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Solvability of Functional Equations' Classes Arising in Dynamic Programming Using Fixed-Point Technique

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Abstract: This paper is devoted to the existence, uniqueness, and iterative approximation of solutions for the classes of functional equations arising in dynamic programming of multistage decision processes. The findings presented here extend and integrate several results from the existing literature. Illustrative examples are also provided to emphasize the significance of the main results. The approach is based on fixed point techniques applied in suitable function spaces. Furthermore, our results unify a variety of known theorems within a broader and more flexible framework.

Keywords: *functional equations; dynamic programming; fixed point; iterative approximation.*

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1 Introduction

In 1922, Banach [1] proved his celebrated fixed point theorem, commonly known as the Banach contraction principle. Bellman [2, 3] introduced and explored the existence of solutions for a class of functional equations arising in dynamic programming. Since then, many researchers (see [4–11]) have studied the existence and uniqueness of solutions to functional equations by modifying the conditions of Bellman's equations in the context of multistage decision processes.

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Recent advances in the study of functional equations, fractional differential models, and fixed point theory have led to significant developments in both theory and applications across various mathematical and engineering domains. In particular, methods for solving Volterra-type and fractional integro-differential equations have been extensively developed using novel decomposition and numerical strategies [12–14]. Parallel to this, fixed point techniques in generalized metric spaces have emerged as powerful tools for addressing nonlinear problems in functional analysis and dynamic systems [15–17]. These tools have found fruitful applications in the analysis of reaction-diffusion systems, blow-up phenomena, and inverse problems under integral constraints [18–24]. Building on this growing body of work, our paper aims to explore the solvability of functional equations arising in dynamic programming by applying fixed point techniques in suitable function spaces.

In this work, we present and analyze the existence, uniqueness, and iterative approximation of solutions for the following classes of functional equations and systems of functional equations arising in dynamic programming:

$$f(x) = \operatorname{opt}_{y\in D}\left\{u(x,y) + \operatorname{opt}\left\{p_i(x,y) + A_i(x,y,f(a_i(x,y))) : i = 1,2,3\right\}\right\}, \tag{1}$$

$$\begin{aligned} f(x) = \operatorname{opt}_{y\in D}\big\{&u(x,y) + r(x,y)f(c(x,y)) + \operatorname{opt}\{p_i(x,y)f(s(x,y)),\\ &t_i(x,y) + q_i(x,y)A_i(x,y,f(a_i(x,y))) : i = 1,2,3\}\big\}, \end{aligned} \tag{2}$$

$$f(x) = \operatorname{opt}_{y\in D}\left\{p(x,y) + \operatorname{opt}\left\{u_i(x,y) + A_i(x,y,g(a_i(x,y))) : i = 1,2,3\right\}\right\}, \tag{3a}$$

$$f(x) = \operatorname{opt}_{y\in D}\left\{q(x,y) + \operatorname{opt}\left\{v_i(x,y) + B_i(x,y,f(b_i(x,y))) : i = 1,2,3\right\}\right\}. \tag{3b}$$

Here, opt denotes either sup or inf. The variables x and y represent the state and decision vectors, respectively. The mappings s, c, and a_i $(i = 1,2,3)$ denote process transformations, and the functions $f(x)$ and $g(x)$ represent the optimal return functions with initial state x.

The structure of this paper is as follows. In Section 2, we introduce basic concepts, notations, and useful lemmas. In Section 3, we establish the existence, uniqueness, and iterative approximation of solutions to functional equation (1) in the spaces $BC(S)$ and $B(S)$.

2 Preliminaries

In this section, we introduce notations, definitions, and some preliminary results that will be used in the remainder of the paper. Let $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{R}^- = (-\infty,0]$. For every $t \in \mathbb{R}$, let $[t]$ denote the greatest integer less than or equal to t. Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|')$ be real Banach spaces. Let $S \subseteq X$ denote the state space and $D \subseteq Y$ denote the decision space. We define the following function classes:

$$\begin{aligned} \Phi_1 &= \left\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \varphi \text{ is right-continuous at } t = 0\right\},\\ \Phi_2 &= \left\{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \mid \varphi \text{ is non-decreasing}\right\},\\ \Phi_3 &= \{\varphi \in \Phi_1 \mid \varphi(0) = 0\},\\ \Phi_4 &= \{\varphi \in \Phi_1 \cap \Phi_2 \mid \varphi(t) < t \text{ for all } t > 0\},\\ \Phi_5 &= \{\varphi \in \Phi_2 \mid \varphi(t) < t \text{ for all } t > 0\}. \end{aligned}$$

We also define the following function spaces:

$$\begin{aligned} B(S) &= \{f : S \to \mathbb{R} \mid f \text{ is bounded}\}, \\ BC(S) &= \{f \in B(S) \mid f \text{ is continuous}\}, \\ BB(S) &= \{f : S \to \mathbb{R} \mid f \text{ is bounded on bounded subsets of } S\}. \end{aligned}$$

It is easy to verify that both $(B(S), \|\cdot\|_1)$ and $(BC(S), \|\cdot\|_1)$ are Banach spaces under the supremum norm $\|f\|_1 = \sup_{x\in S} |f(x)|$. For every positive integer k and for $f, g \in BB(S)$, define the pseudometric

$$d_k(f, g) = \sup\left\{|f(x) - g(x)| : x \in \overline{B}(0, k)\right\},$$

where $\overline{B}(0, k) = \{x \in S : \|x\| \le k\}$. We then define a complete metric d on $BB(S)$ by

$$d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f, g)}{1 + d_k(f, g)}.$$

The collection $\{d_k\}_{k\ge1}$ forms a countable family of pseudometrics on $BB(S)$. A sequence $\{f_n\} \subseteq BB(S)$ is said to converge to $f \in BB(S)$ if for every $k \in \mathbb{N}$, we have $d_k(f_n, f_m) \to 0$ as $n, m \to \infty$. Therefore, $(BB(S), d)$ is a complete metric space.

Definition 2.1 A metric space (M, ρ) is said to be *metrically convex* if for any $x, y \in M$, there exists $z \in M$, with $z \ne x, y$, such that $\rho(x, y) = \rho(x, z) + \rho(z, y)$. Intuitively, every Banach space is metrically convex.

Lemma 2.1 (Menger [5]) *If M is a complete metrically convex metric space, then for every $\alpha \in (0, 1)$ and for any $x, y \in M$, there exists $z \in M$ such that $\rho(x, z) = \alpha\rho(x, y)$ and $\rho(z, y) = (1 - \alpha)\rho(x, y)$.*

Lemma 2.2 ([5]) *Assume that M is a complete metrically convex metric space and that $f : M \to M$ is a mapping satisfying*

$$\rho(fx, fy) \le K\rho(x, y) \tag{4}$$

for some constant $K < \infty$. Define the function $\phi : [0, b) \to [0, b)$ by

$$\phi(t) = \sup\{\rho(fx, fy) : x, y \in M,\ \rho(x, y) = t\}.$$

Then we have

(a) ϕ is subadditive; that is, for all $s, t > 0$ such that $s + t < b$, we have

$$\phi(s + t) \le \phi(s) + \phi(t).$$

(b) ϕ is upper semicontinuous from the right on $[0, b)$.

Proof. (a) Let $x, y \in M$ with $\rho(x, y) = s + t$. Since M is metrically convex, there exists $z \in M$ such that $\rho(x, z) = s$ and $\rho(z, y) = t$. Then

$$\rho(fx, fy) \le \rho(fx, fz) + \rho(fz, fy) \le \phi(s) + \phi(t). \quad \Diamond$$

Taking the supremum over all such $x, y \in M$ with $\rho(x, y) = s + t$ gives the result in (a).

(b) From part (a), for $t > t_0$ and $t - t_0 < b$, we have

$$\phi(t) \leq \phi(t - t_0) + \phi(t_0) \leq K(t - t_0) + \phi(t_0)$$

since $\phi(t - t_0) \leq K(t - t_0)$ by the assumption in equation (4). Therefore,

$$\limsup_{t \to t_0^+} \phi(t) \leq \phi(t_0),$$

proving that ϕ is upper semicontinuous from the right at t_0.

Lemma 2.3 *Let M be a complete metric space, and let $f : M \to M$ be a mapping satisfying*

$$\rho(fx, fy) \leq \varphi\left(\rho(x, y)\right), \tag{5}$$

where $\varphi : \overline{P} \to \mathbb{R}^+$ is upper semicontinuous from the right on $\overline{P}$ and satisfies $\varphi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$. Then f has a unique fixed point $x_0 \in M$, and for all $x \in M$, the sequence $f^n x \to x_0$ as $n \to \infty$.

Proof. Fix $x \in M$ and define $c_n = \rho(f^n x, f^{n-1}x)$. From the contractive condition, we have

$$c_{n+1} = \rho(f^{n+1}x, f^n x) = \rho(f(f^n x), f(f^{n-1}x)) \leq \varphi(c_n),$$

which implies that $\{c_n\}$ is a decreasing sequence. Since $\varphi(t) < t$ for all $t > 0$, the sequence must converge to a limit $c \geq 0$. Assume, by contradiction, that $c > 0$. Then, taking limits, we get

$$c = \lim_{n \to \infty} c_n \leq \limsup_{t \to c^+} \varphi(t) \leq \varphi(c),$$

which contradicts $\varphi(c) < c$. Therefore, $c = 0$. Next, we show that $\{f^n x\}$ is a Cauchy sequence. Suppose not. Then there exist $\varepsilon > 0$ and sequences $\{m_k\}$, $\{n_k\}$ with $m_k > n_k \geq k$ such that

$$d_k := \rho(f^{m_k}x, f^{n_k}x) \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Choose m_k as the smallest index greater than n_k such that this holds. Since $c_n \to 0$, we may assume

$$\rho(f^{m_k - 1}x, f^{n_k}x) < \varepsilon.$$

Then

$$d_k \leq \rho(f^{m_k}x, f^{m_k-1}x) + \rho(f^{m_k-1}x, f^{n_k}x) \leq c_{m_k} + \varepsilon.$$

Thus, $d_k \to \varepsilon^+$ as $k \to \infty$. However, consider

$$\begin{aligned}
d_k &= \rho(f^{m_k}x, f^{n_k}x) \\
&\leq \rho(f^{m_k}x, f^{m_k+1}x) + \rho(f^{m_k+1}x, f^{n_k+1}x) + \rho(f^{n_k+1}x, f^{n_k}x) \\
&\leq c_{m_k+1} + c_{n_k+1} + \varphi(d_k) \leq 2c_k + \varphi(d_k).
\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\varepsilon \leq \varphi(\varepsilon),$$

which contradicts $\varphi(\varepsilon) < \varepsilon$ for $\varepsilon > 0$. Hence, $\{f^n x\}$ is a Cauchy sequence. By the completeness of M, there exists a limit point $x_0 \in M$ such that $f^n x \to x_0$. Since f is continuous (implied by the inequality and structure of the argument), we have

$$fx_0 = \lim_{n\to\infty} f(f^n x) = \lim_{n\to\infty} f^{n+1} x = x_0,$$

so x_0 is a fixed point of f. To prove uniqueness, suppose x_0 and x_1 are both fixed points. Then

$$\rho(x_0, x_1) = \rho(fx_0, fx_1) \le \varphi(\rho(x_0, x_1)) < \rho(x_0, x_1),$$

which is a contradiction unless $\rho(x_0, x_1) = 0$, hence $x_0 = x_1$.

Lemma 2.4 ([5]) *Let (M, ρ) be a complete metrically convex metric space, and let $f : M \to M$ be a mapping satisfying*

$$\rho(fx, fy) \le \varphi\left(\rho(x, y)\right) \quad \text{for all } x, y \in M, \tag{6}$$

where $\varphi : \overline{P} \to \mathbb{R}^+$ is a function such that $\varphi(t) < t$ for all $t \in \overline{P} \setminus \{0\}$, with $P = \{\rho(x, y) : x, y \in M\}$ and $\overline{P}$ being its closure. Then f has a unique fixed point $u \in M$, and for every $x \in M$, we have

$$\lim_{n\to\infty} f^n(x) = u.$$

Proof. Let $\phi : [0, b) \to [0, b)$ be defined as in Lemma 2.2:

$$\phi(t) = \sup\{\rho(fx, fy) : x, y \in M,\ \rho(x, y) = t\}.$$

By Lemma 2.2, ϕ is upper semicontinuous from the right on $[0, b)$. Furthermore, for all $t \in [0, b)$,

$$\phi(t) \le \varphi(t).$$

If $P = [0, b]$ with $b < \infty$, we extend ϕ by defining $\phi(b) := \varphi(b)$. Then we have

$$\rho(fx, fy) \le \phi\left(\rho(x, y)\right) \quad \text{for all } x, y \in M.$$

Now, since ϕ is upper semicontinuous from the right and satisfies $\phi(t) < t$ for $t > 0$, the conditions of Lemma 2.1 (the generalized fixed point theorem) are satisfied for ϕ. Hence, f has a unique fixed point $u \in M$, and the sequence $\{f^n(x)\}$ converges to u for every $x \in M$.

Lemma 2.5 ([8]) *Let $\{a_i, b_i : 1 \le i \le n\} \subseteq \mathbb{R}$. Then*

$$\left|\text{opt}\{a_i : 1 \le i \le n\} - \text{opt}\{b_i : 1 \le i \le n\}\right| \le \max\left\{|a_i - b_i| : 1 \le i \le n\right\}. \tag{$*$}$$

Here, opt *denotes either the supremum or infimum.*

Proof. The inequality clearly holds for $n = 1$. Assume that $(*)$ holds for some $n \in \mathbb{N}$. Consider the case $n + 1$:

$$\text{opt}\{a_i : 1 \le i \le n+1\} = \text{opt}\left\{\text{opt}\{a_i : 1 \le i \le n\},\ a_{n+1}\right\},$$

and similarly for the b_i's. Using the inductive hypothesis and Lemma 2.1 from [9], we obtain

$$\begin{aligned}
&\left|\text{opt}\{a_i : 1 \le i \le n+1\} - \text{opt}\{b_i : 1 \le i \le n+1\}\right| \\
&\quad \le \max\left\{\left|\text{opt}\{a_i : 1 \le i \le n\} - \text{opt}\{b_i : 1 \le i \le n\}\right|, |a_{n+1} - b_{n+1}|\right\} \\
&\quad \le \max\left\{|a_i - b_i| : 1 \le i \le n+1\right\}.
\end{aligned}$$

Hence, by induction, the inequality holds for all $n \in \mathbb{N}$.

Lemma 2.6 *Let $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$. Then*

$$\max\{|a_i + b_i| : i = 1, 2, 3\} \le \max\{|a_i| : i = 1, 2, 3\} + \max\{|b_i| : i = 1, 2, 3\}.$$

Proof. By the triangle inequality, we have for each $i = 1, 2, 3$:

$$|a_i + b_i| \le |a_i| + |b_i|.$$

Taking the maximum over i, we obtain

$$\max\{|a_i + b_i| : i = 1, 2, 3\} \le \max\{|a_i| + |b_i| : i = 1, 2, 3\}.$$

Finally, since for each i,

$$|a_i| + |b_i| \le \max\{|a_i| : i = 1, 2, 3\} + \max\{|b_i| : i = 1, 2, 3\},$$

we conclude

$$\max\{|a_i + b_i| : i = 1, 2, 3\} \le \max\{|a_i| : i = 1, 2, 3\} + \max\{|b_i| : i = 1, 2, 3\}.$$

Lemma 2.7 *[8]*

(i) Let $A : S \times D \to \mathbb{R}$ be a mapping such that $\operatorname{opt}_{y\in D} A(x_0, y)$ is bounded for some $x_0 \in S$. Then

$$\left|\operatorname{opt}_{y\in D} A(x_0, y)\right| \le \sup_{y\in D} |A(x_0, y)|.$$

(ii) Let $A, B : S \times D \to \mathbb{R}$ be mappings such that both $\operatorname{opt}_{y\in D} A(x_1, y)$ and $\operatorname{opt}_{y\in D} B(x_2, y)$ are bounded for some $x_1, x_2 \in S$. Then

$$\left|\operatorname{opt}_{y\in D} A(x_1, y) - \operatorname{opt}_{y\in D} B(x_2, y)\right| \le \sup_{y\in D} |A(x_1, y) - B(x_2, y)|.$$

Proof. **(i)** If $\sup_{y\in D} |A(x_0, y)| = +\infty$, the inequality holds trivially. Otherwise, assume $\sup_{y\in D} |A(x_0, y)| < \infty$. Then, for all $y \in D$,

$$-|A(x_0, y)| \le A(x_0, y) \le |A(x_0, y)|.$$

Taking the infimum and supremum over $y \in D$, we obtain

$$\inf_{y\in D} A(x_0, y) \ge -\sup_{y\in D} |A(x_0, y)|, \quad \sup_{y\in D} A(x_0, y) \le \sup_{y\in D} |A(x_0, y)|.$$

Hence,

$$\left|\operatorname{opt}_{y\in D} A(x_0, y)\right| \le \sup_{y\in D} |A(x_0, y)|.$$

(ii) If $\sup_{y\in D} |A(x_1, y) - B(x_2, y)| = +\infty$, the inequality holds trivially. Otherwise, for all $y \in D$,

$$|A(x_1, y) - B(x_2, y)| \le \sup_{y\in D} |A(x_1, y) - B(x_2, y)|.$$

Thus,

$$B(x_2, y) - \sup_{y\in D} |A(x_1, y) - B(x_2, y)| \le A(x_1, y) \le B(x_2, y) + \sup_{y\in D} |A(x_1, y) - B(x_2, y)|.$$

Taking $\operatorname{opt}_{y\in D}$ on both sides yields

$$\left|\operatorname{opt}_{y\in D} A(x_1, y) - \operatorname{opt}_{y\in D} B(x_2, y)\right| \le \sup_{y\in D} |A(x_1, y) - B(x_2, y)|.$$

3 Existence and Uniqueness of Solutions in $BC(S)$ and $B(S)$

We now discuss the existence and uniqueness of solutions to the functional equation (1) in the spaces $BC(S)$ and $B(S)$.

Theorem 3.1 *Let $u, p_i : S \times D \to \mathbb{R}$, $a_i : S \times D \to S$, and $A_i : S \times D \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, 3$ be given mappings. Let $\varphi \in \Phi_3$ and $\psi \in \Phi_4$. Assume the following conditions hold:*

(C1) *The mappings u, p_i, and A_i are bounded for each $i = 1, 2, 3$.*

(C2) *For every $x_0 \in S$, the limits*

$$u(x, y) \to u(x_0, y), \quad p_i(x, y) \to p_i(x_0, y), \quad a_i(x, y) \to a_i(x_0, y)$$

hold uniformly for $y \in D$ as $x \to x_0$, for all $i = 1, 2, 3$.

(C3) *For all $x, x_0 \in S$, $y \in D$, and $z \in \mathbb{R}$,*

$$\max \{|A_i(x, y, z) - A_i(x_0, y, z)| : i = 1, 2, 3\} \leq \varphi (\|x - x_0\|) .$$

(C4) *For all $x \in S$, $y \in D$, and $z, z_0 \in \mathbb{R}$,*

$$\max \{|A_i(x, y, z) - A_i(x, y, z_0)| : i = 1, 2, 3\} \leq \psi (\|z - z_0\|) .$$

Then the functional equation (1) admits a unique solution $w \in BC(S)$, and for every $h \in BC(S)$, the sequence $\{H^n h\}_{n\geq 1}$ converges to w, where the operator $H : BC(S) \to BC(S)$ is defined by

$$Hh(x) = \text{opt}_{y\in D} \{u(x, y) + \text{opt} \{p_i(x, y) + A_i(x, y, h(a_i(x, y))) : i = 1, 2, 3\}\} , x \in S. \tag{8}$$

Proof. Let $x_0 \in S$ and $h \in BC(S)$. By condition (C1), it is clear that Hh is bounded. From (C2), along with $\varphi \in \Phi_3$ and $\psi \in \Phi_4$, we know that for any $\epsilon > 0$, there exist $\delta_1, \delta_2, \delta_3 > 0$ such that

$$\varphi(\|x - x_0\|) < \frac{\epsilon}{4} \quad \text{for } x \in S \text{ with } \|x - x_0\| < \delta_1, \tag{9}$$

$$\psi(\delta_1) < \frac{\epsilon}{4}, \tag{10}$$

$$|u(x, y) - u(x_0, y)| < \frac{\epsilon}{4} \quad \text{for } (x, y) \in S \times D \text{ with } \|x - x_0\| < \delta_1, \tag{11}$$

$$\max \{|p_i(x, y) - p_i(x_0, y)| : i = 1, 2, 3\} < \frac{\epsilon}{4} \quad \text{for } \|x - x_0\| < \delta_1, \tag{12}$$

$$|h(x) - h(x_0)| < \delta_1 \quad \text{for } x \in S \text{ with } \|x - x_0\| < \delta_2, \tag{13}$$

$$\max \{|a_i(x, y) - a_i(x_0, y)| : i = 1, 2, 3\} < \delta_2 \quad \text{for } \|x - x_0\| < \delta_3. \tag{14}$$

From (10), (13), and (14), we deduce

$$\psi \left(\sup_{y\in D} \{\max \{|h(a_i(x, y)) - h(a_i(x_0, y))| : i = 1, 2, 3\}\} \right) < \frac{\epsilon}{4}, \quad \text{for } \|x - x_0\| < \delta_3. \tag{15}$$

Let $\delta = \min\{\delta_1, \delta_3\}$. Using (C3), (C4), (9), and (15), we get for all $x \in S$ with $\|x - x_0\| < \delta$:

$$\begin{aligned}
&|Hh(x) - Hh(x_0)| \\
&= \big|\mathrm{opt}_{y\in D}\left\{u(x,y) + \mathrm{opt}\left\{p_i(x,y) + A_i(x,y,h(a_i(x,y)))\right\}\right\} \\
&\quad - \mathrm{opt}_{y\in D}\left\{u(x_0,y) + \mathrm{opt}\left\{p_i(x_0,y) + A_i(x_0,y,h(a_i(x_0,y)))\right\}\right\}\big| \\
&\le \sup_{y\in D}\left\{|u(x,y) - u(x_0,y)| + \max_{i=1,2,3}\Big[|p_i(x,y) - p_i(x_0,y)|\right. \\
&\quad \left. + |A_i(x,y,h(a_i(x,y))) - A_i(x_0,y,h(a_i(x_0,y)))|\Big]\right\} \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

This proves that Hh is continuous at x_0, and hence H maps $BC(S)$ into itself. Now, let $h, g \in BC(S)$ and fix $x \in S$. Suppose $\mathrm{opt}_{y\in D} = \sup_{y\in D}$. Then there exist $y, z \in D$ such that

$$\begin{aligned}
Hh(x) &< u(x,y) + \mathrm{opt}\left\{p_i(x,y) + A_i(x,y,h(a_i(x,y)))\right\} + \epsilon, \\
Hg(x) &< u(x,z) + \mathrm{opt}\left\{p_i(x,z) + A_i(x,z,g(a_i(x,z)))\right\} + \epsilon, \\
Hh(x) &\ge u(x,z) + \mathrm{opt}\left\{p_i(x,z) + A_i(x,z,h(a_i(x,z)))\right\}, \\
Hg(x) &\ge u(x,y) + \mathrm{opt}\left\{p_i(x,y) + A_i(x,y,g(a_i(x,y)))\right\}.
\end{aligned} \tag{16}$$

Using (16) and condition (C4), we have

$$\begin{aligned}
|Hh(x) - Hg(x)| &< \max_{i=1,2,3}\{|A_i(x,y,h(a_i(x,y))) - A_i(x,y,g(a_i(x,y)))|, \\
&\qquad |A_i(x,z,h(a_i(x,z))) - A_i(x,z,g(a_i(x,z)))|\} + \epsilon \\
&\le \psi\left(\max_{i=1,2,3}\{|h(a_i(x,y)) - g(a_i(x,y))|, |h(a_i(x,z)) - g(a_i(x,z))|\}\right) + \epsilon \\
&\le \psi(\|h - g\|_1) + \epsilon.
\end{aligned}$$

Taking the supremum over $x \in S$, we obtain

$$\|Hh - Hg\|_1 \le \psi(\|h - g\|_1) + \epsilon. \tag{17}$$

A similar argument holds if $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. Letting $\epsilon \to 0^+$ in (17), we conclude

$$\|Hh - Hg\|_1 \le \psi(\|h - g\|_1).$$

By Lemma 2.4 (a variant of Boyd and Wong's fixed point theorem [6]), the operator H has a unique fixed point $w \in BC(S)$, and for every $h \in BC(S)$, the sequence $\{H^n h\}_{n\ge1}$ converges to w. Clearly, w is the unique solution of the functional equation (8) in $BC(S)$.

Theorem 3.2 *Let $u, p_i : S \times D \to \mathbb{R}$, $a_i : S \times D \to S$, and $A_i : S \times D \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, 3$ be given mappings. Suppose that $\psi \in \Phi_5$, and that conditions* (C1) *and* (C4) *hold. Then the functional equation* (1) *admits a unique solution $w \in BC(S)$, and the sequence $\{H^n h\}_{n\ge1}$ converges to w for every $h \in BC(S)$, where the operator H is defined by (8).*

Remark 3.1

1. If $u = 0$ and $p_3 = A_3 = 0$, then Theorem 3.2 reduces to Theorem 3.2 of Pathak and Deepmala [25].
2. Theorem 3.2 generalizes and strengthens the classical result of Bellman [2].

Example 3.1 We present an example that demonstrates how Theorem 3.1 generalizes and unifies several previous results, including those found in [2, 4, 8, 10, 11, 25, 26]. Let $X = Y = \mathbb{R}$, $S = [1, \infty)$, and $D = \mathbb{R}^+ = [0, \infty)$. Define

$$\varphi(t) = 3t, \quad \psi(t) = \frac{t}{4}.$$

Then Theorem 3.1 guarantees that the functional equation

$$\begin{aligned} f(x) = \operatorname{opt}_{y \in D} \Bigg\{ 1 + \frac{1}{x^2 + \frac{1}{2}y} + \operatorname{opt} \Bigg\{ &\frac{x^2}{x + y^2} + \frac{1}{x^2 + y^2 + 1} + \frac{1}{3} \sin\big(f(2x^2 y)\big), \\ &\frac{x + y}{1 + 3(x + y)} + \frac{1}{x^2 + y} + \frac{1}{3 + |f(\sin(2x) + 3y)|}, \\ &\frac{x^3}{x + y^4} + \frac{1}{1 + x^2 + 2y^2} + \frac{f\big(5 + \sin(7x - 3y)\big)}{3 + 3f\big(5 + \sin(7x - 3y)\big)^2} \Bigg\} \Bigg\}, \quad \forall x \in S, \end{aligned}$$

possesses a unique solution in $B(S)$. To see this, define the following mappings:

$$\begin{aligned} &u(x, y) = 1 + \frac{1}{x^2 + \frac{1}{2}y}, \\ &p_1(x, y) = \frac{x^2}{x + y^2}, \quad p_2(x, y) = \frac{x + y}{1 + 3(x + y)}, \quad p_3(x, y) = \frac{x^3}{x + y^4}, \end{aligned}$$

and

$$\begin{aligned} A_1(x, y, z) &= \frac{1}{x^2 + y^2 + 1} + \frac{1}{3 \sin z}, \\ A_2(x, y, z) &= \frac{1}{x^2 + y} + \frac{1}{3 + |z|}, \\ A_3(x, y, z) &= \frac{1}{1 + x^2 + 2y^2} + \frac{z}{3 + 3z^2}. \end{aligned}$$

Then the following relations hold:

$$\begin{aligned} |A_i(x, y, z) - A_i(x_0, y, z)| &\le 3|x - x_0| \quad \text{for all } i = 1, 2, 3, \\ |A_i(x, y, z) - A_i(x, y, z_0)| &\le \frac{1}{4}|z - z_0| \quad \text{for all } i = 1, 2, 3. \end{aligned}$$

Thus, all the assumptions of Theorem 3.2 are satisfied, and hence the functional equation possesses a unique solution in $B(S)$.

4 Conclusion

In this paper, we established new existence, uniqueness, and iterative approximation results for the classes of functional equations arising in dynamic programming. By employing fixed point techniques and suitable contractive conditions, we extended and unified several known results in the literature. The general framework presented here accommodates a broad range of applications and improves upon classical formulations such as Bellman's equations. A detailed example was also provided to illustrate the applicability of our main theorem.

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