(t\mathrm{u}_0^-, t\mathrm{v}_0^-) < \inf_{(\mathrm{u},\mathrm{v})\in N_{\lambda_1,\lambda_2}} J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v}) = \gamma^-_{\lambda_1,\lambda_2}.
$$

This a contradiction, consequently, $(\mathrm{u}_n^-, \mathrm{v}_n^-) \to (\mathrm{u}_0^-, \mathrm{v}_0^-) \in W(\mathcal{U})$, and $\lim_{n\to+\infty} J_{\lambda_1,\lambda_2}(\mathrm{u}_n^-, \mathrm{v}_n^-) = J_{\lambda_1,\lambda_2}(\mathrm{u}_0^-, \mathrm{v}_0^-) = \gamma^-_{\lambda_1,\lambda_2}$. After that, we infer that $(\mathrm{u}_0^-, \mathrm{v}_0^-)$ is a minimization of J_{λ_1,λ_2} on $N^-_{\lambda_1,\lambda_2}$.

Proof of Theorem 1.1 From Theorem 4.1 and Theorem 4.2, there are $(u^+, v^+) \in N^+_{\lambda_1,\lambda_2}$ and $(u^-, v^-) \in N^-_{\lambda_1,\lambda_2}$ for every $\lambda_1 + \lambda_2 \in (0, min\,\{K_1, K_2\})$ such that

$$
J_{\lambda_1,\lambda_2}(\mathrm{u}_0^-, \mathrm{v}_0^-) = \inf_{(\mathrm{u},\mathrm{v})\in N^-_{\lambda_1,\lambda_2}}(\mathrm{u},\mathrm{v}) \text{ and } J_{\lambda_1,\lambda_2}(\mathrm{u}_0^+, \mathrm{v}_0^+) = \inf_{(\mathrm{u},\mathrm{v})\in N^+_{\lambda_1,\lambda_2}}(\mathrm{u},\mathrm{v}).
$$

Then the system (1) admits $(\mathrm{u}_0^-, \mathrm{v}_0^-) \in N^-_{\lambda_1,\lambda_2}$ and $(\mathrm{u}_0^+, \mathrm{v}_0^+) \in N^+_{\lambda_1,\lambda_2}$ as two solutions in $W(\mathcal{U})$; thanks to Lemma (3.2), it follows that $N^-_{\lambda_1,\lambda_2} \cap N^+_{\lambda_1,\lambda_2} = \emptyset$. Then $(\mathrm{u}_0^-, \mathrm{v}_0^-) \neq (\mathrm{u}_0^+, \mathrm{v}_0^+)$. Following this, we show that $(\mathrm{u}_0^\pm, \mathrm{v}_0^\pm)$ are non-negative in $\mathcal{U}$. For that, we introduce the truncation function $h_{+,i}(\mathrm{y}, s) : \partial\mathcal{U} \times \mathbb{R} \to \mathbb{R}$ defined by

$$
h_{+,i}(\mathrm{y}, s) = h_i(\mathrm{y}, s) \text{ if } s > 0, \text{ and } h_{+,i}(\mathrm{y}, s) = 0, \text{ if } s < 0, \text{ with } i = 1, 2.
$$

We set $H_{+,i}(\mathrm{y},s)=\int_0^s h_i(\mathrm{y},t)dt$, and the C^1-functional $J^+_{\lambda_1,\lambda_2}:W\to\mathbb{R}$ is given by

$$\begin{aligned}&J_{\lambda_1,\lambda_2}(\mathrm{u},\mathrm{v})\\&=\int_{\mathcal{U}}\frac{1}{p(\mathrm{y})}\left(|\ \mathrm{Du}(\mathrm{y})|^{p(\mathrm{y})}+|\mathrm{Dv}(\mathrm{y})|^{p(\mathrm{y})}\right)\mathrm{dy}+\int_{\mathcal{U}}\frac{1}{q(\mathrm{y})}\mu_1(\mathrm{y})|\mathrm{Du}(\mathrm{y})|^{q(\mathrm{y})}\\&+\int_{\mathcal{U}}\frac{1}{q(\mathrm{y})}|\mu_2(\mathrm{y})\mathrm{Dv}(\mathrm{y})|^{q(\mathrm{y})}\mathrm{dy}-\int_{\partial\mathcal{U}}H_{+,1}(\mathrm{y},\mathrm{u}(\mathrm{y}))+H_{+,2}(\mathrm{y},\mathrm{v}(\mathrm{y}))\mathrm{d}\nu.\end{aligned}$$

Then by Proposition 2.3, we get that for all $(\mathrm{u}_0^-,\mathrm{v}_0^-)=\min(0,(\mathrm{u},\mathrm{v}))$,

$$\begin{aligned}0&=\left\langle J^+_{\lambda_1,\lambda_2}(u_-,v_-),(u_-,v_-)\right\rangle\\&\geq p^-\rho_{p(.)}(Du_-,Dv_-)+\frac{\mu_0}{D^{p^+}(c+1)^{p^+}}\rho_{q(.)}(u_-,v_-)\\&\geq\rho_{p(.)}(u_-,v_-)\\&\geq\|(u_-,v_-)\|^{p^-}.\end{aligned}$$

Hence $\|(u_-,v_-)\|^{p^-}=0$ and thus, $(\mathrm{u},\mathrm{v})=(u_+,v_+)$, then, choosing $(\mathrm{u},\mathrm{v})=(\mathrm{u}_0^-,\mathrm{v}_0^-)$ and $(\mathrm{u},\mathrm{v})=(\mathrm{u}_0^+,\mathrm{v}_0^+)$, we conclude $(\mathrm{u}_0^\pm,\mathrm{v}_0^\pm)$ is a non-negative solution of system (1).

5 Conclusion

The Nehari manifold method is a powerful variational tool for proving the existence (and sometimes multiplicity) of solutions to nonlinear coupled elliptic systems under Neumann boundary conditions. Its effectiveness arises from converting the PDE problem into a constrained minimization problem in an appropriate Sobolev space. The method filters out trivial or non-physical solutions by exploiting the geometry of the energy functional and the nonlinearity.

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The Duffing Oscillator: Metamorphoses of 1 : 2 Resonance and Its Interaction with the Primary Resonance

J. Kyzioł and A. Okniński *

Politechnika Świętokrzyska, Al. 1000–lecia PP 7, 25-314 Kielce, Poland

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Abstract: We investigate the 1 : 2 resonance in the periodically forced asymmetric Duffing oscillator, it is created via the period-doubling of the primary 1 : 1 resonance or forming independently and coexisting with the primary resonance. We compute the steady-state asymptotic solution – the amplitude-frequency response function. Working in the framework of differential properties of implicit functions, we discover and describe complicated metamorphoses of the 1 : 2 resonance and its interaction with the primary resonance.

Keywords: *Duffing equation; resonances; singularities; metamorphoses.*

Mathematics Subject Classification (2020): 34C05, 34C25, 34E05, 37G35, 70K20, 70K30, 70K50.

1 Introduction and Motivation

A period-doubling cascade of bifurcations is a typical route to chaos in nonlinear dynamical systems. A generic example is the asymmetric Duffing oscillator governed by the non-dimensional equation

$$\ddot{y} + 2\zeta\dot{y} + \gamma y^3 = F_0 + F\cos(\Omega t), \tag{1}$$

which has a single equilibrium and a one-well potential [1], where ζ, γ, F_0, F are parameters and Ω is the angular frequency of the periodic force.

Szemplińska-Stupnicka elucidated the period-doubling scenario in the dynamical system (1) in a series of far-reaching papers [2–4]; see also [1] for a review and further results.

* Corresponding author: mailto:fizao@tu.kielce.pl

The main idea introduced in [2] consists in perturbing the main steady-state (approximate) asymptotic solution of Eq.(1)

$$y_0(t) = A_0 + A_1 \cos(\Omega t + \theta) \tag{2}$$

as

$$y(t) = y_0(t) + B \cos\left(\tfrac{1}{2}\Omega t + \varphi\right), \tag{3}$$

substituting $y(t)$ into Eq.(1) and considering the condition $B \neq 0$. In papers [1–4], the authors found several conditions guaranteeing the formation and stability of solution (3) and used them to study the period-doubling phenomenon.

In our recent work, we studied the period-doubling scenario using the period-doubling condition determined in [1–4] as an implicit function. More precisely, using the formalism of differential properties of implicit functions [5, 6], we derived analytic formulas for the birth of period-doubled solutions [7].

The motivation of this work stems from two observations: (i) In some cases, 1 : 2 resonance, coexisting with 1 : 1 resonance, is not created via the period doubling of 1 : 1 resonance, and (ii) 1 : 2 resonance depends on the parameters in a more complicated way than the primary resonance.

Thus, the present work aims to study the metamorphoses of the 1 : 2 resonance coexisting with the primary 1 : 1 resonance.

The paper is structured as follows. Section 2 describes the amplitude-frequency response, an implicit function, for the 1 : 2 resonance for Eq.(1). In Section 3, we derive equations to compute singular points and vertical tangencies of the response function. Section 4 presents an example of the metamorphoses of 1 : 2 resonance based on computed singular points. Section 5 provides an example of similar metamorphoses for a different dynamical system, suggesting a greater generality of our results. We summarize our results in the last section.

2 The 1 : 2 Resonance: Steady-State Solution

Since the stable 1 : 2 resonance can coexist with the stable primary 1 : 1 resonance, we assume the following steady-state solution of Eq.(1):

$$y(t) = B_0 + B \cos\left(\tfrac{1}{2}\Omega t + \varphi\right), \tag{4}$$

which can be computed by proceeding as in [8, 9]. More exactly, we get

$$\frac{3}{2}\gamma B_0 B^2 + \gamma B_0^3 + \frac{3}{2}\gamma B_0 C^2 - F_0 + \frac{3}{4}\gamma B^2 C \cos 2\varphi = 0, \tag{5a}$$

$$\zeta B \Omega - 3\gamma B_0 B C \sin 2\varphi = 0, \tag{5b}$$

$$\frac{1}{4}B\Omega^2 - \frac{3}{4}\gamma B^3 - 3\gamma B_0^2 B - \frac{3}{2}\gamma B C^2 - 3\gamma B_0 B C \cos 2\varphi = 0, \tag{5c}$$

where

$$C = \frac{4F}{3\gamma B^2 - 4\Omega^2}, \quad \left(3\gamma B^2 - 4\Omega^2 \neq 0\right). \tag{5d}$$

We note that in papers [1,2], a form describing a combination of 1 : 1 and 1 : 2 resonances was assumed (see Eq.(8.5.20) in [1] and Eq. (8a) in [2]), and thus, different equations for the asymptotic solution were obtained.

Assuming $B \neq 0$, we get from Eqs.(5b), (5c),

$$\begin{aligned} S_1\left(B_0, B, \Omega; \zeta, \gamma, F\right) &= \zeta^2\Omega^2 + \left(\tfrac{1}{4}\Omega^2 - \tfrac{3}{4}\gamma B^2 - 3\gamma B_0^2 - \tfrac{3}{2}\gamma C^2\right)^2 \\ &\quad -9\gamma^2 B_0^2 C^2 = 0. \end{aligned} \tag{6}$$

Moreover, equations (5a) and (5c) lead to

$$\begin{aligned} S_2\left(B_0, B, \Omega; \zeta, \gamma, F_0, F\right) &= -B^2\Omega^2 + 3B^4\gamma - 12B^2\gamma B_0^2 + 6B^2\gamma C \\ &\quad -16\gamma B_0^4 - 24\gamma B_0^2 C^2 + 16 B_0 F_0 = 0. \end{aligned} \tag{7}$$

Equation (7) is quadratic concerning B_0^2. Therefore, we solve this equation for B_0^2,

$$\begin{aligned} &B_0^2 = -\tfrac{1}{4}B^2 + \tfrac{1}{12\gamma}\Omega^2 \pm \frac{\sqrt{f\left(B, \Omega; \zeta, \gamma, F\right)}}{3\gamma\left(3\gamma B^2 - 4\Omega^2\right)^2},\ \left(\gamma > 0,\ 3\gamma B^2 - 4\Omega^2 \neq 0\right) \\ &f\left(B, \Omega; \zeta, \gamma, F\right) = -81\Omega^2\zeta^2\gamma^4 B^8 + 108\gamma^3\left(4\Omega^4\zeta^2 - 3F^2\gamma\right)B^6 \\ &-108\Omega^2\gamma^2\left(8\Omega^4\zeta^2 - 9F^2\gamma\right)B^4 + 96\gamma\Omega^4\left(8\Omega^4\zeta^2 - 9F^2\gamma\right)B^2 \\ &-256\zeta^2\Omega^{10} + 192F^2\gamma\Omega^6 - 576F^4\gamma^2. \end{aligned} \tag{8}$$

We substitute the following expression for B_0:

$$B_0\left(B, \Omega\right) = \sqrt{-\frac{1}{4}B^2 + \frac{1}{12\gamma}\Omega^2 + \frac{\sqrt{f\left(B, \Omega; \zeta, \gamma, F\right)}}{3\gamma\left(3\gamma B^2 - 4\Omega^2\right)^2}} \tag{9}$$

to Eq.(7) (it turns out that we have to choose the plus sign) to get a complicated but valuable implicit non-polynomial function $L\left(B, \Omega; \zeta, \gamma, F_0, F\right) = 0$,

$$L\left(B, \Omega; \zeta, \gamma, F_0, F\right) = S_2\left(B_0\left(B, \Omega\right), B, \Omega; \zeta, \gamma, F_0, F\right). \tag{10}$$

3 Vertical Tangencies and Singular Points

Equations for vertical tangencies read

$$L\left(B, \Omega; \zeta, \gamma, F_0, F\right) = 0, \tag{11a}$$

$$\frac{\partial L\left(B, \Omega; \zeta, \gamma, F_0, F\right)}{\partial B} = 0, \tag{11b}$$

while equations for singular points are

$$L\left(B, \Omega; \zeta, \gamma, F_0, F\right) = 0, \tag{12a}$$

$$\frac{\partial L\left(B, \Omega; \zeta, \gamma, F_0, F\right)}{\partial B} = 0, \tag{12b}$$

$$\frac{\partial L\left(B, \Omega; \zeta, \gamma, F_0, F\right)}{\partial \Omega} = 0. \tag{12c}$$

Equations (11), (12) can be solved numerically, yet simplify greatly for $B = 0$.

3.1 Vertical tangencies, $B = 0$

We check that

$$\left[\partial L\left(B, \Omega; \zeta, \gamma, F_0, F\right)/\partial B\right]_{B=0} = 0. \tag{13}$$

Therefore, we obtain a simplified equation for vertical tangencies

$$L\left(0,\Omega;\zeta,\gamma,F_0,F\right)=0. \tag{14}$$

We can solve equation (14) for Ω, getting

$$\begin{aligned} f_1\left(\Omega;\zeta,\gamma,F\right) &= \Omega^{12}+16\zeta^2\Omega^{10}-12\gamma F^2\Omega^6+36\gamma^2F^4=0, && (15a)\\ f_2\left(\Omega;\zeta,\gamma,F_0,F\right) &= \sum_{k=0}^{18} a_k\Omega^{2k}=0, && (15b)\end{aligned}$$

and obtaining non-zero coefficients a_k as shown in Table 1 below.

For example, to find vertical tangencies, we can choose values of ζ, γ, F_0, and F, solve Eq.(15b), and accept all solutions with $\Omega > 0$. Alternatively, we set ζ, γ, and F_0 and solve Eq.(15a), accepting solutions with $\Omega > 0$.

k	a_k
18	1
17	$48\zeta^2$
16	$768\zeta^4$
15	$36\gamma F^2-3456\gamma F_0^2+4096\zeta^6$
14	$1152\zeta^2\gamma F^2+165\,888\zeta^2\gamma F_0^2$
13	$9216\gamma F^2\zeta^4$
12	$756\gamma^2F^4-248\,832\gamma^2F^2F_0^2+2985\,984\gamma^2F_0^4$
11	$24\,192\zeta^2\gamma^2F^4+1990\,656\gamma^2F^2F_0^2\zeta^2$
10	$193\,536\gamma^2F^4\zeta^4$
9	$3456\gamma^3F^6-2239\,488\gamma^3F^4F_0^2$
8	$165\,888\gamma^3F^6\zeta^2$
6	$-31\,104\gamma^4F^8+4478\,976\gamma^4F_0^2F^6$
5	$2488\,320\gamma^4F^8\zeta^2$
3	$-933\,120F^{10}\gamma^5$
0	$4665\,600F^{12}\gamma^6$

Table 1: Non-zero coefficients a_k of the polynomial (15b).

3.2 Singular points, $B = 0$

Moreover, we can significantly simplify the equation for singular points (12) in the case $B = 0$. We note that for $B = 0$, Eq.(14) solves first two of equations (12), thus to find a solution of Eq.(12c), it suffices to demand that the equation (15b) has a double root (equation (15a) has no physical double roots). To this end, we request that the discriminant of the polynomial $f_2(\Omega)$ vanishes or solve the equivalent set of equations

$$\begin{aligned} f_2\left(\Omega;\zeta,\gamma,F_0,F\right) &= 0, && (16a)\\ \frac{\partial f_2\left(\Omega;\zeta,\gamma,F_0,F\right)}{\partial\Omega} &= 0. && (16b)\end{aligned}$$

By solving Eqs.(16) for F_0, F, we obtain clear and simplified equations for singular points, making the computations easier,

$$p\left(Z\right)=d_{12}Z^{12}+d_{10}Z^{10}+d_8Z^8+d_6Z^6++d_4Z^4+d_2Z^2+d_0=0, \tag{17}$$

$$d_{12} = 466\,560\gamma^6,\ d_{10} = -233\,280\gamma^5\Omega^2,\ d_8 = 20\,736\gamma^4\Omega^2\left(2\Omega^2 + 11\zeta^2\right),$$
$$d_6 = -1728\gamma^3\Omega^4\left(2\Omega^2 + 41\zeta^2\right),\ d_4 = 54\gamma^2\Omega^4\left(3\Omega^4 + 112\zeta^2\Omega^2 + 448\zeta^4\right),$$
$$d_2 = -3\gamma\Omega^6\Omega^4 + 72\zeta^2\Omega^2 + 896\zeta^4,\ d_0 = 4\zeta^2\Omega^6\left(\Omega^4 + 20\zeta^2\Omega^2 + 64\zeta^4\right),$$

where $Z = \dfrac{F}{\Omega^2}$, and

$$q\,(T) = e_2T^2 + e_0 = 0, \tag{18}$$

$$e_2 = 1728\gamma\Omega^6 + 41\,472\gamma\zeta^2\Omega^4 + 1603\,584\gamma\zeta^4\Omega^2 + 22\,118\,400\gamma\zeta^6,$$
$$\begin{aligned} e_0 &= -\Omega^{14} + \left(-534Z^2\gamma + 592\zeta^2\right)\Omega^{12} + \begin{pmatrix} 15\,768Z^4\gamma^2 - 31\,968Z^2\zeta^2\gamma \\ +20\,128\zeta^4 \end{pmatrix}\Omega^{10} \\ &+ \left(-247\,968Z^6\gamma^3 + 627\,552Z^4\zeta^2\gamma^2 - 499\,008Z^2\zeta^4\gamma + 195\,072\zeta^6\right)\Omega^8 \\ &+ \begin{pmatrix} 1736\,640Z^8\gamma^4 - 3805\,056Z^6\zeta^2\gamma^3 + 2596\,608Z^4\zeta^4\gamma^2 \\ -958\,464Z^2\zeta^6\gamma + 458\,752\zeta^8 \end{pmatrix}\Omega^6 \\ &+ \begin{pmatrix} -4199\,040Z^{10}\gamma^5 + 9642\,240Z^8\zeta^2\gamma^4 + 7617\,024Z^6\zeta^4\gamma^3 \\ -40\,255\,488Z^4\zeta^6\gamma^2 + 16\,465\,920Z^2\zeta^8\gamma \end{pmatrix}\Omega^4 \\ &+ \left(-19\,906\,560Z^{10}\zeta^2\gamma^5 - 76\,308\,480Z^8\zeta^4\gamma^4 + 23\,224\,320Z^6\zeta^6\gamma^3\right)\Omega^2 \\ &-99\,532\,800Z^{10}\zeta^4\gamma^5, \end{aligned}$$

where $T = F_0\Omega$.

4 Numerical Verification

In this section, we shall compute singular points and vertical tangencies for the implicit function $L\,(B, \Omega; \zeta, \gamma, F_0, F) = 0$, Eq.(10), for chosen values of ζ, γ, and Ω. In the case $B = 0$, we use Eqs. (16) in the reduced forms (17) and (18) to compute singular points and apply Eqs.(15) to compute vertical tangencies. Then, in the case $B \neq 0$, we use Eqs.(12) and (11), respectively.

In what follows, we arbitrarily choose $\zeta = 0.09$, $\gamma = 0.3$, $\Omega = 1.5$.

4.1 Singular points and vertical tangencies, $B = 0$

Thus, we choose the singular point as $(B,\, \Omega) = (0,\, 1.5)$. We must compute the parameters F_0 and F, for which the selected point is singular.

Therefore, for $\zeta = 0.09$, $\gamma = 0.3$, $\Omega = 1.5$, and $B = 0$, we solve the equation $p\,(Z) = 0$ with $p\,(Z)$ defined in Eq.(17), obtaining four positive roots, $Z = 0.198\,445,\ 0.509\,084,\ 1.095\,980,\ 1.259\,811$. We check, however, that only $Z = 1.095\,980$ leads to a solution of (12). Since $Z = F/\Omega^2$, we get, for $Z = 1.095\,980$, $F = 2.465\,955$. We now solve $q\,(T) = 0$ with $q\,(T)$ defined in Eq.(18), obtaining, for $Z = 1.095\,980$, one positive root $T = 0.112\,203$. Since $T = F_0\Omega$, we get $F_0 = 0.074\,802$.

Now, we check that for $\gamma = 0.3$, $F_0 = 0.074\,802$, $F = 2.465\,955$, equations (12), solved numerically, yield indeed $\zeta = 0.09$ and an isolated singular point $(B,\, \Omega) = (0,\, 1.5)$; see Fig.1 and Subsection 4.4 for description.

To find vertical tangencies, we set, for example, $\zeta = 0.082$ and use just computed parameter values $\gamma = 0.3$, $F_0 = 0.074\,802$, $F = 2.465\,955$. Solving equation (15b), we get $(B,\, \Omega) = (0,\, 1.474\,612)$, $(0,\, 1.527\,914)$ (all solutions of Eq.(15a) are complex); see red boxes in Fig.1.

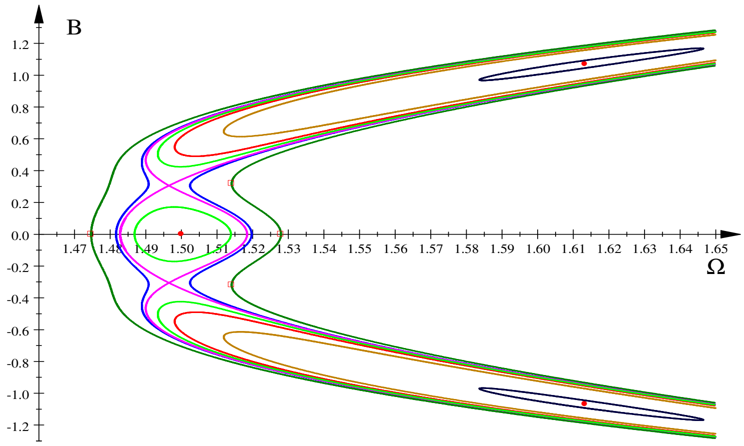


Figure 1: Sequential metamorphoses of amplitude-frequency implicit function $L(B, \Omega; \zeta, \gamma, F_0, F) = 0$, describing $1:2$ resonance.

4.2 Singular points and vertical tangencies, $B \neq 0$

We work with parameter values computed in the preceding subsection, i.e., $\gamma = 0.3$, $F_0 = 0.074\,802$, $F = 2.465\,955$. We have shown in the prior section that equations (12) have, for $\zeta = 0.09$, an isolated singular point $(B, \Omega) = (0, 1.5)$.

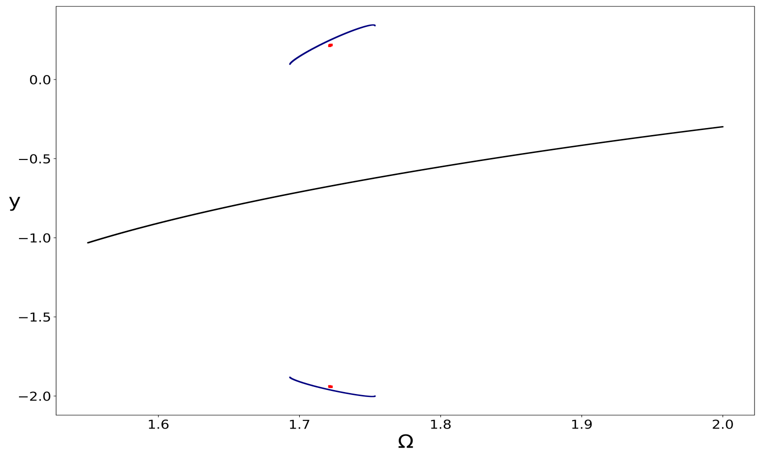
Moreover, equations (12) have other singular points for $\gamma = 0.3$, $F_0 = 0.074\,802$, $F = 2.465\,955$, and $B \neq 0$. Solving numerically Eqs.(12), we get (i) $\zeta = 0.086\,504$ and a pair of self-intersections $(B, \Omega) = (\pm 0.305\,755, 1.\,496\,498)$, (ii) a pair of isolated points, $\zeta = 0.108\,010$, $(B, \Omega) = (\pm 1.\,069\,257, 1.\,613\,277)$, see Fig.1 where we show all singular points.

We also compute vertical tangencies for $B \neq 0$. Solving equations (11a) and (11b) numerically for γ, F_0, F listed above and $\zeta = 0.082$, we get $(B, \Omega) = (\pm 0.318\,833, 1.\,514\,052\,)$; see red boxes in Fig.1.

4.3 Bifurcation diagrams

We have computed bifurcation diagrams solving Eq.(1) numerically – obtaining $y(t)$ as a function of Ω; for computational details, see Appendix A. Comparison with Fig.1 reveals which branches are stable. Colors in the bifurcation diagrams correspond to those in Fig.1, resonance $1:1$ is black.

We show, in Fig.2, the birth of $1:2$ resonance ($\zeta = 0.114\,69$, red; top figure), its growth ($\zeta = 0.114$, navy; top figure), and one period-doubling ($\zeta = 0.0977$, sienna; bottom figure).



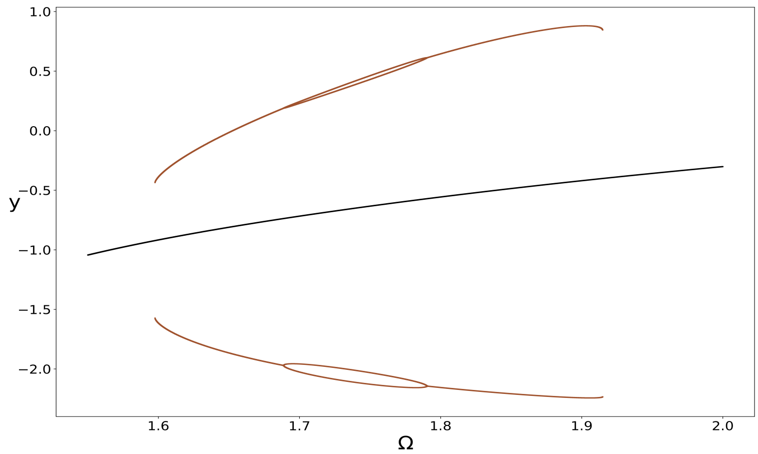


Figure 2: Bifurcation diagrams, $\zeta = 0.114\,69$ (two red boxes), $\zeta = 0.114$ (navy) – top, $\zeta = 0.0977$ (sienna) – bottom.

Figure 3 displays the first point of contact of $1 : 2$ resonance ($\zeta = 0.094\,185$, red circle; top figure) with $1 : 1$ resonance (black).

This singular point grows into an oval ($\zeta = 0.091$, green; bottom figure); a double period-doubling cascade forms and breaks.

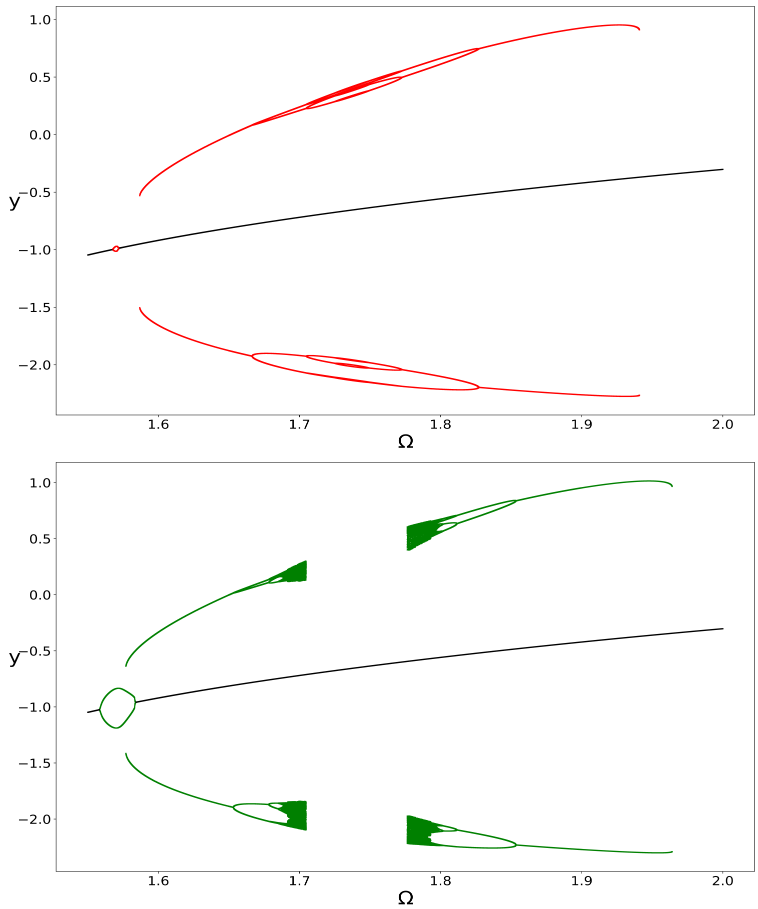


Figure 3: Bifurcation diagrams, $\zeta = 0.094\,185$ (top), $\zeta = 0.091$ (bottom).

Finally, in Fig.4, we display the formation of a double self-intersection ($\zeta = 0.090\,14$, magenta; top figure) and its subsequent breaking ($\zeta = 0.90$, blue; bottom figure).

We note that resonance $1:1$ and left resonance $1:2$ merge (top figure), then connected branches split in another way (bottom figure).

Section 4.4 describes in detail the complicated metamorphoses of $1:2$ resonance and its interaction with $1:1$ resonance.

4.4 Description of transmutations of $1:2$ resonance

We describe the metamorphoses of $1:2$ resonance for $\gamma = 0.3$, $F_0 = 0.074\,802$, $F = 2.465\,955$ and descending values of ζ. In what follows, ζ_n denotes the values of ζ computed by integrating the Duffing equation (1) numerically, while ζ_a means values of ζ computed from the analytical condition (12) for singular points. We have computed the function $L\,(B, \Omega; \zeta, \gamma, F_0, F)$, Eq.(10), from the asymptotic solutions (4) and (5).

In Fig.1, the transmutations of function $L\,(B, \Omega; \zeta, \gamma, F_0, F)$ are shown, and in bifurcation diagrams, Figs 2, 3 and 4, the metamorphoses of the solutions of the Duffing equation (1) are presented.

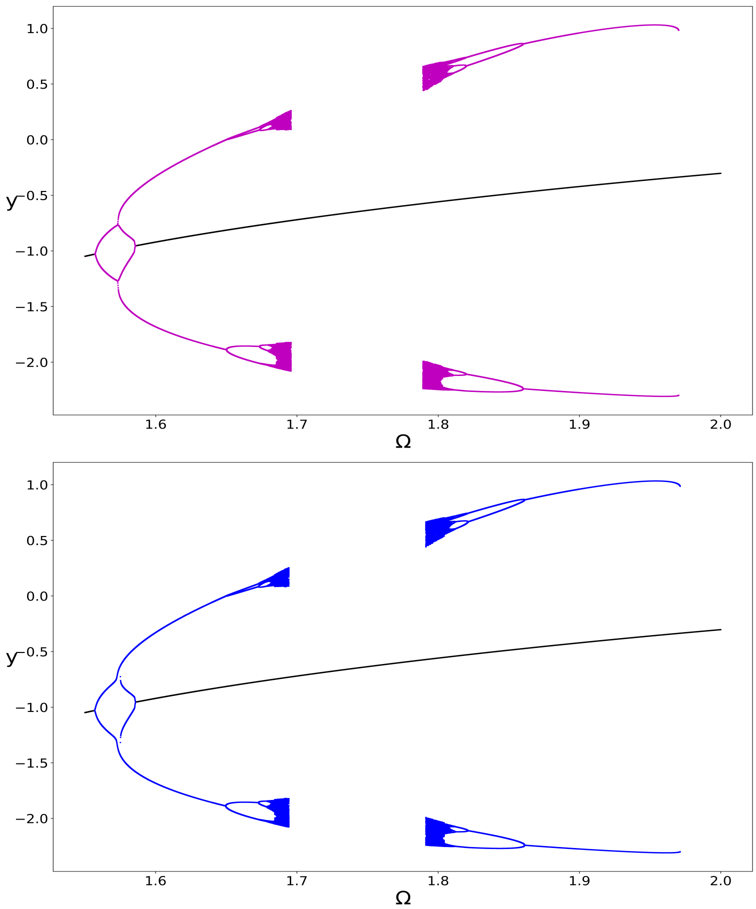


Figure 4: Bifurcation diagrams, $\zeta = 0.090\,14$ (top), $\zeta = 0.90$ (bottom).

Parameters in Fig.1 are $\gamma = 0.3$, $F_0 = 0.07480195$, $F = 2.465\,954$, and $\zeta_a = 0.108\,010$ (two red dots), $\zeta = 0.107\,2$ (navy), 0.095 (sienna), $\zeta_a = 0.09$ (red, a dot and two branches), $\zeta = 0.088$ (light green, an oval and two branches), $\zeta_a = 0.086\,504$ (magenta, two self-intersections), $\zeta = 0.086$ (blue), 0.082 (green).

We note good qualitative correspondence between predicted transmutations shown in Fig.1 and the metamorphoses of the solutions of the Duffing equation documented in Figs.2, 3, and 4.

We describe these changes as follows.

1. For $\zeta_n = 0.114\,69$, the $1:2$ resonance appears for the first time at $\Omega_n = 1.722$; see two red boxes (singular, two isolated points) in the top Fig.2. The corresponding analytical values are $\zeta_a = 0.108\,010$, $\Omega_a = 1.613$; see two red dots in Fig.1. Next, for descending values of ζ, the $1:2$ resonance grows and transforms rapidly.
   For $\zeta_n = 0.114$, $1:2$ resonance becomes larger; see navy lines in top Fig.2. Then, for $\zeta_n = 0.097\,7$, there is a first period-doubling of $1:2$ resonance; see two sienna branches in bottom Fig.2.

2. For $\zeta_n = 0.0941\,85$ ($\zeta_a = 0.09$), a new isolated point of $1:2$ resonance (red) appears on $1:1$ resonance (black) at $\Omega_n = 1.57$ ($\Omega_a = 1.5$); see small red circle in top Fig.3 or red dot in Fig.1. The primary $1:1$ resonance is black.
   It is the first contact of these resonances. Note that the $1:2$ resonance has been subject to more period-doubling.

3. The singular, isolated point gives rise to an oval – a period-doubling of $1:1$ resonance; see the light green line in Fig.1 and the green oval in bottom Fig.3. The $1:1$ resonance is black.
   There is still no other contact between the primary $1:2$ resonance and the $1:1$ resonance.
   There are already two whole cascades of period-doublings of $1:2$ resonance that have been disrupted and moved away; we call them left and right; see green lines in Fig.3).

4. For $\zeta_n = 0.090\,14$ ($\zeta_a = 0.086\,504$), there are two self-intersections (two singular points) at $\Omega_n = 1.573$ ($\Omega_a = 1.496$) and resonance $1:1$ and left resonance $1:2$ merge; see magenta lines in Fig.1 and top Fig.4. The right $1:2$ resonance stays separated. The primary $1:1$ resonance is black.

5. For $\zeta < \zeta_n = 0.090\,14$ ($\zeta_a < 0.086\,504$), connected branches split in another way; see blue lines in Fig.1 and Fig.4. The primary $1:1$ resonance (black) absorbs the left $1:2$ resonance with the whole cascade of period-doubling; there is also another branch of $1:1$ resonance, with one period-doubling. The right $1:2$ resonance stays unchanged.

Summing up, the separate $1:2$ resonance, after the first contact with $1:1$ resonance, splits into two parts, left and right, then the left resonance $1:2$ merges with the primary $1:1$ resonance and splits in another way.

5 Summary and Most Important Findings

Based on the amplitude-frequency steady-state implicit equation (10) computed for Eq.(1), we studied the metamorphoses of the resonance $1:2$ and its interaction with the primary resonance $1:1$.

Working within the formalism of the differential properties of implicit functions, we derived formulas to compute singular points of the amplitude-frequency function $L(B,\Omega;\zeta,\gamma,F_0,F)$ defined in (10), see Section 3. In Section 4, we have computed singular points for arbitrary parameters $\zeta = 0.09$, $\gamma = 0.3$, $\Omega = 1.5$.

It should be stressed that the dynamics of the initial equation (1) change in the neighborhood (in the parameter space) of singular points.

We also computed bifurcation diagrams by solving Eq.(1) numerically, see Figs.2, 3, and 4, and obtained good agreement with the amplitude-frequency profiles of Fig.1.

The most significant achievements of this work are:

1. The semi-analytic and numerical procedures to compute singular points of the amplitude-frequency implicit function (10) that indicate the birth of $1:2$ resonance, correspond to the first contact of resonances $1:2$ and $1:1$, and merging of $1:2$ and $1:1$ resonances.

2. Discovery and detailed description of complicated transmutations of 1 : 2 resonance that resulted in the first contact with the primary 1 : 1 resonance, disruption of the 1 : 2 resonance into two parts, left and right, merging of the left 1 : 2 resonance with the primary 1 : 1 resonance, and breaking again; the right 1 : 2 resonance effectively not evolving.

A Computational details

Nonlinear equations were solved numerically using the computational engine Maple 4.0 from Scientific WorkPlace 4.0. Figure 1 was plotted with the computational engine MuPAD 4.0 from Scientific WorkPlace 5.5. Bifurcation diagrams in Figs.2, 3, and 4 were computed by integrating numerically Eq.(1) running DYNAMICS [11], as well as our programs written in Pascal and Python [12].

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Implementation of Recurrent Neural Network and Kalman Filter Method to Predict Hypertension Case in East Java Province

T. Mahyuvi [1], I. Indasah [1*], S. Suwarto [2], B. M. Suhita [3], T. Herlambang [4]

[1] *Post Graduate Program of Public Health, Universitas STRADA Indonesia, Kediri, Indonesia.*
[2] *Master Program in Community Development, Universitas Sebelas Maret.*
[3] *Post Graduate Program of Nursery, Universitas STRADA Indonesia, Kediri.*
[4] *Departement of Information Systems, Universitas Nahdlatul Ulama Surabaya, Surabaya, Indonesia.*

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Abstract: An individual's health condition can be judged by their activities and lifestyle. People who live in big cities like Surabaya, tend to be more prone to stress if they do not manage it well. This high level of stress triggers the onset of hypertension or high blood pressure. Hypertension is a disease related to a person's blood pressure, where the blood pressure exceeds the limits of normal conditions. The number of people affected by this disease, especially in East Java province, is quite high with most sufferers being of productive age. Therefore, it is necessary to take systematic, scientific, and technology-based steps relevant in addressing this case. By utilizing advances in information technology, the increase in number of these cases can be monitored using prediction methods based on data analysis, statistics and mathematics. This research used the Recurrent Neural Network (RNN) and Kalman Filter methods to predict the increase in cases of hypertension in East Java. The simulation results showed that the Recurrent Neural Network (RNN) method produced the best error value (RMSE) of 0.0009, while the Kalman Filter method produced the best error value (RMSE) of 234.213.

Keywords: *hypertension; East Java; prediction; RNN; Kalman filter.*

Mathematics Subject Classification (2020): 62J05, 70-10, 90Bxx.

* Corresponding author: `mailto:indasah.strada@gmail.com`

1 Introduction

Hypertension often called high blood pressure is one type of dangerous disease occurring in Indonesia. According to WHO (World Health Organization), hypertension (high blood pressure) is defined as a medical condition occurring when the pressure in the blood vessels is equal to or more than 140/90 mmHg [1]. Hypertension occurs due to increased blood pressure in the arteries. Hypertension can increase the risk of various diseases such as heart failure, stroke, aneurysm, heart attack, and kidney damage [2].

According to the 2018 Riskesdas, the prevalence of hypertension in Indonesia was 34.11% with an estimated number of hypertension cases in Indonesia of 63,309,620 people, while the mortality rate in Indonesia due to hypertension was 427,218 deaths. East Java is in the sixth position. The highest number of patients in East Java is in Surabaya City, which has 313,960 residents [3]. If this situation is not well taken care of, it is likely that more and more people in East Java province will get hypertensive disease.

There are many ways to minimize the spread of hypertension, one of which is by using machine learning technology-based forecasting methods. Machine learning is a computing paradigm allowing computer systems to learn from previous data and experience without having to be explicitly programmed. This approach allows the system to find patterns, identify trends, and make decisions automatically, with the aim of solving problems or performing specific tasks [4]. Thus, the decision-making process by the respective party can be more focused and efficient.

In this study, the authors used the Recurrent Neural Network (RNN) method. Recurrent Neural Network (RNN) is a Neural Network (NN) having feedback to the neuron itself and other neurons so that the flow of information from the input has many directions (multi directional) [5]. In the previous research in 2024, this Recurrent Neural Network (RNN) method was used to forecast IDX30 blue chip stock prices [6]. The Kalman Filter is a method introduced by R.E Kalman. The Kalman Filter method is an algorithm that can be used to estimate discrete linear systems [7]. In the previous research in 2024, this Kalman Filter method was used to forecast chili prices in Ponorogo [8].

2 Research Methods

The time-series data used in this study come from the SIPTM (Non-Communicable Disease Information System) data of East Java province having 3 columns and 114 rows with a time span from 2020 to 2022. The data in this study were analyzed using the Python programming language with the dataset as in Table 1 below.

Year	Name of Regency/City	Number of Hypertension Cases
02/01/2020	Pacitan	170
11/01/2020	Ponorogo	115
25/01/2020	Trenggalek	302
02/02/2020	Tulungagung	74
11/02/2020	Blitar	448
25/02/2020	Kediri	189
02/03/2020	Malang	1235
11/03/2020	Lumajang	106
21/03/2020	Jember	278
29/03/2020	Banyuwangi	556
02/04/2020	Bondowoso	4429
...	...	...
18/12/2022	Batu	910

Table 1: Hypertension dataset of East Java province.

After obtaining and exploring the data, the next step is to carry out the stages of the research methodology one by one as shown in Figure 1 below.

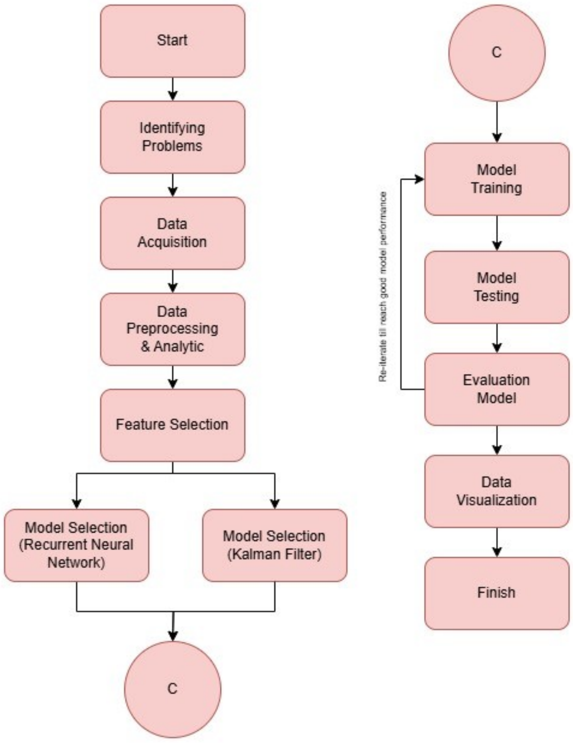

Figure 1: Research methods.

1. Problem Identification: This research raises a case study on the prediction of hypertension cases in East Java.

2. Data Acquisition: The data used in this study is secondary data sourced from SIPTM (Non-Communicable Disease Information System) of East Java province with a time span from 2020 to 2022.

3. Data Preprocessing and Analytics: Data preprocessing is basically used to see the initial condition of the dataset obtained from the source. At this stage, the dataset is identified starting from the data type to missing values that may occur.

4. Feature Selection: This research uses a univariate table model with 1 target column, with numerical data that has no empty/null values.

5. Model Selection: Recurrent Neural Network (RNN) is one part of deep networks, learning unsupervised by using previous data samples as a source of learning. In the RNN, there are recurrent connections occurring in each neuron and layer that can form cycles in the RNN architecture, and this makes it possible to model

time-related behavior such as time-series [9]. The mathematical function of the Recurrent Neural Network (RNN) method is as follows [10]- [11]:

$$\hat{Y} = f^0 \left(\beta_0 + \sum_{j=1}^{p} \left(\beta_j f^h \left(\gamma_{j0} + \gamma_{dj} + \sum_{i=1}^{q} \gamma_{ji} X_i \right) \right) \right), \tag{1}$$

where
$\hat{Y}$ is the output variable; β_0 is the bias weight at the hidden layer; β_j is the weight for the j-th hidden neuron, $j = 1, 2, \ldots, p$; γ_{j0} is the weight of the i-th input neuron to the j-th hidden neuron; γ_{dj} is the delay weight or context neuron; γ_{j0} is the bias weight at the input layer, $i = 1, 2, \ldots, n$; X_i is the input neuron, i $= \cos 1, 2, .., n$; $f^h(x)$ is the activation function at the hidden layer; $f^0(x)$ is the activation function at the output layer. Meanwhile, Kalman Filter is an assimilation method. Assimilation is a way of improving estimates by combining state space models with observational data [12]. Kalman Filter was first introduced by R.E. Kalman and is a combination of model and measurement algorithms. Below is the equation of the Kalman Filter algorithm.

System Model and Measurement Model
$x_{k+1} = A_k x_k + B_k u_k + G_k w_k$ $z_k = H_k x_k + v_k$ $x_0 \sim N(\bar{x}_0, P_{x_0}); w_k \sim N(0, Q_k); v_k \sim N(0, R_k)$
Initialization
$\hat{x}_0 = \bar{x}_0$ $p_0 = p_{x_0}$
Prediction stage
Estimation : $\hat{x}^-_{k+1} = A_k \hat{x} + B_k u_k$ Error Covariance : $P^-_k = A_k P_k A^T_k + G_k Q_k G^T_k$
Correction stage
Kalman Gain : $K_{k+1} = P^T_{k+1} H^T_{k+1} (H_{k+1} P^-_{k+1} H^T_{k+1} + R_{k+1})^{-1}$ Estimation : $\hat{x}_{k+1} = \hat{x}^-_{k+1} + K_{k+1}(z_{k+1} - H_{k+1}\hat{x}^-_{k+1})$ Error Covariance : $P_{k+1} = [I - K_{k+1} H_{k+1}] P^-_{k+1}$

Table 2: Kalman Filter Algorithm [13].

The prediction stage is influenced by the dynamics of the system by predicting the state variables using the state variable estimation equation, and the level of accuracy is calculated using the error covariance equation. At the correction stage, the state variable estimation results obtained in the prediction stage are corrected using the measurement model. One part of this stage is to determine the Kalman Gain matrix used to minimize the error covariance.

6. Model Training: At this stage, the selected values of the features are trained using the Recurrent Neural Network (RNN) and Kalman Filter based on the division of training data and testing data to obtain the error value.

7. Model Testing: Model testing of learning outcomes with the prepared test data.

8. Model Evaluation: At the evaluation stage, the model trained and tested is calculated for accuracy based on the resulting error value. This research uses the Root

Mean Square Error (RMSE) method to calculate the error value generated by the model. The function of the Root Mean Square Error (RMSE) is as follows:

$$RMSE = \sqrt{\frac{\Sigma\left(y_i - \hat{y}_i\right)^2}{n}}, \tag{2}$$

where n is the quantity of data; y_i is the actual value at the i-th data; $\hat{y}_i$ is the predicted value at the i-th data.

3 Result and Discussion

From the testing simulation conducted with the division in proportion of training data and testing data, the following results can be seen in Figures 2–4.

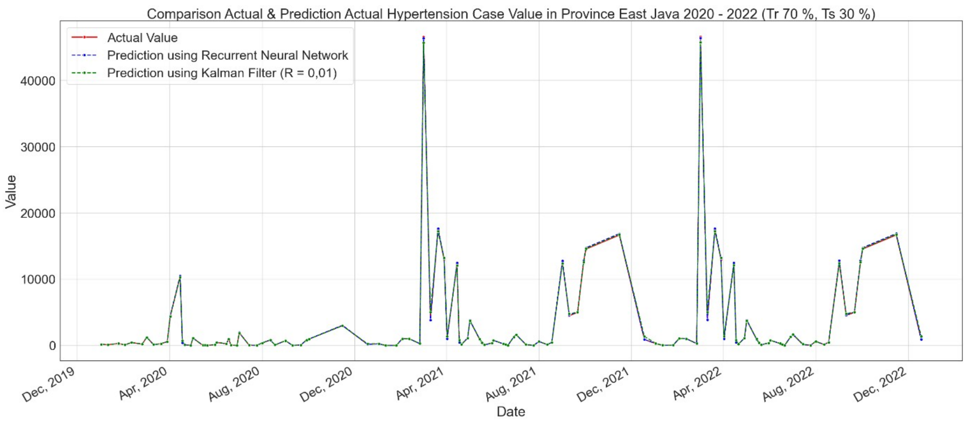

Figure 2: First Simulation Plot of RNN - Kalman Filter (70% training data, 30% testing data).

The first simulation was conducted using the Recurrent Neural Network (RNN) method with the parameters of the hidden layer = 50 and epoch = 100 and a ratio of 70% training data and 30% testing data. The results of the first simulation provide a good and accurate prediction value as shown by the Recurrent Neural Network (RNN) prediction graph (blue line) touching the actual value graph (red line). The resulting RMSE value is 0.0020. As for the prediction results of the Kalman Filter method (green line) with the parameter value R = 0.01, it produces an RMSE value of 234.213. The prediction results of the two methods show that the Recurrent Neural Network (RNN) method yields better prediction results.

The second simulation was conducted using the Recurrent Neural Network (RNN) method with the parameters of the hidden layer = 50 and epoch = 100 and a ratio of 80% training data and 20% testing data. The second simulation results provide a good and accurate prediction value as shown by the Recurrent Neural Network (RNN) prediction graph (blue line) touching the actual value graph (red line). The resulting RMSE value is 0.0009. As for the prediction results of the Kalman Filter method (green line) with the parameter value R = 0.01, it produces an RMSE value of 234.228. The prediction results of the two methods show that the Recurrent Neural Network (RNN) method yields better prediction results under the condition that the error value decreases considerably.

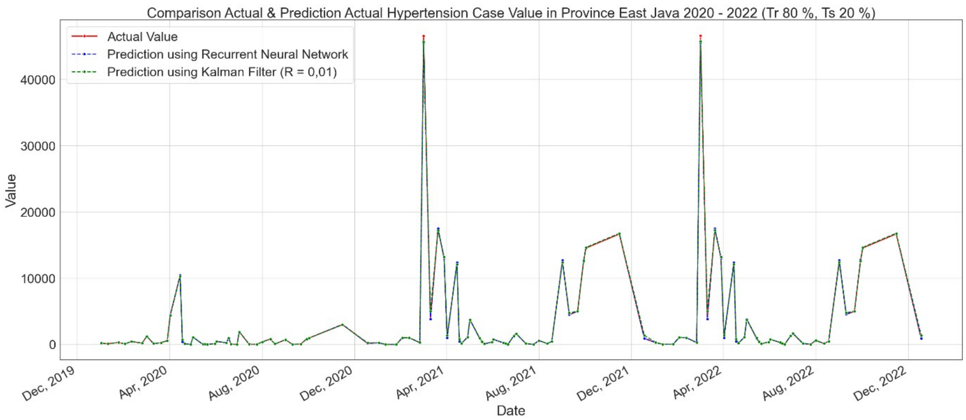

Figure 3: Second Simulation Plot of RNN - Kalman Filter (80% training data, 20% testing data).

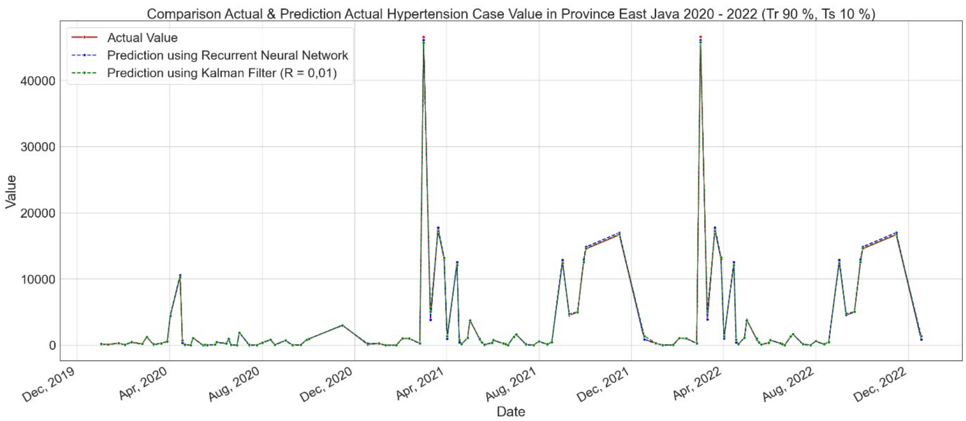

Figure 4: Third Simulation Plot of RNN - Kalman Filter (90% training data, 10% testing data).

The third simulation was conducted using the Recurrent Neural Network (RNN) method with the parameters of the hidden layer = 50 and epoch = 100 and a ratio of 90% training data and 10% testing data. The third simulation results provide a good and accurate prediction value as shown by the Recurrent Neural Network (RNN) prediction graph (blue line) touching the actual value graph (red line). The resulting RMSE value is 0.0037. As for the prediction results of the Kalman Filter method (green line) with the parameter value R = 0.01, it produces an RMSE value of 234.249. The prediction results of the two methods show that the Recurrent Neural Network (RNN) method still yields better prediction results even though the error value increases as compared to the first and second simulations.

Based on the simulation results obtain above, a recapitulation of the simulation results of the Recurrent Neural Network (RNN) method with the parametersof the hidden layer

= 50 and epoch = 100 and Kalman Filter with a value of R = 0.01, with a parameter value of R = 0.01 can be seen in the table below.

Comparison of Training Data (DTr) and Testing Data (DTs)	RMSE Value by Recurrent Neural Network (RNN)	RMSE Value by Kalman Filter R = 0.01
70% : 30%	0.0020	234.213
80% : 20%	0.0009	234.228
90% : 10%	0.0037	234.249

Table 3: RMSE Value Comparison.

The above table shows the simulation results generated by the Recurrent Neural Network (RNN) method with hidden layer parameters = 50 and epoch = 100 and the Kalman Filter method with a value of R = 0.01, from the first simulation to the third simulation. For the overall simulation results of the Recurrent Neural Network (RNN) method with hidden layer parameters = 50 and epoch = 100, there are dynamics of the resulting error value with an error value range of 0.0011 to 0.0028. For the Kalman Filter method with a value of R = 0.01, the range of error values is from 0.005 to 0.021.

Based on the results of the data analysis and simulation above, it can be seen that the target column containing the value of the incidence rate of hypertension disease spread in districts / cities in East Java is predicted to be close to the actual value by both methods. Thus, it is easier for the respectibe stakeholders to make strategic decisions.

In addition, this shows that the Recurrent Neural Network (RNN) and Kalman Filter methods have a high accuracy and are able to handle univariate data. Thus, these methods can be recommended for using in other prediction case studies with more complex datasets.

4 Conclusion

Based on the results of the simulations conducted, it can be concluded that the results of the first to third simulations using the Recurrent Neural Network (RNN) from the first simulation to the third simulation with a parameter value of hidden layer = 50 and epoch = 100 and Kalman Filter with a value of R = 0.01, provided a satisfactory prediction error value (RMSE). In the second simulation, the Recurrent Neural Network (RNN) method yields the best error value (RMSE) of 0.0009 with a division of 80% training data and 20% testing data. The best error value by the Kalman Filter method with a value of R = 0.01, was obtained in the first simulation with a division of 70% training data and 30% testing data, with an RMSE value of 234.213. These results prove that the Recurrent Neural Network (RNN) method provides good and consistent prediction results.

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Generalized n-Characteristic, Coincidence and Fixed Point Theorems for a Class of Pairs of Morphisms

C. Matmat *

Department of Mathematics, University Constantine 1 Frères Mentouri, BP 325 Route Ain El Bey, Constantine, Algeria.

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Abstract: This paper is devoted to the construction and study of a topological invariant for a class of pairs of morphisms $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$, where Top denotes the category of Hausdorff topological spaces and continuous single valued maps and X, Y, Z represent subsets of $\mathbb{R}^{n+1}$ such that X, Y contain the sphere S^n. This invariant termed as a generalized n-characteristic of the pair (f, g), is derived using homotopy methods serving as a valuable tool in coincidence point theory. The paper establishes several properties of this invariant, extends it to a class of admissible multivalued mappings, and presents a fixed point theorem among its results.

Keywords: *homotopy; topological invariant; n-connected spaces; multivalued mappings.*

Mathematics Subject Classification (2020): 70K99, 93B25.

1 Introduction

The concept of topological invariants, degrees, characteristics, or generalized characteristics, has been extensively explored by numerous authors for various classes of single-valued and multivalued mappings (see, for example, [7, 10, 13]. By exploration of various topological techniques, this concept serves as a powerful tool in analyzing and proving results in fixed point theory. This provides practical applications across diverse fields such as nonlinear analysis [6], economics [3], biology [5] and physics [9]. In some cases, they are very useful to prove the existence of solutions for linear or semi-linear dynamical systems [8].

Moreover, topological invariants are highly instrumental for the study of bifurcations and nonlinear dynamical systems (see [9]). In the case when nonlinearities are not smooth enough, they help to identify fixed points and their stability, which are critical

* Corresponding author: mailto:c.matmat@yahoo.fr

in understanding bifurcations. Also, they offer deep insights into the global structure and complexity of systems, making it possible to fully comprehend qualitative transitions as parameters change. With these invariants, researchers can not only detect and classify bifurcations, but also develop predictive theories and models for a wide range of dynamical phenomena.

Let $S^n \subset X \subseteq Y \subset \mathbb{R}^{n+1}$, where S^n is the unit sphere of $\mathbb{R}^{n+1}$ and X is an n-connected space, and let Z be an arbitrary non empty subset of $\mathbb{R}^{n+1}$.

In the present paper, we build a topological invariant for a class of pairs of morphisms $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$ such that the maps f, g do not coincide on the sphere S^n. This class of morphisms is denoted M_Z. This construction can be seen as a natural extension of the characteristic $\chi_{S^n}(f)$ defined on the sphere S^n (see [4], [14]). Indeed, recall briefly that the degree of continuous maps f in $Mor_{Top}(S^n, S^n)$ is closely related to the group $\pi_n(S^n)$ and it is often used in analysis. Consider a continuous map $f : S^n \to S^n$ and let γ_n be a generator of the group $\pi_n(S^n) \simeq \mathbb{Z}$ (see [16]), then $f_*(\gamma_n) = \alpha\gamma_n$, where $\alpha \in \mathbb{Z}$. The number α is called the degree of f and it is denoted by $deg f$.

Since the space $\mathbb{R}^{n+1} \setminus \{\theta\}$ is homotopy equivalent to the sphere S^n, one may define the degree of morphisms f in $Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. It is called the characteristic (or rotation) of the vector field f, and it is denoted by $\chi_{S^n}(f)$. Using algebraic and geometric topological methods, different generalizations for topological degree are given for single valued maps and multivalued maps (see [13, 14]).

This paper is divided into three sections. After the introduction, in Section 2, we build and develop some properties of the generalized n-characteristic, in particular, we show that it is a homotopy invariant. We also show that it can be used in coincidence and fixed points theories. In Section 3, we build and study a generalized n-mult characteristic for a class of multivalued mappings. As application, a fixed point theorem for this class is provided.

2 Generalized n-Characteristic of a Pair of Morphisms

Let Top be the category of T_2-topological spaces and continuous single valued maps and G_d be the category of graded groups and homomorphisms of degree zero. In what follows, $\pi : Top \to G_d$ stands for the covariant functor of the homotopy which assigns to a T_2-topological space X a graded group $\{\pi_n(X)\}_{n\geq 0} \in Obj(G_d)$, and to a given morphism $f \in Mor_{Top}(X, Y)$ a homomorphism of degree zero $\pi_n(f) \in Mor_{G_d}(\{\pi_n(X)\}_{n\geq 0}, \{\pi_n(Y)\}_{n\geq 0})$.

For given topological spaces X, Y in the category Top, the set of all continuous maps from X to Y is denoted by $C(X, Y)$. For the sake of easy reference, we recall some terminology and facts concerning n-connected spaces.

Definition 2.1 (see [12], [4]) A path-connected space X is n-connected if its n-first homotopy groups $\pi_k(X)$ $(0 < k \leq n)$ are trivial.

Thus, 0-connected means path connected and 1-connected means simply connected. The examples of n-connected spaces are:

- The Euclidean space $\mathbb{R}^n$ is 1-connected.
- The unit sphere S^n is (n-1)-connected.
- The unit ball $B^{n+1} = \{x \in \mathbb{R}^{n+1}, \|x\| \leq 1\}$ is an n-connected space.

- Every CW complex X, with exactly one 0 cell and all other cells having dimensions greater than n, is n-connected.

Because of its size (n-connected spaces can also be produced from other spaces) and the intriguing properties and applications that have been developed for it, the class of n-connected spaces is quite significant. Examples include those of Whitehead, Hurewicz, and Frendenthal (refer to [1, 11, 12, 16]), which, when applied to various problems, yield remarkable results in the area of algebraic topology.

Let M_Z be the class of maps defined in the Introduction. Our goal in this section is to define for the elements (f, g) of M_Z a topological invariant.

Let $(f, g) \in M_Z$, which means that $(f, g) \in Mor_{Top}(X, Z) \times Mor_{Top}(Y, Z)$ such that $S^n \subset X \subseteq Y \subset \mathbb{R}^{n+1}$, X is an n-connected space and Z is an arbitrary non empty subset of $\mathbb{R}^{n+1}$. We can get the pair of morphisms $(\tilde{f}, \tilde{g}) \in Mor_{Top}(S^n, Z)$, where $\tilde{f}, \tilde{g} : S^n \to Z$ are the restrictions of f, g, respectively, on the sphere S^n. The fact that $\mathrm{Coinc}((f, g), S^n) = \emptyset$ entails $(\tilde{f} - \tilde{g}) \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. This allows us to give the following definition.

Definition 2.2 The generalized n-characteristic of the pair $(f, g) \in M_Z$ is defined as the homomorphism $\xi(f, g) = \pi_n(\tilde{f} - \tilde{g}) = (\tilde{f} - \tilde{g})_{*,n} \in Mor_{G_d}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\}))$.

Let us give some properties of this generalized n-characteristic.

Proposition 2.1 *Let $(f, g) \in M_Z$ so that the generalized n-characteristic $\xi(f, g)$ is not trivial, then the equation $f(x) - g(x) = \theta$ admits at least one solution in $X \setminus S^n$, i.e., there exists a coincidence point of f, g in $X \setminus S^n$.*

Proof. For the reason that $(f, g) \in M_Z$, we have $\mathrm{Coinc}((f, g), S^n) = \emptyset$. Suppose that $f(x) \neq g(x)$ for all $x \in X \setminus S^n$, so $(f - g) \in Mor_{Top}(X, \mathbb{R}^{n+1} \setminus \{\theta\})$. Thus, we have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{f-g} & \mathbb{R}^{n+1} \setminus \{\theta\} \\ & \nwarrow_{i} & \uparrow \tilde{f} - \tilde{g} \\ & & S^n. \end{array}$$

Applying the co-variant functor π to the diagram above and using the fact that X is an n-connected space, we deduce that $(\tilde{f} - \tilde{g})_{n,*}$ is a trivial homomorphism.

Corollary 2.1 *Let $X \subseteq Z \subseteq \mathbb{R}^{n+1}$, $i \in Mor_{Top}(X, Z)$ be a canonical injection, and $(i, g) \in M_Z$ so that $\xi(i, g)$ is not trivial, then the equation $x - g(x) = \theta$ admits at least one solution in $X \setminus S^n$.*

Proof. We apply Proposition 2.1 to the pair (i, g).

Let us show that the generalized n-characteristic is a homotopy invariant. First, we introduce the following definition.

Definition 2.3 Two pairs (f, g), (f', g') in M_Z are said to be homotopic if there exists a pair of continuous maps $(F, G) \in Mor_{Top}(X \times [0, 1], Z) \times Mor_{Top}(Y \times [0, 1], Z)$ such that the following conditions are satisfied:

1. $\mathrm{Coinc}((F, G), S^n \times [0, 1]) = \emptyset$.
2. $F(x, 0) = f(x), F(x, 1) = f'(x)$, for all $x \in X$.

3. $G(y,0) = g(y), G(y,1) = g'(y)$, for all $y \in Y$.

In this case, (F,G) is said to be the homotopy between the pairs (f,g) and (f',g').

Proposition 2.2 *If (f,g), (f',g') are some homotopic elements of M_Z, then $\xi(f,g) = \xi(f',g')$.*

Proof. Let (F,G) be the homotopy between (f,g) and (f',g'). One can consider the morphism $\tilde{F} - \tilde{G} \in Mor_{Top}(S^n \times [0,1], \mathbb{R}^{n+1} \setminus \{\theta\})$, where $\tilde{F}, \tilde{G}$ are the restrictions of F, G to $S^n \times [0,1]$. Using the fact that $\tilde{F}$, $\tilde{G}$ are, respectively, homotopies between $\tilde{f}$, $\tilde{f}'$ and $\tilde{g}$, $\tilde{g}'$, one deduces that $\tilde{F} - \tilde{G}$ is continuous and it is a homotopy between the morphisms $(\tilde{f} - \tilde{g}), (\tilde{f}' - \tilde{g}') \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. Therefore, we have

$$\xi(f,g) = (\tilde{f} - \tilde{g})_{n,*} = (\tilde{f}' - \tilde{g}')_{n,*} = \xi(f',g').$$

Proposition 2.3 *Let X_0 be a subset of $\mathbb{R}^{n+1}$ which contains S^n, and $\varphi \in Mor_{Top}(X_0, X)$ so that $\varphi(S^n) = \tilde{\varphi}(S^n) \subseteq S^n$ and $(f,g) \in M_Z$, then $(f,g) \circ \varphi = (f \circ \varphi, g \circ \varphi) \in M_Z$ and $\xi((f,g) \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*}$.*

Proof. Let us consider $f \circ \varphi \in Mor_{Top}(X_0, Z)$ and $\tilde{f} \circ \tilde{\varphi} \in Mor_{Top}(S^n, \mathbb{R}^{n+1} \setminus \{\theta\})$. If we suppose that there exists $x_0 \in S^n$ such that $f \circ \varphi(x_0) = g \circ \varphi(x_0)$, then we obtain $y_0 = \varphi(x_0) \in \mathrm{Coinc}((f,g), S^n)$. It is impossible because $(f,g) \in M_Z$. We deduce that $\mathrm{Coinc}(f \circ \varphi, g \circ \varphi) = \emptyset$, then $(f \circ \varphi, g \circ \varphi) \in M_Z$. Furthermore,

$$\xi(f \circ \varphi, g \circ \varphi) = (\widetilde{f \circ \varphi} - \widetilde{f \circ \varphi})_{n,*} = ((\tilde{f} - \tilde{g}) \circ \tilde{\varphi})_{n,*} = (\tilde{f} - \tilde{g})_{n,*} \circ \tilde{\varphi}_{n,*}.$$

Thus, $\xi(f \circ \varphi, g \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*}$.

Corollary 2.2 *Let $S^n \subset X_0 \subset \mathbb{R}^{n+1}$, $\varphi \in Mor_{Top}(X_0, X)$ with $\varphi(S^n) = S^n$, and the restriction $\tilde{\varphi} : S^n \to S^n$ induces an epimorphism in the n-th group of homotopy, then if $(f,g) \in M_Z$, we have the equivalence*

$$\xi(f,g) \textit{ is not trivial if only if } \xi((f,g) \circ \varphi) \textit{ is not trivial.}$$

Proof. Suppose that $\xi((f,g) \circ \varphi) = \xi(f,g) \circ \tilde{\varphi}_{n,*} = 0$. Using the fact that $\varphi_{n,*}$ is an epimorphism, we can deduce that

$$\xi(f,g) \circ \tilde{\varphi}_{n,*}(\pi_n(S^n)) = \xi(f,g)(\pi_n(S^n)) = 0_{\pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})}.$$

Then $\xi(f,g) = 0$. On the other hand, suppose that $\xi(f,g) = 0$, then $\xi(f,g) \circ \tilde{\varphi}_{n,*} = 0$, so $\xi((f,g) \circ \varphi) = 0$.

Proposition 2.4 *Let $(f',g') \in Mor_{Top}(X', Z') \times Mor_{Top}(Y', Z')$ satisfying the conditions*

1. *$S^n \subset X' \subseteq Y' \subset \mathbb{R}^{n+1}$ and $Z' \subseteq \mathbb{R}^{n+1}$,*

2. *$\mathrm{Coinc}((f',g'), S^n) = \emptyset$.*

Let (f,g) be an element of M_Z, then the product pair $(f \times f', g \times g') \in M_{Z \times Z'}$ and $\xi(f \times f', g \times g') = J \circ (\xi(f,g) \times \xi(f',g')) \circ I$, where I, J stand for the product isomorphisms.

Proof. Using the fact that (f, g) and (f', g') are in M_Z, we can check that $(f \times f', g \times g') \in M_{Z \times Z'}$. Furthermore, denote by P_r the continuous projection, we can built the following diagram:

$$\begin{array}{ccc}
S^n & \overset{\tilde{f}'-\tilde{g}'}{\longrightarrow} & \mathbb{R}^{n+1} \setminus \{\theta\} \\
Pr_{r_2}^{X'} \uparrow & (I) & \uparrow Pr_{r2}^{Y'} \\
S^n \times S^n & \overset{\widetilde{f \times f'} - \widetilde{g \times g'}}{\longrightarrow} & \mathbb{R}^{n+1} \setminus \{\theta\} \times \mathbb{R}^{n+1} \setminus \{\theta\} \\
Pr_{r1}^{X} \downarrow & (II) & \downarrow Pr_{r1}^{Y} \\
S^n & \overset{\tilde{f}-\tilde{g}}{\longrightarrow} & \mathbb{R}^{n+1} \setminus \{\theta\}.
\end{array}$$

From the commutativity of the two squares $(I), (II)$ of the diagram above, one can deduce the equality $j \circ (\widetilde{f \times f'} - \widetilde{g \times g'}) = ((\tilde{f} - \tilde{g}) \circ \ (\tilde{f}' - \tilde{g}')) \circ i$, where $i = Pr_{r1}^{X} \times Pr_{r2}^{X'}$ and $j = Pr_{r1}^{Y} \times Pr_{r2}^{Y'}$. Since the induced homomorphisms I, J of i, j, respectively, are both isomorphisms (see [15]), we deduce the equality

$$\xi(f \times f', g \times g') = J^{-1} \circ (\xi(f, g) \times \xi(f', g')) \circ I.$$

Let us consider the case $X = Y = Z = B^{n+1} = \{x \in \mathbb{R}^{n+1} / \|x\| \leq 1\}$ and let $C_\theta : B^{n+1} \to B^{n+1}$ be the constant map, where $C_\theta(x) = \theta$ for every element $x \in B^{n+1}$.

Proposition 2.5 *Let $(Id_{B^{n+1}}, g)$ be an element of $M_{B^{n+1}}$. Then for C_θ as defined above, we have $\xi(Id_{B^{n+1}}, g) = \xi(Id_{B^{n+1}}, C_\theta)$ and it is not trivial.*

Proof. Let $F : B^{n+1} \times [0, 1] \to B^{n+1}$ and $G : B^{n+1} \times [0, 1] \to B^{n+1}$ be given by the rules $F(x, t) = x$ and $G(x, t) = tg(x)$ for every $(x, t) \in B^{n+1} \times [0, 1]$. The pair (F, G) satisfies the conditions of Definition 2.3, then we deduce that it is a homotopy between $(Id_{B^{n+1}}, C_\theta)$ and $(Id_{B^{n+1}}, g)$. Hence, from Proposition 2.2, one gets $\xi(Id_{B^{n+1}}, C_\theta) = \xi(Id_{B^{n+1}}, g)$. We conclude the proof by remarking that $\xi(Id_{B^{n+1}}, C_\theta) = i_*$, where $i : S^n \to \mathbb{R}^{n+1} \setminus \{\theta\}$ is a homotopy equivalence (see [4]).

Corollary 2.3 *If $(Id_{B^{n+1}}, g)$ is an element of $M_{B^{n+1}}$, then the equation $x - g(x) = \theta$ admits at least one solution in the interior of B^{n+1}.*

Proof. From Proposition 2.5, we have $\xi(Id_{B^{n+1}}, g)$ is not trivial. The result is obtained by applying Proposition 2.1 to the pair $(Id_{B^{n+1}}, g)$.

Proposition 2.6 *If (f, g) in $M_{B^{n+1}}$ is such that $f(x) - g(x)$ and x are not in the opposite sense for all vector x in S^n, then $\xi(f, g)$ is not trivial.*

Proof. First, we can verify that the pair $(Id, g - f)$ is in $M_{B^{n+1}}$. Let $F : B^{n+1} \times [0, 1] \to B^{n+1}$ and $G : B^{n+1} \times [0, 1] \to B^{n+1}$ be continuous maps defined by $F(x, t) = (1 - t)f(x) + tId_{B^{n+1}}(x)$ and $G(x, t) = g(x) - tf(x)$ for every $(x, t) \in B^{n+1} \times [0, 1]$. Since $f(x) - g(x)$ and x are not in the opposite sense for all vectors $x \in S^n$, one can show that $\mathrm{Coinc}((F, G), S^n \times [0, 1]) = \emptyset$. Furthermore, we verify that the pair (F, G) is a homotopy between the pairs (f, g) and $(Id, g - f)$. Thus, $\xi(f, g) = \xi(Id, g - f)$. We end the proof by remarking that $\xi(f, g)$ is not trivial because $\xi(Id, g - f)$ is not trivial.

Corollary 2.4 *Let $(f, g) \in M_{B^{n+1}}$ so that $f(x) - g(x)$ and x are not in opposite sense for all $x \in S^n$, then f and g have at least in $B^{n+1} \setminus S^n$ a coincidence point.*

Proof. It is a consequence of Proposition 2.6 and Proposition 2.1.

3 A Generalized n-Mult Characteristic for a Class of Multivalued Mapping

Let Z_1, Z_2 be two arbitrary subsets of $\mathbb{R}^{n+1}$ such that $Z_1 \subseteq Z_2 \subset \mathbb{R}^{n+1}$ and $F : Z_1 \to Z_2$ is an upper semi continuous multivalued mapping.

Definition 3.1 F is said to be admissible (respectively, strongly admissible) if there exist an n-connected space X such that $S^n \subset X \subset \mathbb{R}^{n+1}$ and a pair $(p, q) \in Mor_{Top}(X, Z_1) \times Mor_{Top}(X, Z_2)$ such that p is surjective and $q \circ p^{-1}(x) \subseteq F(x)$ (respectively, $q \circ p^{-1}(x) = F(x)$) for all $x \in Z_1$. In this case, the pair (p, q) is called a representation of the multivalued mapping F on the n-connected space X and will be denoted by $(p, q, X) \subseteq F$, or $(p, q) \subseteq F$ if we do not to specify X.
The class of admissible multivalued mappings from Z_1 to Z_2 is denoted by $\mathcal{A}(Z_1, Z_2)$.

Let us consider the set $\mathrm{Coinc}(p, q) = \{x \in X/p(x) = q(x)\}$ and the set

$$\mathcal{AM}(Z_1, Z_2) = \{F \in \mathcal{A}(Z_1, Z_2)/\mathrm{Coinc}((p, q), S^n) = \emptyset, \forall (p, q) \subseteq F\}.$$

In the case where $Z_1 = Z_2 = Z$, the space $\mathcal{AM}(Z_1, Z_2)$ will be denoted by $\mathcal{AM}(Z)$.
For any representation (p, q) of F, the restrictions of p, q on the sphere S^n are denoted by $\tilde{p}, \tilde{q}$.

Definition 3.2 The generalized n-mult characteristic of $F \in \mathcal{AM}(Z_1, Z_2)$ is denoted and defined by

$$\chi_{\mathcal{AM}}(F) = \{\pi_n(\tilde{p} - \tilde{q}) \in Mor_{Top}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})), \forall (p, q) \subseteq F\}.$$

Lemma 3.1 *For every admissible representation (p, q) of F, we have*

$$p(\mathrm{Coinc}(p, q)) = \{z \in Z_1/z \in q \circ p{-}1(z)\}.$$

Proof. The proof is obvious.
We have the following theorem.

Theorem 3.1 *Let F be a multivalued mapping satisfying the following conditions:*

1. *$F \in \mathcal{AM}(Z)$,*
2. *$\chi_{\mathcal{AM}}(F) \neq \{0\}$, where 0 is the zero homomorphism,*

so there exists at least a representation (p, q) of F and an element $z \in Z \setminus p(S^n)$ such that $z \in F(z)$, i.e., a fixed point of F.

Proof. Because $\chi_{\mathcal{AM}}(F) \neq \{0\}$, then by definition, there exists at least a representation (p, q) of F on an n-connected space X such that $\pi_n(\tilde{p} - \tilde{q}) \neq \{0\}$. By Proposition 2.1, the equation $p(x) - q(x) = \theta$ has at least one solution x_0 in $X \setminus S^n$. We put $z = p(x_0)$, so we obtain $z \in q(p^{-1}(z)) = F(z)$.

Definition 3.3 Let F, G be two elements of $\mathcal{AM}(Z)$. F, G are said to be homotopic if there exist two representations (p, q), (p', q') on the same n-connected space X which contains S^n, respectively, and a pair of continuous maps

$$(H, H') \in Mor_{Top}(X \times [0, 1], Z) \times Mor_{Top}(X \times [0, 1], Z)$$

such that

1. $Coin((H,H'),S^n \times [0,1]) = \emptyset$,
2. $H(x,0) = p(x), H(x,1) = p'(x)$, $\forall x \in X$,
3. $H'(x,0) = q(x), H'(x,1) = q'(x)$, $\forall x \in X$.

In this case, the pair $H = (H,H')$ is called a homotopy between F and G.

We have the following proposition.

Proposition 3.1 *If F,G are two homotopic elements of $\mathcal{AM}(Z)$, then $\chi_{\mathcal{AM}}(F) \cap \chi_{\mathcal{AM}}(G) \neq \emptyset$.*

Proof. Since F,G are homotopic, there exist representations (p,q), (p',q') on an n-connected space X such that they are homotopic in the sense of Definition 2.3. So, by Proposition 2.2, $\xi(p,q) = \xi(p',q')$. But $\xi(p',q') \in \chi_{\mathcal{AM}}(G)$, so $\chi_{\mathcal{AM}}(F) \cap \chi_{\mathcal{AM}}(G) \neq \emptyset$.

Remark 3.1 The proposition above can be formulated as follows. If F,G are two homotopic elements of $\mathcal{AM}(Z_1,Z_2)$, then there exist two representations (p,q), (p',q') of F and G, respectively, on the same n-connected space X which contains S^n such that $\pi_n(\tilde{p} - \tilde{q}) = \pi_n(\tilde{p}' - \tilde{q}')$.

Proposition 3.2 *Assume that $F,G : Z_1 \to Z_2$ are two upper semi continuous multivalued mappings such that the following conditions are satisfied:*

1. $F(x) \subset G(x)$, $\forall x \in Z_1$,

2. $F \in \mathcal{AM}(Z_1,Z_2)$.

Then $G \in \mathcal{AM}(Z_1,Z_2)$ and $\chi_{\mathcal{AM}}(F) \subset \chi_{\mathcal{AM}}(G)$.

Proof. Indeed, $\forall (p,q) \subseteq F$, we have $(p,q) \subseteq G$.

Let us consider the case $Z_1 = Z_2 = B^{n+1}$ and let $\mathcal{AM}(B^{n+1})$. Consider the set $\mathcal{AM}_{B^{n+1}}(B^{n+1}) \subset \mathcal{AM}(B^{n+1})$, where $\mathcal{AM}_{B^{n+1}}(B^{n+1})$ contains all admissible multivalued mappings F of $\mathcal{AM}(B^{n+1})$ which have at least a representation (p,q) on the unit ball B^{n+1}. Denote

$$\chi_{\mathcal{AM}}(F,B^{n+1}) = \{\xi(p,q) = \pi_n(\tilde{p} - \tilde{q}) \in Mor_{Top}(\pi_n(S^n), \pi_n(\mathbb{R}^{n+1} \setminus \{\theta\})), \forall (p,q,B^{n+1}) \subseteq F\}.$$

It is easy to see that $\chi_{\mathcal{AM}}(F,B^{n+1}) \subset \chi_{\mathcal{AM}}(F)$.

Proposition 3.3 *If $F \in \mathcal{AM}(B^{n+1})$ is a multivalued mapping defined by $F(x) = \{f(x)\}$ for every $x \in B^{n+1}$, then $\chi_{\mathcal{AM}}(F) \neq \{0\}$.*

Proof. In this case, we take $p = Id_{B^{n+1}}$, $q = f$ and $X = B^{n+1}$, we get $(Id_{B^{n+1}}, f)$ is one representation of F. Because $F \in \mathcal{AM}(B^{n+1})$, so Coinc$(Id_{B^{n+1}}, f) = \emptyset$ on the sphere S^n. By Proposition 2.5, we have $\xi(Id_{B^{n+1}}, f) = \xi(Id_{B^{n+1}}, C_\theta)$ and it is not trivial. But $\xi(Id_{B^{n+1}}, f) \in \chi_{\mathcal{AM}}(F)$, so $\chi_{\mathcal{AM}}(F) \neq \{0\}$.

Proposition 3.4 *If $F \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ is such that there exists a representation (p,q) of f satisfying the condition $p(x) - q(x)$ and x are not in opposite sense for every $x \in S^n$, then $\chi_{\mathcal{AM}}(F) \neq \{0\}$.*

Proof. We apply Proposition 2.6 for the pair (p,q), we obtain $\xi(p,q) \neq 0$. But $\xi(p,q) \in \chi_{\mathcal{AM}}(F)$, so $\chi_{\mathcal{AM}}(F) \neq \{0\}$.

Corollary 3.1 *Let $F \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ be such that there exists a representation $(p,q) \in \mathcal{AM}_{B^{n+1}}(B^{n+1})$ of F satisfying the condition $p(x)-q(x)$ and x are not in opposite sense for every $x \in S^n$, then there exists $z \in B^{n+1} \backslash p(S^n)$ such that $z \in F(z) = q \circ p^{-1}(z)$.*

Proof. By Proposition 3.4, we have $\xi(p,q) \neq 0$. But $\xi(p,q) \in \chi_{\mathcal{AM}}(F)$, so by Theorem 3.1, we obtain the result.

4 Conclusion

This paper introduces a topological invariant designed for the pairs of mappings defined on n-connected spaces containing the sphere S^n, under the condition that they differ on this sphere. The invariant's definition indicates that it is a group homomorphism, establishes its novelty, and extends the definition given for the sphere S^n. Among the properties that are provided, we show its homotopic invariance and utilize it to establish theorems in coincidence and fixed point theories when these properties are not verified on S^n. In Section 3, leveraging the aforementioned construction, we define a generalized n-mult characteristic for a class of admissible multivalued mappings. Various properties of this n-mult characteristic are delineated, notably including the fixed point theorem for this class of multivalued mappings. In conclusion, recognizing the effectiveness of topological invariants in solving diverse nonlinear problems, we anticipate that the introduced topological invariant will also find applications, serving as a potent tool for addressing various mathematical and scientific challenges across different domains.

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Limit Cycles for a Class of Generalized Liénard Polynomial Differential Systems via the First-Order Averaging Method

A. Menaceur [1*] and A. Makhlouf [2]

[1] *Laboratory of Analysis and Control of Differential Equations "ACED", Department of Mathematics, University of Guelma, P.O. Box 401, Guelma 24000, Algeria.*
[2] *Department of Mathematics, Badji-Mokhtar University, BP12 El Hadjar, Annaba, Algeria.*

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Abstract: In this paper, using the averaging method of first order, we compute the maximum number of limit cycles that can bifurcate from the periodic orbits of the center $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2q-1}$ with p and q being positive integers, under perturbation in the particular class of the generalized Liénard polynomial differential systems.

Keywords: *limit cycle; averaging method; periodic orbit; polynomial differential system.*

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1 Introduction

The second part of the Hilbert's 16th problem [9,17] aims to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree. The limit cycles problem and the center problem are concentrated on specific classes of systems. For instance, much has been written on Kukles systems, Duffing systems, Mathieu differential equations, Kolmogorov systems (see for example, [5, 10, 11, 15]) and Liénard systems, that is, systems of the form

$$\dot{x} = y, \ \dot{y} = -x - f(x),$$

where $f(x)$ is a polynomial in the variable x of degree m. The motivation in the Liénard family is because it is one of the most important families related to the Hilbert's

* Corresponding author: menaceur.amor@univ-guelma.dz

16th problem. Moreover, some classes of Liénard systems appear in applied sciences. Bifurcation of limit cycles in Liénard systems have been tackled by several authors and by using different approaches. See, for example, [1, 4, 7].

In [13], Lliber et al. proved that the generalized Liénard polynomial differential system

$$\dot{x} = y,\ \dot{y} = -g(x) - f(x)y,$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m, respectively, can have $\left[\frac{n+m-1}{2}\right]$ limit cycles, where [.] denotes the integer part function. In [14], Llibre and Makhlouf proved that the maximum number of limit cycles of the following generalized Liénard polynomial differential system:

$$\dot{x} = -y^{2p-1},\ \dot{y} = x^{2q-1} - \varepsilon f(x)y^{2m-1},$$

is at most $\left[\frac{n}{2}\right]$, where p, q and m are positive integers, ε is a small parameter and $f(x)$ is a polynomial of degree n. In [2], Benterki et al. studied the maximum number of crossing limit cycles of planar piecewise differential systems formed by linear Hamiltonian saddles.

In this paper, we want to study the maximum number of limit cycles of the following generalized Liénard polynomial differential system:

$$\dot{x} = -y^{2p-1},\ \dot{y} = x^{2q-1} - \varepsilon f(x, y)y^{2m-1}, \tag{1}$$

where p, q and m are positive integers, ε is a small parameter and $f(x, y)$ is a polynomial of degree n. Clearly, system (1) with $\varepsilon = 0$ is a Hamiltonian system with the Hamiltonian

$$H(x, y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}.$$

More precisely, our main results are as follows.

Theorem 1.1 *For the sufficiently small $|\varepsilon|$, system (1) has at most*

$$\mu = \left[\frac{n}{2}\right] \max\{p, q\}$$

limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}$, $\dot{y} = x^{2q-1}$, using averaging theory of first order, where [.] denotes the integer part function.

The proof of Theorem 1.1 is given in Section 3.

Theorem 1.2 *Consider system (1) with $q = lp$, l is a positive integer, then for $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized Liénard polynomial differential system (1) bifurcating from the periodic orbits of the center $\dot{x} = -y^{2p-1}, \dot{y} = x^{2lp-1}$, using the averaging theory of first order, is*

$$\textbf{(a)}\ \mu_1 = \frac{1}{2}\left(\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right] + 3\right)\right) \quad \textit{if } \left[\frac{n}{2}\right] \leq l - 1,$$

$$\textbf{(b)}\ \mu_2 = l\left[\frac{n}{2}\right] - \frac{l(l-3)+2}{2} \quad \textit{if } \left[\frac{n}{2}\right] \geq l.$$

The proof of Theorem 1.2 is given in Section 4.

2 Preliminaries

2.1 First order averaging method

The averaging theory is also an interesting method to research the limit cycles. Essentially, we have to look for the zeros of some specific function associated to the initial system.

Theorem 2.1 *Consider the following two initial value problems:*

$$\dot{x} = \varepsilon R(t,x) + \varepsilon^2 G(t,x,\varepsilon), x(0) = x_0 \tag{2}$$

and

$$\dot{y} = \varepsilon f^{\,0}(y), y(0) = x_0, \tag{3}$$

where x, y *and* $x_0 \in D$ *which is an open domain of* $\mathbb{R}$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, R *and* G *are periodic functions with their period* T *with its variable* t, *and* $f^0(y)$ *is the average function of* $R(t,y)$ *with respect to* t, *i.e.,*

$$f^0(y) = \tfrac{1}{T}\int_0^T R(t,y)dt.$$

Assume that

(i) R, $\dfrac{\partial R}{\partial x}$, $\dfrac{\partial^2 R}{\partial x^2}$, G *and* $\dfrac{\partial G}{\partial x}$ *are well defined, continuous and bounded by a constant independent of* $\varepsilon \in (0,\varepsilon_0]$ *in* $[0,\infty)\times D$.

(ii) T *is a constant independent of* ε.

(iii) $y(t)$ *belongs to* D *on the time scale* $1/\varepsilon$. *Then the following statements hold:*

(a) *On the time scale* $\dfrac{1}{\varepsilon}$, *we have*

$$x(t) - y(t) = O(\varepsilon), \text{ as } \varepsilon \longrightarrow 0.$$

(b) *If* p *is an equilibrium point of the averaged system* (3) *such that*

$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \neq 0, \tag{4}$$

then system (2) *has a* T*-periodic solution* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \longrightarrow 0$.

(c) *If* (4) *is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *of equation* (2) *according to* (t,x) *is asymptotically stable for all* ε *sufficiently small; if* (4) *is positive, then it is unstable.*

For more information about the averaging theory, see [6, 16, 18].

2.2 (p.q)-trigonometric functions

Following Lyapunov [12], let $u(\theta) = Cs\theta$ and $v(\theta) = Sn\theta$ be the solutions of the following initial value problem:

$$\begin{aligned} \dot{u} &= -v^{2p-1}, \dot{v} = u^{2q-1}, \\ u(0) &= \sqrt[2q]{\frac{1}{p}} \text{ and } v(0) = 0. \end{aligned}$$

Moreover, they satisfy the following properties:

(a) The functions $Cs\theta$ and $Sn\theta$ are T–periodic with

$$T = 2p^{\frac{-1}{2q}} q^{\frac{-1}{2p}} \frac{\Gamma(\frac{1}{2p})\Gamma(\frac{1}{2q})}{\Gamma(\frac{1}{2p} + \frac{1}{2q})},$$

where Γ is the gamma function.

(b) For $p = q = 1$, we have $Cs\theta = \cos\theta$ and $Sn\theta = \sin\theta$.

(c) $pCs^{2p}\theta + qSn^{2q}\theta = 1$.

(d) Let $Cs\theta$ and $Sn\theta$ be the $(1.q)$-trigonometrical functions, for i and j being both even (see [8]),

$$\int_0^T Cs^i\theta Sn^j\theta d\theta = 2q^{-\frac{j+1}{2}} \frac{\Gamma(\frac{i+1}{2q})\Gamma(\frac{j+1}{2})}{\Gamma(\frac{i+1}{2q} + \frac{j+1}{2})}. \tag{5}$$

2.3 Descartes theorem

The purpose of the Descartes theorem is to provide an insight into how many real roots a polynomial $P(x)$ may have.

Theorem 2.2 *[3] Consider the real polynomial*

$$p(x) = a_{l_1}x^{l_1} + a_{l_2}x^{l_2} + ... + a_{l_k}x^{l_k}$$

with $0 \le l_1 < l_2 < ... < l_k$ and $a_{l_i} \ne 0$ being real constants for $i \in \{1, 2, 3, ..., k\}$. When $a_{l_i}a_{l_{i+1}} < 0$, we say that a_{l_i} and $a_{l_{i+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $k - 1$ positive real roots.

3 Proof of Theorem 1.1

We shall need the first order averaging theory to prove Theorem 1.1. We write system (1) in (p, q)-polar coordinates (r, θ), where $x = r^pCs\theta$ and $y = r^qSn\theta$. In this way, system

(1) will be written in the standard form for applying the averaging theory. If we write $f(x,y)=\sum_{i+j=0}^{n} a_{i,j}x^i y^j$, then system (1) becomes

$$\begin{cases} \dot{r} = -\varepsilon r^{2q(m-1)+1} \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)} \\ \dot{\theta} = r^{pq-p-q} - \varepsilon r^{2q(m-1)} p \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^{i+1} \left(Sn\theta\right)^{j+2m-1}. \end{cases} \tag{6}$$

Treating θ as the independent variable, we get from system (6) the following:

$$\frac{dr}{d\theta} = \varepsilon R(r,\theta) + O(\varepsilon^2),$$

where

$$R(r,\theta) = -r^{-2pq+q(2m-1)+p+1} \sum_{i+j=0}^{n} a_{i,j} r^{pi+qj} \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)}.$$

Using the notation introduced in Section 2, we have

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{i+j=0}^{n} \left(a_{i,j} r^{pi+qj} \int_0^T \left(Cs\theta\right)^i \left(Sn\theta\right)^{j+2(m+p-1)} \right),$$

we write

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{i+j=0}^{n} a_{i,j} I_{i,j+2(m+p-1)} r^{pi+qj},$$

where

$$I_{i,j} = \int_0^T Cs^i\theta Sn^j\theta d\theta.$$

It is known that

$$\begin{aligned} I_{i,j} &= 0 \text{ if } i \text{ or } j \text{ is odd}, \\ I_{i,j} &> 0 \text{ if } i \text{ and } j \text{ are even}. \end{aligned}$$

Hence

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} a_{2s,2k} I_{2s,2k+2(m+p-1)} r^{2(ps+qk)}. \tag{7}$$

For the simplicity of calculation, let $\lambda_{s,k} = a_{2s,2k} I_{2s,2k+2(m+p-1)}$, therefore, (7) can be reduced to

$$f^0(r) = -\frac{r^{-2pq+q(2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2(ps+qk)}. \tag{8}$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$N(r) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2(ps+qk)},$$

then, to find the real positive roots of $N(r)$, we must find the zeros of a polynomial in the variable $t = r^2$,

$$M(t) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} t^{ps+qk}. \tag{9}$$

Now, we expand the polynomial (9) as follows:

$$\begin{aligned}
M(t) \; = \; & \lambda_{0,0} \\
& +\lambda_{1,0}t^{p} + \lambda_{0,1}t^{q} \\
& +\lambda_{2,0}t^{2p} + \lambda_{1,1}t^{p+q} + \lambda_{0,2}t^{2q} \\
& +... \\
& +\lambda_{d,0}t^{dp} + \lambda_{d-1,1}t^{(d-1)p+q} + \lambda_{d-2,2}t^{(d-2)p+2q} \\
& +... + \lambda_{2,d-2}t^{2p+(d-2)q} + \lambda_{1,d-1}t^{p+(d-1)q} + \lambda_{0,d}t^{qd} \\
& +... \\
& +\lambda_{\left[\frac{n}{2}\right],0}t^{\left[\frac{n}{2}\right]p} + \lambda_{\left[\frac{n}{2}\right]-1,1}t^{(\left[\frac{n}{2}\right]-1)p+q} + \lambda_{\left[\frac{n}{2}\right]-2,2}t^{(\left[\frac{n}{2}\right]-2)p+2q} \\
& +... + \lambda_{2,\left[\frac{n}{2}\right]-2}t^{2p+(\left[\frac{n}{2}\right]-2)q} + \lambda_{1,\left[\frac{n}{2}\right]-1}t^{p+(\left[\frac{n}{2}\right]-1)q} + \lambda_{0,\left[\frac{n}{2}\right]}t^{\left[\frac{n}{2}\right]q}.
\end{aligned}$$

So, the degree of $M(t)$ is bounded by $\mu = \left[\frac{n}{2}\right] \max\{p, q\}$, we conclude that $f^0(r)$ has at most μ positive roots r. Hence, Theorem 1.1 is proved.

4 Proof of Theorem 1.2

Consider the polynomial differential system (1) with $q = lp$, from equation (8), we obtain

$$f^0(r) = -\frac{r^{lp(-2p+2m-1)+p+1}}{T} \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2p(s+lk)}. \tag{10}$$

As we all know, the number of positive roots of $f^0(r)$ is equal to that of

$$G(r) = \sum_{s+k=0}^{\left[\frac{n}{2}\right]} \lambda_{s,k} r^{2p(s+lk)}. \tag{11}$$

We write (11) as follows:

$$\begin{aligned} G(r) &= \lambda_{0,0} + (\lambda_{1,0}r^{2p} + \lambda_{0,1}r^{2pl}) + (\lambda_{2,0}r^{4p} + \lambda_{1,1}r^{(l+1)2p} + \lambda_{0,2}r^{4lp}) \\ &+(\lambda_{3,0}r^{6p} + \lambda_{2,1}r^{(l+2)2p} + \lambda_{1,2}r^{(1+2l)2p} + \lambda_{0,3}r^{6lp}) \\ &+(\lambda_{4,0}r^{8p} + \lambda_{3,1}r^{(l+3)2p} + \lambda_{2,2}r^{(2+2l)2p} + \lambda_{1,3}r^{(1+3l)2p} + \lambda_{0,4}r^{8lp}) \\ &+... \\ &+[\lambda_{[\frac{n}{2}]-2,0}r^{\left([\frac{n}{2}]-2\right)2p} + \lambda_{[\frac{n}{2}]-3,1}r^{\left([\frac{n}{2}]+l-3\right)2p} + \lambda_{[\frac{n}{2}]-4,2}r^{\left([\frac{n}{2}]+2l-4\right)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-3}r^{\left(1+([\frac{n}{2}]-3)l\right)2p} + \lambda_{0,([\frac{n}{2}]-2)}r^{\left([\frac{n}{2}]-2\right)2lp}] \\ &+[\lambda_{[\frac{n}{2}]-1,0}r^{\left([\frac{n}{2}]-1\right)2p} + \lambda_{[\frac{n}{2}]-2,1}r^{\left([\frac{n}{2}]+l-2\right)2p} + \lambda_{[\frac{n}{2}]-3,2}r^{\left([\frac{n}{2}]+2l-3\right)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-2}r^{\left(1+([\frac{n}{2}]-2)l\right)2p} + \lambda_{0,[\frac{n}{2}]-1}r^{\left([\frac{n}{2}]-1\right)2lp}] \\ &+[\lambda_{[\frac{n}{2}],0}r^{[\frac{n}{2}]2p} + \lambda_{[\frac{n}{2}]-1,1}r^{\left([\frac{n}{2}]+l-1\right)2p} + \lambda_{[\frac{n}{2}]-2,2}r^{\left([\frac{n}{2}]+2l-2\right)2p} \\ &+... + \lambda_{1,[\frac{n}{2}]-1}r^{\left(1+([\frac{n}{2}]-1)l\right)2p} + \lambda_{0,[\frac{n}{2}]}r^{[\frac{n}{2}]2lp}]. \end{aligned} \tag{12}$$

Let us write (12) as

$$\begin{aligned} G(r) &= [\lambda_{0,0} + \lambda_{1,0}r^{2p} + \lambda_{2,0}r^{4p} + \\ &... + \lambda_{[\frac{n}{2}]-2,0}r^{\left([\frac{n}{2}]-2\right)2p} + \lambda_{[\frac{n}{2}]-1,0}r^{\left([\frac{n}{2}]-1\right)2p} + \lambda_{[\frac{n}{2}],0}r^{[\frac{n}{2}]2p}] \\ &+[\lambda_{0,1}r^{2lp} + \lambda_{1,1}r^{(l+1)2p} + \lambda_{2,1}r^{(l+2)2p} + \\ &... + \lambda_{[\frac{n}{2}]-2,1}r^{\left(l+[\frac{n}{2}]-2\right)2p} + \lambda_{[\frac{n}{2}]-1,1}r^{\left(l+[\frac{n}{2}]-1\right)2p}] \\ &+[\lambda_{0,2}r^{4lp} + \lambda_{1,2}r^{(2l+1)2p} + \lambda_{2,2}r^{(2l+2)2p} + \\ &... + \lambda_{[\frac{n}{2}]-3,2}r^{(2l+[\frac{n}{2}]-3)2p} + \lambda_{[\frac{n}{2}]-2,2}r^{(2l+[\frac{n}{2}]-2)2p}] \\ &+... + [\lambda_{0,([\frac{n}{2}]-2)}r^{\left(([\frac{n}{2}]-2)l\right)2p} + \lambda_{1,[\frac{n}{2}]-2}r^{\left(1+([\frac{n}{2}]-2)l\right)2p} \\ &+\lambda_{2,[\frac{n}{2}]-2}r^{\left(2+([\frac{n}{2}]-2)l\right)2p}] \\ &+[\lambda_{0,[\frac{n}{2}]-1}r^{\left([\frac{n}{2}]-1\right)2lp} + \lambda_{1,[\frac{n}{2}]-1}r^{\left(1+([\frac{n}{2}]-1)l\right)2p}] \\ &+\lambda_{0,[\frac{n}{2}]}r^{[\frac{n}{2}]l}. \end{aligned} \tag{13}$$

To find the number of positive roots of polynomials $G(r)$, we distinguish two cases.

Case 1: For $\left[\frac{n}{2}\right] \leq l-1$, the number of terms in polynomial (13) is

$$\left(\left[\frac{n}{2}\right]+1\right) + \left[\frac{n}{2}\right] + \left(\left[\frac{n}{2}\right]-1\right) + ... + 2 + 1 = \frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right).$$

Now, we shall apply the Descartes theorem introduced in Section 2, we can choose the appropriate coefficients $\lambda_{i,j}$ in order that the simple positive root number of $G(r)$ is at most

$$\begin{aligned}\mu_1 &= \frac{1}{2}\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right)-1\\ &= \frac{1}{2}\left(\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+3\right)\right).\end{aligned}$$

Hence (a) of Theorem 1.2 is proved.

Case 2: For $\left[\frac{n}{2}\right]\geq l$, the number of terms in polynomial (13) is

$$\begin{aligned}&\left(\left[\frac{n}{2}\right]+1\right)+\left[\frac{n}{2}\right]+\left(\left[\frac{n}{2}\right]-1\right)+...+2+1\\ &-\left(\left[\frac{n}{2}\right]-l+1\right)-\left(\left[\frac{n}{2}\right]-l\right)-\left(\left[\frac{n}{2}\right]-l-1\right)-...-2-1\\ =\ &\frac{1}{2}[\left(\left[\frac{n}{2}\right]+2\right)\left(\left[\frac{n}{2}\right]+1\right)-\left(\left[\frac{n}{2}\right]-l+1\right)\left(\left[\frac{n}{2}\right]-l+2\right)]\\ =\ &l\left[\frac{n}{2}\right]-\frac{l(l-3)}{2},\end{aligned}$$

by the Descartes theorem introduced in Section 2, we can choose the appropriate coefficients $\lambda_{i,j}$ in order that the simple positive root number of $G(r)$ is at most

$$\begin{aligned}\mu_2 &= l\left[\frac{n}{2}\right]-\frac{l(l-3)}{2}-1\\ &= l\left[\frac{n}{2}\right]-\frac{l(l-3)+2}{2}.\end{aligned}$$

Hence (b) of Theorem 1.2 is proved.

Example 4.1 We consider system (1), where $p=1$, $q=3$, $m=2$ and

$$f(x,y)=-3.6x^2+2.4xy+0.635y^2+0.5.$$

In this case, $n=2$, $l=3$ and $Cs\theta$ and $Sn\theta$ are T-periodic functions with period $T=8.413\,1$. From equation (10), we obtain

$$f^0(r)=-\frac{r^5}{T}(\lambda_{0.0}+\lambda_{1.0}r^2+\lambda_{0.1}r^6),$$

where $\lambda_{s.k}=a_{2s,2k}I_{2s,2k+4}$.

Using (5), we get

$$I_{0,4}=0.63098, I_{2,4}=0.15115 \text{ and } I_{0,6}=0.19718.$$

So

$$f^0(r)=-\frac{r^5}{8.413\,1}\left(0.315\,49-0.544\,14r^2+0.125\,21r^6\right).$$

This polynomial has two positive real roots, $r_1=0.8$ and $r_2=1.3$. According to statement (a) of Theorem 1.2, the system has exactly two limit cycles bifurcating from the periodic orbits of the center $\dot{x}=-y$, $\dot{y}=x^5$, using the averaging theory of first order.

Example 4.2 We consider system (1), where $p = 1$, $q = 2$, $m = 3$ and

$$f(x,y) = 1.5x^5 + 2xy^4 - 56.095x^4 + 13.575x^2y^2 - 0.46834y^4 + 21.227x^2 + y^2 + 2.7x - 1.$$

In this case, $n = 5$, $l = 2$ and $Cs\theta$ and $Sn\theta$ are T-periodic functions with period $T = 7.4163$. From equation (10), we obtain

$$f^0(r) = -\frac{r^8}{T}(\lambda_{0.0} + \lambda_{1.0}r^2 + (\lambda_{0.1} + \lambda_{2.0})\, r^4 + \lambda_{1.1}r^6 + \lambda_{0.2}r^8),$$

where $\lambda_{s,k} = a_{2s,2k} I_{2s,2k+6}$.

Using (5), we get

$$\begin{array}{rcl} I_{0.6} & = & 0.48158, I_{2.6} = 8.6894 \times 10^{-2}, I_{0.8} = 0.22474, \\ I_{4.6} & = & 3.2105 \times 10^{-2}, I_{2.8} = 3.5780 \times 10^{-2} \text{ and } I_{0.10} = 0.10645. \end{array}$$

So

$$f^0(r) = -\frac{r^8}{7.4163}(-0.48158 + 1.8445r^2 - 1.5762r^4 + 0.48571r^6 - 4.9855 \times 10^{-2}r^8).$$

This polynomial has four positive real roots, $r_1 = 0.6, r_2 = 1.4, r_3 = 1.85$ and $r_4 = 2$. According to statement (b) of Theorem 1.2, the system has exactly four limit cycles bifurcating from the periodic orbits of the center $\dot{x} = -y$, $\dot{y} = x^3$, using the averaging theory of first order.

5 Concluding Remarks

The second part of the Hilbert's 16th problem concerns the maximum number of limit cycles of all planar polynomial vector fields of degree n. One way to produce limit cycles is by perturbing a Hamiltonian system which has a center, in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits in the original system. In this work, by using the averaging theory of the first order, we have proved upper bounds for the maximum number of limit cycles bifurcating from the periodic orbits of the Hamiltonian system with the Hamiltonian $H(x,y) = \frac{1}{2q}x^{2q} + \frac{1}{2p}y^{2p}$, where p and q are positive integers. We will continue our research on the maximum number of limit cycles for differential systems that model phenomena in biology, physics, etc, using the higher-order averaging theory.

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On Existence and Uniqueness of Solution of Heat Equations in Quasi-Metric Spaces

A. H. Zakiyudin , K. Fahim *, M. Yunus , Sunarsini , I. G. N. R. Usadha and Sadjidon

Department of Mathematics, Faculty of Science and Analytical Data, Institut Teknologi Sepuluh Nopember, Sukolilo, Surabaya, 60111, East Java, Indonesia

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Abstract: In this paper, we prove the existence and uniqueness of solutions to heat equations in quasi-metric spaces by applying the $\phi G-$contraction in this setting. This type of contraction is analogous to the ψF-contraction introduced by Secelean et al. in 2019. In the ψF-contraction, we have $F : \mathbb{R}^+ \to \mathbb{R}$ as an increasing mapping and $\psi : (-\infty, \mu) \to \mathbb{R}$ for some μ in $\mathbb{R}^+ \cup \{\infty\}$ as an increasing and continuous function such that $\psi(t) < t$ for every t in $(-\infty, \mu)$. Meanwhile, in the ϕG-contraction, we have G as a strictly increasing mapping from $\mathbb{R}^+ \cup \{0\}$ to $\mathbb{R}^+ \cup \{0\}$. Also $\phi : (-\infty, \mu) \to \mathbb{R}^+ \cup \{0\}$ as a strictly increasing and continuous function satisfying $\phi(t) < t$ for all t and $\phi(0) = 0$. This approach also provides a framework for solving nonlinear equations.

Keywords: *fixed point theories; heat equations; quasi-metric spaces.*

Mathematics Subject Classification (2020): 46, 47H10, 47H09, 54E50, 93B28.

1 Introduction

In the last century, fixed points have become an important topic in pure and applied mathematics, as well as in nonlinear dynamics, see, for example, [1–6]. The concept of fixed points and their associated mappings is crucial in investigating the existence and uniqueness of solutions to various mathematical models. The study of fixed points began with the Banach Contraction Principle in complete metric spaces in 1922 (see [7]). Afterward, many researchers introduced other types of fixed points in complete metric spaces and found their applications both in pure and applied mathematics.

* Corresponding author: mailto:fahim@its.ac.id

Many authors have also studied fixed point results in other spaces, including quasi-metric spaces which were first introduced by Wilson in 1931 (see [8]). In the last decade, many authors have found applications of fixed points in quasi-metric spaces, for example, in software engineering [9], the problems of existence and uniqueness of solution of boundary value problems [10–15], nonlinear fractional differential equations [16], and stability analysis of a solution for the fractional-order models on rabies transmission dynamics [17]. More recently, Secelean et al. [18, 19] introduced the ψF-contraction on quasi-metric spaces and obtained the fixed point results on that space. They also found its applications in fractals theory. Following this, Zakiyudin and Fahim [20] introduced the ϕG-contraction on quasi-metric space which is analogous to the ψF-contraction and obtained similar fixed point result. They also found its application to the nonhomogeneous Cauchy equation (see [20]). Other fixed point results and their applications can be seen in [1–5, 21–24].

Our purpose in the present paper is to find an application of ϕG-contraction, specifically for heat equations. For this purpose, let $\Omega \subset \mathbb{R}^n$ be a closed set and let $H = L^2(\Omega)$, where $L^2(\Omega)$ is the space of measurable functions for which the square of the absolute value is Lebesgue integrable. Additionally, let $T \in (0, +\infty)$ and $w = w(t)$ be a function valued in the Banach space H. Consider a Lipschitz function $f : H \to H$, meaning there exists a constant $C \in \mathbb{R}^+$ such that for all $x, y \in H$, one has

$$\|f(x) - f(y)\|_H \leq C\|x - y\|_H.$$

Now, consider the heat equation

$$\begin{cases} \dfrac{dw}{dt} = \Delta w(t) + f(w(t)), & 0 \leq t \leq T, \\ w(0) = w_0, \end{cases} \tag{1}$$

where $w_0 \neq 0, w_0 \in H$, and Δ satisfies the Neumann or Dirichlet Boundary Condition. Then, for $T > 0$ and $p \in [1, +\infty)$, we define $L^p_\lambda(0, T; H)$ as the space

$$L^p_\lambda(0, T; H) = \{f : [0, T] \to H : \int_0^T e^{-\lambda t}\|f(t)\|^p_H \, dt < \infty\}$$

equipped with the norm

$$\|f\|_{L^p_{\lambda^*}(0,T;H)} = \left(\int_0^T e^{-\lambda t}\|f(t)\|^p_H \, dt\right)^{1/p}.$$

In this work, we study the existence and uniqueness of solutions for the heat equation (1) in the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ defined below:

$$X^K_{p,\lambda} = \{g \in L^p_\lambda(0, T; H) : g(\mathbf{0}) = w_0 \text{ and } \|g\|_{L^p_\lambda(0,T;H)} \leq K\}$$

with the mapping

$$\rho^K_{p,\lambda}(g, h) = \begin{cases} K, & \text{for } \|g\|_{L^p_\lambda(0,T;H)} = K \text{ and } \|g - h\|_{L^p_\lambda(0,T;H)} > K, \\ \|g - h\|_{L^p_\lambda(0,T;H)}, & \text{for other } g \text{ and } h, \end{cases}$$

where $K \in (0, +\infty)$ and $p \in [1, +\infty)$.

In order to do this, some concepts on quasi-metric spaces, such as the definition and some properties of this space, for example, forward convergence, forward Cauchy sequence, forward completeness, forward Picard operator, and their analogs for the backward are presented in Section 2. Then we provide the definition of ϕG-contraction in quasi-metric spaces, and the fixed point result is presented in Theorem 2.1. In addition, we give the required conditions for the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ in Lemmas 2.1 and 2.2. In Section 3, we present our result for the heat equation, it begins with Lemma 3.1 which states that there exists a C_0-contractive semigroup with Δ as its infinitesimal generator. Furthermore, we introduce the definition of the mild solution of (1) in Definition 3.1. Afterward, we define a mapping Υ and state that Υ is a self-mapping in Lemma 3.2. Additionally, we prove an inequality that will be used to establish the existence and uniqueness of solution of (1) in Lemma 3.3. Finally, using the provided lemmas, we state Theorem 3.1 which asserts that the mild solution of the heat equation (1) exists and is unique. We end this paper with the conclusion in Section 4.

2 Preliminaries

In this section, we present some preliminaries on quasi-metric spaces and provide the result of our previous work about the ϕG-contraction in such spaces. In addition, we introduce the auxiliary spaces.

2.1 Quasi-metric spaces

Let X be a non-empty set. These preliminaries on quasi-metric spaces are discussed in [25].

Definition 2.1 A quasi-metric $\rho : X \times X \to [0, +\infty)$ is a mapping satisfying the following conditions:

(ρ_1) $\rho(x, y) = 0$ if and only if $x = y$;

(ρ_2) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality).

A pair (X, ρ) denotes a quasi-metric space.

Example 2.1 *For $a > 0$, let $X := \mathbb{R}$ and $\rho : X \times X \to [0, +\infty)$ be defined as*

$$\rho(x, y) := \begin{cases} x - y, & x \geq y, \\ a(y - x), & x < y. \end{cases}$$

Then (X, ρ) is a quasi-metric space.

Example 2.2 *Let $a > 0$ and consider a decreasing function $g : \mathbb{R} \to \mathbb{R}$, let $X = \mathbb{R}$ and $\rho : X \times X \to [0, +\infty)$ be defined as*

$$\rho(x, y) := \begin{cases} x - y, & x \geq y, \\ a\left(f(x) - f(y)\right), & x < y. \end{cases}$$

Then (X, ρ) is a quasi-metric space.

Definition 2.2 If (X, ρ) is a quasi-metric space and (x_n) is a sequence on X, then

1. The sequence (x_n) is forward convergent (for short, f-convergent) to $x \in X$ if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x, x_n) < \varepsilon$ for all $n \geq k$.

2. The sequence (x_n) is backward convergent (for short, b-convergent) to $x \in X$ if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon$ for all $n \geq k$.

Definition 2.3 If (X, ρ) is a quasi-metric space and (x_n) is a sequence on X, then

1. The sequence (x_n) is forward Cauchy (for short, f-Cauchy) if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ for all $m \geq n \geq k$.

2. The sequence (x_n) is backward Cauchy (for short, b-Cauchy) if for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m \geq n \geq k$.

Definition 2.4 If (X, ρ) is a quasi-metric space, then (X, ρ) is forward complete (for short, f-complete) if every f-Cauchy sequence is f-convergent, and (X, ρ) is backward complete (for short, b-complete) if every b-Cauchy sequence is b-convergent.

Definition 2.5 If (X, ρ) is a quasi-metric space and $T : X \to X$ is a mapping, then

1. T is a forward Picard Operator (for short, f-P.O.) if there exists a unique $\eta \in X$ such that $T\eta = \eta$ and the sequence $x_n := T^n x_0$ for $x_0 \in X, n \in \mathbb{N}$ is f-convergent to η.

2. T is a backward Picard Operator (for short, b-P.O.) if there exists a unique $\eta \in X$ such that $T\eta = \eta$ and the sequence $x_n := T^n x_0$ for $x_0 \in X, n \in \mathbb{N}$ is b-convergent to η.

2.2 The ϕG-contraction

In the following, we present a fixed point result concerning ϕG-contraction obtained by Zakiyudin and Fahim in [20].

First, we denote by $\mathcal{G}$ a family of all strictly increasing functions $G : [0, +\infty) \to [0, +\infty)$. Let $\mu \in (0, \infty]$, and we denote by Φ_μ a family of all strictly increasing and continuous functions $\phi : [0, \mu) \to [0, +\infty)$ that satisfy $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. We write Φ_μ as Φ_G if $\mu \geq \sup_{x \in [0,+\infty)} G(x)$.

Definition 2.6 Let (X, ρ) be a quasi-metric space, $G \in \mathcal{G}$, and $\phi \in \Phi_G$. A mapping $\Upsilon : X \to X$ is called

(i) forward $\phi G-$contraction if

$$\Upsilon x \neq \Upsilon y \quad \Longrightarrow \quad G(\rho(\Upsilon x, \Upsilon y)) \leq \phi(G(\rho(x, y))),$$

(ii) backward $\phi G-$contraction if

$$\Upsilon x \neq \Upsilon y \quad \Longrightarrow \quad G(\rho(\Upsilon x, \Upsilon y)) \leq \phi(G(\rho(y, x))).$$

Theorem 2.1 *Let $G \in \mathcal{G}$ and $\phi \in \Phi_G$. Let (X, ρ) be a quasi-metric space such that*

1. *(X, ρ) is $f-$complete;*

2. *Every f-convergent sequence in quasi-metric space (X, ρ) is also $b-$convergent.*

If $\Upsilon : X \to X$ is a forward $\phi G-$contraction, then Υ is a forward Picard operator.

2.3 The auxiliary spaces

In the following, we present some properties of the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ obtained by Zakiyudin and Fahim in [20].

Lemma 2.1 *The space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is an f-complete quasi-metric space.*

Proof. First, we prove that $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is a quasi-metric space. It can be observed that $\rho^K_{p,\lambda} : X^K_{p,\lambda} \to [0, +\infty)$ and for $x, y \in X^K_{p,\lambda}$, we have $\rho^K_{p,\lambda}(x, y) = 0$ if and only if $x = y$. Next, to prove the triangle inequality, consider the following conditions:

1. When $\|x\|_{L^p_\lambda(0,T;H)} = K$ and $\|y\|_{L^p_\lambda(0,T;H)} = K$.

   We obtain that $\rho^K_{p,\lambda}(x, y) = K$ and $\rho^K_{p,\lambda}(x, z) = \|x - z\|_{L^p_\lambda(0,T;H)}$. Since $\rho^K_{p,\lambda}(x, z)$ and $\rho^K_{p,\lambda}(z, y)$ are non-negative, one has

$$\rho^K_{p,\lambda}(x, y) \leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y).$$

2. When $x \neq 0$, $y = 0$, and $z = 0$.

   We obtain that $\rho^K_{p,\lambda}(z, y) = 0$, $\rho^K_{p,\lambda}(x, y) = \rho^K_{p,\lambda}(x, z)$, then

$$\rho^K_{p,\lambda}(x, y) \leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y).$$

3. When $x = 0$, $y \neq 0$, and $z = 0$.

   We obtain that $\rho^K_{p,\lambda}(x, z) = 0$, $\rho^K_{p,\lambda}(x, y) = \|y\|_{L^p_\lambda(0,T;H)}$, and $\rho^K_{p,\lambda}(z, y) = \|y\|_{L^p_\lambda(0,T;H)}$, then

$$\rho^K_{p,\lambda}(x, y) \leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y).$$

4. When $x = 0$, $y \neq 0$, and $z \neq 0$.

   We obtain that $\rho^K_{p,\lambda}(x, y) = \|y\|_{L^p_\lambda(0,T;H)}$, $\rho^K_{p,\lambda}(x, z) = \|z\|_{L^p_\lambda(0,T;H)}$, and $\rho^K_{p,\lambda}(z, y) = \|z - y\|_{L^p_\lambda(0,T;H)}$. Then, by using the triangle inequality in the norm, one has

$$\|y\|_{L^p_\lambda(0,T;H)} = \|y - z + z\|_{L^p_\lambda(0,T;H)} \leq \|y - z\|_{L^p_\lambda(0,T;H)} + \|z\|_{L^p_\lambda(0,T;H)},$$

   so

$$\rho^K_{p,\lambda}(x, y) \leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y).$$

5. When $x \neq 0$, $y = 0$, and $z \neq 0$.

   We obtain that $\rho^K_{p,\lambda}(x, y) \leq \|x\|_{L^p_\lambda(0,T;H)}$ and $\rho^K_{p,\lambda}(x, z) = \|x - z\|_{L^p_\lambda(0,T;H)}$.
   If $\|z\|_{L^p_\lambda(0,T;H)} > \frac{1}{2}K$, then

$$\rho^K_{p,\lambda}(x, y) \leq \frac{1}{2}K = \rho^K_{p,\lambda}(z, y) \leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y)$$

   and if $\|z\|_{L^p_\lambda(0,T;H)} \leq \frac{1}{2}K$, then

$$\begin{aligned}\rho^K_{p,\lambda}(x, y) \leq \|x\|_{L^p_\lambda(0,T;H)} &\leq \|x - z\|_{L^p_\lambda(0,T;H)} + \|z\|_{L^p_\lambda(0,T;H)} \\ &\leq \rho^K_{p,\lambda}(x, z) + \rho^K_{p,\lambda}(z, y).\end{aligned}$$

6. When $x \neq 0$, $y \neq 0$, and $z = 0$.
   We obtain that $\rho^K_{p,\lambda}(x,y) = \|x-y\|_{L^p_\lambda(0,T;H)}$, $\rho^K_{p,\lambda}(x,z) = K$, and $\rho^K_{p,\lambda}(z,y) = \|y\|_{L^p_\lambda(0,T;H)}$. Then, by using the triangle inequality in the norm, one has

$$\|x-y\|_{L^p_\lambda(0,T;H)} \leq \|x\|_{L^p_\lambda(0,T;H)} + \|y\|_{L^p_\lambda(0,T;H)} \leq K + \|y\|_{L^p_\lambda(0,T;H)},$$

   so

$$\rho^K_{p,\lambda}(x,y) \leq \rho^K_{p,\lambda}(x,z) + \rho^K_{p,\lambda}(z,y).$$

7. When $x \neq 0$, $y \neq 0$, and $z \neq 0$.
   We obtain that $\rho^K_{p,\lambda}(x,y) = \|x-y\|_{L^p_\lambda(0,T;H)}$, $\rho^K_{p,\lambda}(x,z) = \|x-z\|_{L^p_\lambda(0,T;H)}$, and $\rho^K_{p,\lambda}(z,y) = \|z-y\|_{L^p_\lambda(0,T;H)}$. Then, by using the triangle inequality in the norm, one has

$$\|x-y\|_{L^p_\lambda(0,T;H)} \leq \|x-z\|_{L^p_\lambda(0,T;H)} + \|z-y\|_{L^p_\lambda(0,T;H)},$$

   so

$$\rho^K_{p,\lambda}(x,y) \leq \rho^K_{p,\lambda}(x,z) + \rho^K_{p,\lambda}(z,y).$$

Therefore, the triangle inequality holds. Next, we prove that $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is f-complete. Take any f-Cauchy sequence in $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$, then for every $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\rho^K_{p,\lambda}(x_n, x_m) < \varepsilon, \quad \forall m \geq n \geq M(\varepsilon).$$

Then for every $0 < \varepsilon \leq \frac{1}{2}K$, there exists $M(\varepsilon)$ such that

$$\|x_m - x_n\|_{L^p_\lambda(0,T;H)} < \varepsilon, \quad \forall m \geq n \geq M(\varepsilon).$$

This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M\left(\frac{1}{2}K\right)$ for $\varepsilon > \frac{1}{2}K$. Thus, (x_n) is a Cauchy sequence in $(X^K_{p,\lambda}, \|\cdot\|_{L^p_\lambda(0,T;H)})$. Since this space is complete, (x_n) is also convergent in this space. Therefore, there exists $x \in X^K_{p,\lambda}$ such that for all $\varepsilon > 0$, there exists $M(\varepsilon) \in \mathbb{N}$, where

$$\|x - x_n\|_{L^p_\lambda(0,T;H)} < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

This implies that for every $0 < \varepsilon \leq P$, where

$$P = \begin{cases} \frac{1}{2}K & \text{if } x = 0, \\ \min\left\{\frac{1}{2}K, \frac{1}{2}\|x\|_{L^p_\lambda(0,T;H)}\right\} & \text{if } x \neq 0, \end{cases}$$

there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\rho^K_{p,\lambda}(x, x_n) < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M(P)$ so that (x_n) is f-convergent in the quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$. Therefore, $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is an f-complete quasi-metric space.

Lemma 2.2 *Every f-convergent sequence in the quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ is also b-convergent.*

Proof. Take any f-convergent sequence (x_n) in quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$. Then there exists $x \in X^K_{p,\lambda}$ such that for every $0 < \varepsilon \leq \frac{1}{2}K$, there exists $M(\varepsilon) \in \mathbb{N}$ such that

$$\|x - x_n\|_{L^p_\lambda(0,T;H)} = \rho^K_{p,\lambda}(x, x_n) < \varepsilon, \quad \forall n \geq M(\varepsilon).$$

Since $0 < \varepsilon \leq \frac{1}{2}K$, one has $\rho^K_{p,\lambda}(x_n, x) = \|x - x_n\|_{L^p_\lambda(0,T;H)} < \varepsilon,\ \forall n \geq M(\varepsilon)$. This can be generalized for all $\varepsilon > 0$ by taking $M(\varepsilon) = M\left(\frac{1}{2}K\right)$ for $\varepsilon > \frac{1}{2}K$. Consequently, (x_n) is b-convergent in quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$.

3 The Existence and Uniqueness of Solution of the Heat Equation

In the following section, we establish the existence and uniqueness of solution of the heat equation (1).

First, we define the resolvent set $\sigma(A)$ of a linear operator A as the set of all complex numbers λ for which $\lambda I - A$ is invertible, that is, $(\lambda I - A)^{-1}$ is a bounded linear operator. For $\lambda \in \sigma(A)$, the bounded linear operator $R(\lambda : A) := (\lambda I - A)^{-1}$ is called the resolvent of A. Now, we prove that $\{\mathfrak{R}(t)\}_{t\geq 0}$ is a C_0-contractive semigroup, where Δ is its infinitesimal generator, by using the following lemma.

Lemma 3.1 *There exists a C_0-contractive semigroup $\{\mathfrak{R}(t)\}_{t\geq 0}$, where Δ is its infinitesimal generator.*

Proof. Let $x_n \to x$ in H, where $x_k \in D(\Delta)$ for all $k \in \mathbb{N}$ and $\Delta x_n \to y$ in Y. Since $D(\Delta) = H^2(\Omega)$ is a Hilbert space, $x \in D(\Delta)$ and it follows that

$$\|\Delta x_n - \Delta x\|_H = \|x_n - x\|_{D(\Delta)} \to 0.$$

Therefore, $\Delta x_n \to \Delta x$ in H. We get $y = \Delta x$ since the limit of a sequence in the Hilbert space is unique. Thus Δ is a closed operator and we have $\overline{D(\Delta)} = H$. Then, from [26], for $\lambda \in (0, +\infty)$, one has

$$\|R(\lambda : \Delta)\| \leq \frac{1}{\lambda}|\sin^2(\arg(\lambda))| \leq \frac{1}{\lambda}.$$

By the Hille-Yosida Theorem (see [27]), the operator Δ is the infinitesimal generator of the C_0-contractive semigroup $\{\mathfrak{R}(t)\}_{t\geq 0}$. The properties of C_0-semigroup imply that $u(t) = \mathfrak{R}(t)u_0$ is a unique solution to the differential equation

$$\frac{du}{dt} = \Delta u(t), \quad u(0) = u_0.$$

Next, using the Fourier Transform, we have

$$(\mathcal{F}[u(t)])(\xi) = e^{-4\pi^2|\xi|^2 t}\mathcal{F}[u_0], \quad \xi \in \mathbb{R}^n.$$

Therefore, for all $v \in H$,

$$\mathcal{F}(\mathfrak{R}(t)v) = e^{-4\pi^2|\cdot|^2 t}F[v],$$

and

$$\|\mathfrak{R}(t)v\|_H = \|\mathcal{F}(\mathfrak{R}(t)v)\|_H = \left\|e^{-4\pi^2|\cdot|^2 t}F[v]\right\|_H \le \|F[v]\|_H = \|v\|_H.$$

This is the definition of a mild solution to the heat equation (1).

Definition 3.1 Let $\{\mathfrak{R}(t)\}_{t\ge 0}$ be a C_0-contractive semigroup and let Δ be its infinitesimal generator. A function $u:[0,T]\to H$ is called the mild solution of equation (1) if there exists $\lambda^* = \lambda^*(\rho, u_0, T) > 0$ such that $u \in L^p_{\lambda^*}(0,T;H)$ and satisfies

$$u(t) = \mathfrak{R}(t)u_0 + \int_0^t \mathfrak{R}(t) f(u(s))\, \mathrm{d}s.$$

Next, we define a mapping Υ such that

$$\Upsilon(v)(t) = \mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t) f(v(s))\, \mathrm{d}s. \tag{2}$$

Thus, we specify some properties of Υ below.

Lemma 3.2 *Let $\lambda^* > 0$ such that*

$$\lambda^* \ge \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\, \|v_0\|_H^p + 2^{2p-2}T^p\, \|f(v_0)\|_H^p}{K^p} + 2^{3p-3}C^pT^{p-1},$$

then the mapping Υ is a self-mapping on X^K_{p,λ^*}.

Proof. Let $v \in X^K_{p,\lambda}$, then $\Upsilon(v)(0) = v_0 = w_0$. Since $\mathfrak{R}$ is a C_0-contractive semigroup of $\{\mathfrak{R}(t)\}_{t\ge0}$ and by applying Hölder's inequality, we have

$$\begin{aligned}
\|\Upsilon(v)(t)\|^p_{L^p_\lambda(0,T;H)} &= \left\|\mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t) f(v(s))\ ds\right\|^p_{L^p_\lambda(0,T;H)} \\
&\le 2^{p-1}\left[\|\mathfrak{R}(t)v_0\|^p_{L^p_\lambda(0,T;H)} + \left\|\int_0^t \mathfrak{R}(t) f(v(s))\ ds\right\|^p_{L^p_\lambda(0,T;H)}\right] \\
&= 2^{p-1}\int_0^T e^{-\lambda t}\, \|\mathfrak{R}(t)v_0\|_H^p\ dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t}\left\|\int_0^t \mathfrak{R}(t) f(v(s))\ ds\right\|_H^p\ dt \\
&\le 2^{p-1}\int_0^T e^{-\lambda t}\, \|v_0\|_H^p\ dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t}\left(\int_0^t \|f(v(s))\|_H\ ds\right)^p\ dt \\
&\le \frac{2^{p-1}\, \|v_0\|_H^p}{\lambda} + 2^{p-1}\int_0^T e^{-\lambda t} 2^{p-1}\left(\int_0^T \|f(v(0))\|_H\ ds\right)^p\ dt \\
&\quad + 2^{p-1}\int_0^T e^{-\lambda t} 2^{p-1}\left(\int_0^t C\, \|v(s)-v(0)\|_H\ ds\right)^p dt \\
&\le \frac{2^{p-1}\, \|v_0\|_H^p}{\lambda} + 2^{2p-2}C^p\int_0^T e^{-\lambda t} 2^{p-1}\left(\int_0^t \|v(s)\|_H\ ds)\right)^p\ dt
\end{aligned}$$

$$
\begin{aligned}
&+ 2^{2p-2}C^p \int_0^T e^{-\lambda t} 2^{p-1} \left(\int_0^t \|v(0)\|_H \ ds \right)^p dt \\
&+ 2^{2p-2} \int_0^T e^{-\lambda t} \left(\int_0^T \|f(v(0))\|_H \ ds \right)^p \ dt \\
\leq\ & \frac{2^{p-1}\|v_0\|_H^p}{\lambda} + 2^{3p-3}C^p \int_0^T e^{-\lambda t} \left(\int_0^t \|v(s)\|_H \ ds \right)^p dt \\
&+ \left(2^{3p-3}C^p \|v(0)\|_H^p + 2^{2p-2}\|f(v(0))\|_H^p \right) \frac{T^p}{\lambda} \\
\leq\ & \frac{2^{p-1}\|v_0\|_H^p}{\lambda} + 2^{3p-3}L^p \int_0^T \int_s^T e^{-\lambda t} t^{p-1} \left(\|v(s)\|_H \right)^p \ dt \ ds \\
&+ \left(2^{3p-3}C^p \|v(0)\|_H^p + 2^{2p-2}\|f(v(0))\|_H^p \right) \frac{T^p}{\lambda} \\
\leq\ & \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\|v(0)\|_H^p}{\lambda} + \frac{2^{2p-2}T^p\|f(v(0))\|_H^p}{\lambda} \\
&+ \frac{2^{3p-3}C^pT^{p-1}K^p}{\lambda}.
\end{aligned}
$$

Now, let

$$\lambda = \lambda^* \geq \frac{(2^{p-1} + 2^{3p-3}C^pT^p)\|v_0\|_H^p + 2^{2p-2}T^p\|f(v_0)\|_H^p}{K^p} + 2^{3p-3}C^pT^{p-1},$$

and we obtain that if $v \in X_{p,\lambda^*}^K$, then $\Upsilon(v) \in X_{p,\lambda^*}^K$.

Lemma 3.3 *Let* $\alpha = \left(\frac{T^{p-1}C^p}{\lambda^*} \right)^{\frac{1}{p}}$ *and assume that* λ^* *satisfies* $\lambda^* > T^{p-1}C^p$. *Then for any* $u, v \in X_{p,\lambda^*}^K$, *the following inequality is satisfied:*

$$\rho_{p,\lambda^*}^K(\Upsilon u, \Upsilon v) \leq \alpha \rho_{p,\lambda^*}^K(u, v). \tag{3}$$

Proof. Consider that $v \in X_{p,\lambda^*}^K$, so $\Upsilon v \neq 0$. We can prove this by contradiction. Assume that $\Upsilon v = 0$, it follows that

$$\mathfrak{R}(t)v_0 + \int_0^t \mathfrak{R}(t) f(v(s))\, ds = 0.$$

If we set $t = 0$, this implies $v_0 = 0$, which contradicts the fact that $v_0 \neq 0$. Hence, $\Upsilon v \neq 0$. Subsequently, since S is a C_0-contractive semigroup and by applying Hölder's inequalities, for all $u, v \in X_{p,\lambda^*}^K$, we have

$$
\begin{aligned}
\rho_{p,\lambda}^K(\Upsilon u, \Upsilon v) &= \|\Upsilon u - \Upsilon v\|_{L_\lambda^p(0,T;H)} \\
&\leq \left\| \int_0^t \mathfrak{R}(t) \left[f(u(s)) - f(v(s)) \right] \ ds \right\|_{L_\lambda^p(0,T;H)} \\
&= \left(\int_0^T e^{-\lambda t} \left(\| \int_0^t \mathfrak{R}(t) \left[f(u(s)) - f(v(s)) \right] \|_H \, ds \right)^p \ dt \right)^{1/p} \\
&\leq \left(\int_0^T e^{-\lambda t} \left(\int_0^t \|f(u(s)) - f(v(s))\|_H \, ds \right)^p \ dt \right)^{1/p}
\end{aligned}
$$

$$\leq \left(\int_0^T e^{-\lambda t} \left(\int_0^t C \|u(s) - v(s)\|_H \, ds \right)^p dt \right)^{1/p}$$

$$\leq \left(C^p \int_0^T e^{-\lambda t} t^{p-1} \int_0^t \left(\|u(s) - v(s)\|_H \right)^p \, ds \, dt \right)^{1/p}$$

$$\leq \left(\frac{C^p T^{p-1}}{\lambda} \int_0^T e^{-\lambda s} \left(\|u(s) - v(s)\|_H \right)^p \, ds \right)^{1/p}.$$

Now, let $\lambda = \lambda^* > T^{p-1} C^p$, then we obtain

$$\rho^K_{p,\lambda^*}(\Upsilon u, \Upsilon v) \leq \alpha \rho^K_{p,\lambda^*}(u, v),$$

where

$$\alpha = \left(\frac{T^{p-1} C^p}{\lambda} \right)^{1/p} < 1.$$

Finally, we establish the existence and uniqueness of solution to the heat equation (1) through the following theorem.

Theorem 3.1 *For some λ^*, the heat equation (1) has a unique mild solution in the space $(X^K_{p,\lambda^*}, \rho^K_{p,\lambda^*})$.*

Proof. Let $G(x) = x$ and $\phi(x) = \alpha x = \left(\frac{T^{p-1} C^p}{\lambda^*} \right)^{1/p} x$, and

$$\lambda^* > \max \left\{ \frac{(2^{p-1} + 2^{3p-3} C^p T^p) \|v_0\|^p_H + 2^{2p-2} T^p \|f(v_0)\|^p_H}{K^p} + 2^{3p-3} C^p T^{p-1}, \right.$$
$$\left. T^{p-1} C^p \right\}.$$

Then, by Lemmas 3.2 and 3.3, and also by Definition 2.6, we have that $\Upsilon : X^K_{p,\lambda^*} \to X^K_{p,\lambda^*}$ is a forward ϕG-contraction. The necessary conditions for the quasi-metric space $(X^K_{p,\lambda^*}, \rho^K_{p,\lambda^*})$ in Theorem 2.1 are satisfied by Lemma 2.1 and Lemma 2.2. This implies that Υ is a forward Picard Operator, hence there exists a unique $u \in X^K_{p,\lambda^*} \subset L^p_{\lambda^*}(0, T; H)$ such that

$$u(t) = \mathfrak{R}(t) u_0 + \int_0^t \mathfrak{R}(t) f(u(s)) \, ds.$$

Therefore, $u(t)$ is the unique mild solution of the heat equation (1).

4 Conclusion

In this paper, we have demonstrated the existence and uniqueness of mild solutions to the heat equation (1) in the quasi-metric space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$. This result was obtained by using the fixed point theory of the ϕG-contraction, as stated in Theorem 2.1. We established that the space $(X^K_{p,\lambda}, \rho^K_{p,\lambda})$ satisfies the necessary conditions for Theorem 2.1 to hold. Moreover, we showed that the mapping $\Upsilon(u)(t) = u(t)$, where $u(t)$ is the mild solution of the heat equation, is a ϕG-contraction in this space. The final result, presented in Theorem 3.1, confirms that the mild solution to the heat equation exists and is unique within the given quasi-metric space framework. Future work may explore the extension of these results to more general classes of differential equations.

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