Nonlinear Dynamics and Systems Theory, **25** (6) (2025) 691–704

Fokker-Planck Equation and Its Application in Production Function

S. Sutrima [1*], S. Siswanto [1], S. Wibowo [1] and C.R. Indrati [2]

[1] *Department of Mathematics, Sebelas Maret University, Surakarta, Indonesia,*
[2] *Department of Mathematics, Gadjah Mada University, Yogyakarta, Indonesia.*

Received: February 17, 2025; Revised: November 11, 2025

Abstract: The one-dimensional Fokker-Planck equation (FPE) with drift and diffusion coefficients depending on the space variable is identified by a semigroup approach. The stationary solution u_s of the FPE induces a Hilbert space X, i.e., $L^2(a,b)$ with an inner product weighted by u_s. The backward Fokker-Planck operator A generates a C_0-semigroup in X. The well-posedness for the FPE follows the well-posedness for the Cauchy problem generated by A. The solution u is asymptotically stable with respect to u_s as $t \to \infty$. Furthermore, if the ratio of the drift to diffusion coefficients is nondecreasing, then u is a nonnegative classical solution. As an application, the backward Fokker-Planck operator A confirms the well-posedness for production function equations. In case $X = L^2(0,\infty)$, the operator A has a continuous spectrum generating the Gaus-Weierstrass semigroup.

Keywords: *Fokker-Planck equation; stationary solution; C_0-semigroup; well-posed; production function.*

Mathematics Subject Classification (2020): 35K57, 35Q84, 47D07, 70K20,93-10.

1 Introduction

The Fokker-Planck equation (FPE) which originally describes a Brownian motion of a particle has wide applications leading to many interdisciplinary studies, for example, in solid state physics, quantum optics, chemical physics, theoretical biology, circuit theory, plasma waves, finance and economics. Concretely, the applications of the FPEs are found in differential equations and stochastic processes [1, 2], bilinear control systems

* Corresponding author: mailto:sutrima@mipa.uns.ac.id

and chaotic systems [3,4], production and generation of entropies [5–7], plasma physics, protein folding and fluid [8–10].

There has been a lot of theoretical and numerical research discussing the FPE for various cases and applications. Bluman [11] proposed to construct the exact solutions for the one-dimensional FPE corresponding to a class of non-linear forcing functions using group theoretic methods. A particle approach to solve variational formulation of the FPE with decay was considered in [12]. The analytical solution of the FPE with logarithmic, decreasing and bistable drifts has been investigated in [13–15]. See also the numerical research of the FPE using He's variational iteration method [16], the differential transform method [17], the cubic B-spline scaling functions [18], the Adomian decomposition method [19], and the homotopy perturbation method [20].

The generic form of the one-dimensional FPE for the probability density $u(x,t)$ depending on the space variable x and time t is given by

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[f(x,t)u\right] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x,t)}{2}u\right] \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

where $u(x,t)$ is the unknown function, $f(x,t)$ is the drift coefficient or force, and $D(x,t) > 0$ is the diffusion coefficient [21]. Equation (1) describes a motion for the distribution function $u(x,t)$ and is called the forward Kolmogorov equation. Also, a similar equation, the backward Kolmogorov equation, is given by

$$\frac{\partial u}{\partial t} = -f(x,t)\frac{\partial u}{\partial x} + \frac{D(x,t)}{2}\frac{\partial^2 u}{\partial x^2}. \tag{2}$$

The special case of FPE (1) provides the distribution function $W(v,t)$ for a small particle of mass m immersed in a fluid satisfying

$$\frac{\partial W}{\partial t} = \gamma\frac{\partial (vW)}{\partial v} + \gamma\frac{KT}{m}\frac{\partial^2 W}{\partial v^2},$$

where v is the velocity for the Brownian motion of a particle, t is the time, γ is the fraction constant, K is Boltzmann's constant and T is the temperature of fluid [21]. In this case, the drift coefficient is linear and the diffusion coefficient is a constant.

We note that the partial differential equations in (1) and (2) are nonlinear-parabolic. This nonlinearity affects the difficulties in solving the equations, either theoretically or numerically. For the linear case, the well-posedness for the equations follows Theorems 12, 15 and 16 in Chapter I of [22]. However, we can not explicitly find the sufficiency and necessity of the well-posedness from the theorems. We see that the linearities of the FPE depend on the type of drift and diffusion coefficients. In the fact, if the drift and diffusion coefficients do not depend on u, the equations are linear. The FPE of this type are involved in the advection-diffusion equations, see [23,24]. This allows us to formulate the FPE as an autonomous or non-autonomous abstract Cauchy problem. In this context, the C_0-semigroup or C_0-quasi semigroup approach has proven to be reliable for solving the problems, which have not been discussed in previous studies. Therefore, the infinitesimal generator of the C_0-semigroup or C_0-quasi semigroup can characterize the well-posedness and the properties of solution of the problems.

In this paper, we focus on the well-posedness and the properties of solution of FPE (1), where the drift and diffusion coefficients depend only on the space variable x, and the use of C_0-semigroup approach and its application in the product function given in [23]. The organization of the results is as follows. Section 2 provides the well-posedness and some properties of the solution of FPE (1) equipped with the boundary values. In Section 3, we apply the FPE to solve the product function in two cases of the spectrum of the infinitesimal generator of the related semigroup.

2 Solution of Fokker-Planck equation

In this paper, we study the Fokker–Planck equation in one dimension in a bounded interval with boundary conditions. Let X_t be a diffusion process in the interval $[0, \ell]$ with the drift and diffusion coefficients $f(x)$ and $D(x)$, respectively. We assume that $D(x)$ is positive in $[0, \ell]$. The transition probability density $u(x, t)$ is the solution of an initial and boundary value problem of the FPE

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)u\right] + \frac{\partial^2}{\partial x^2}\left[\frac{D(x)}{2}u\right], \quad 0 < x < \ell, \tag{3a}$$

with the boundary and initial conditions

$$J(u(0)) = J(u(\ell) = 0, \tag{3b}$$

$$u(x, 0) = u_0(x), \quad 0 \le x \le \ell, \tag{3c}$$

where

$$J(u) := f(x)u(x) - \frac{1}{2}\frac{d}{dx}\left[D(x)u(x)\right], \quad 0 \le x \le \ell.$$

We can rewrite problem (3) as a Cauchy problem and use a strongly continuous semigroup (C_0-semigroup) to solve. Indeed, problem (3) is the Cauchy problem

$$\dot{u}(t) = \mathcal{A}u(t), \quad t \ge 0, \qquad u(0) = u_0, \tag{4}$$

where $\mathcal{A}$ is a generator of the Fokker–Planck (forward Kolmogorov) equation (3) that is defined by

$$\mathcal{A}u := -\frac{d}{dx}\left[f(x)u\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[D(x)u\right] \tag{5}$$

on a domain

$$\mathcal{D}(\mathcal{A}) := \{u \in C^2(0, \ell) : J(u(0)) = J(u(\ell)) = 0\}.$$

Henceforth, we call the generator $\mathcal{A}$ the Fokker-Planck operator.

To solve the Fokker–Planck problem (3), we first transform the equation into the backward Kolmogorov equation generated by an operator A. Note that the boundary conditions for the operator A are simpler than those for the operator $\mathcal{A}$. Due to this fact, we first find the stationary solution of problem (3). Let $u_s(x)$ be the stationary solution of (3), i.e., the long-time limit of $u(x, t)$ as $t \to \infty$, which follows from the equation

$$-\frac{d}{dx}\left[f(x)u_s(x) - \frac{1}{2}\frac{d}{dx}\left[D(x)u_s(x)\right]\right] = 0.$$

Integrating the both sides with respect to x yields $J(u_s) = C$ for some constant C. However, the boundary conditions give a zero flux

$$J(u_s) = f(x)u_s(x) - \frac{1}{2}\frac{d}{dx}\left[D(x)u_s(x)\right] = 0. \tag{6}$$

Therefore, we have

$$\frac{2f(x)}{D(x)}dx = \frac{du_s(x)}{u_s(x)} + \frac{dD(x)}{D(x)}$$

that gives the stationary solution

$$u_s(x) = N^{-1}Y(x), \quad x \in [0, \ell], \tag{7}$$

where

$$Y(x) = \frac{1}{D(x)} \exp\left(2\int_0^x \frac{f(\tau)}{D(\tau)}d\tau\right)$$

and N is the normalization constant,

$$N = \int_0^\ell \frac{1}{D(x)} \exp\left(2\int_0^x \frac{f(\tau)}{D(\tau)}d\tau\right) dx.$$

Further, from (6), if $u = vu_s$, we have

$$J(u) = J(vu_s) = -\frac{1}{2}D(x)u_s(x)\frac{dv}{dx}.$$

This confirms that if $J(u(0)) = J(u(\ell)) = 0$, then $v'(0) = v'(\ell) = 0$. Therefore, when setting $u(x,t) = v(x,t)u_s(x)$, the problem (3) leads to the Cauchy problem

$$\dot{v}(t) = Av(t), \quad t \geq 0, \qquad v(0) = v_0(x) = u_s^{-1}(x)u_0(x), \tag{8}$$

where A generates the backward Kolmogorov equation given by

$$Av := f(x)\frac{dv}{dx} + \frac{D(x)}{2}\frac{d^2v}{dx^2} \tag{9}$$

on the domain $\mathcal{D}(A) = \{v \in C^2(0,\ell) : v'(0) = v'(\ell) = 0\}$. In other words, solving problem (3) is sufficient to solve problem (8). The sufficiency for the Cauchy problem (8) is well-posed is that A is the infinitesimal generator of a C_0-semigroup.

Lemma 2.1 *If $u(x) = v(x)u_s(x)$ for $v \in \mathcal{D}(A)$, then $\mathcal{A}u = u_sAv$ and both have common eigenvalues.*

Proof. For $u(x) = v(x)u_s(x)$ and from (6), we obtain

$$\begin{aligned}
\mathcal{A}u &= \frac{d}{dx}\left[-f(x)v(x)u_s(x) + \frac{1}{2}\frac{d}{dx}\left[D(x)v(x)u_s(x)\right]\right] \\
&= \frac{d}{dx}\left[\left[-f(x)u_s(x) + \frac{1}{2}\frac{d}{dx}[D(x)u_s(x)]\right]v(x) + \frac{1}{2}[D(x)u_s(x)]\frac{dv}{dx}\right] \\
&= \frac{1}{2}\frac{d}{dx}[D(x)u_s(x)]\frac{dv}{dx} + \frac{1}{2}[D(x)u_s(x)]\frac{d^2v}{dx^2} \\
&= u_s(x)\left[f(x)\frac{dv}{dx} + \frac{D(x)}{2}\frac{d^2v}{dx^2}\right] = u_sAv.
\end{aligned}$$

Therefore, if λ is an eigenvalue of $\mathcal{A}$ corresponding to the eigenfunction u, then λ is the eigenvalue of A with the eigenfunction v.

Henceforth, let X be the Hilbert space $L^2(0,\ell)$, the space of square-integrable functions in the interval $(0,\ell)$, with respect to the weighted inner product

$$\langle v, w\rangle_{u_s} := \int_0^\ell v(x)w(x)u_s(x)\,dx.$$

The operator A in the Hilbert space X gives the following.

Lemma 2.2 *The operator A has the following properties:*

(a) $-A$ is a positive operator on X.

(b) The null space of A consists of constants.

(c) A is self-adjoint with a discrete spectrum in the Hilbert space X.

Proof. (a) Using an integration by parts and the fact that the stationary solution u_s is positive, for $v \in \mathcal{D}(A)$, we obtain

$$\langle -Av, v\rangle_{u_s} = \int_0^\ell (-Av)vu_s\,dx = \int_0^\ell -\mathcal{A}(u_s)v\,dx = \frac{1}{2}\int_0^\ell |v'|^2 Du_s\,dx. \tag{10}$$

Therefore, $-A$ is positive.

(b) Let v be in the null space of A. From (10), we deduce that

$$\int_0^\ell |v'|^2 Du_s\,dx = 0.$$

Since $u_s, D > 0$, this implies that v is a constant.

(c) For all $v, w \in \mathcal{D}(A)$, again using the integration by parts and the stationary solution, we obtain

$$\int_0^\ell v\mathcal{A}(wu_s)\,dx = \left[\frac{Du_s}{2}(vw' - wv')\right]_0^\ell + \int_0^\ell \left(fv' + \frac{D}{2}v''\right) wu_s\,dx = \int_0^\ell Avwu_s\,dx.$$

Therefore, Lemma 2.1 gives the self-adjointness of A,

$$\langle Av, w\rangle_{u_s} = \int_0^\ell Avwu_s\,dx = \int_0^\ell v\mathcal{A}(wu_s)\,dx = \int_0^\ell vAwu_s\,dx = \langle v, Aw\rangle_{u_s}.$$

Since A is self-adjoint in the Hilbert space $L^2(0,\ell)$, where $[0,\ell]$ is a bounded interval, A has a discrete spectrum [28]. Moreover, the eigenvalues λ_n of $-A$ are real and nonnegative such that $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. Also, the eigenfunctions ϕ_n corresponding to the eigenvalues λ_n form an orthonormal basis in X. From (b), we obtain that $\lambda_0 = 0$ corresponds to $\phi_0 = 1$.

As a consequence of Lemma 2.2, the Fokker-Planck operator $\mathcal{A}$ is nonpositive, self-adjoint with discrete spectrum and its null space is the one-dimensional space with the basis $\{u_s\}$.

Theorem 2.1 *The operator A generates a contraction of C_0-semigroup $T(t)$ given by*

$$[T(t)v](x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\langle v, \phi_n\rangle_{u_s}\phi_n(x), \quad v \in L^2(0,\ell). \tag{11}$$

Proof. Since A is self-adjoint, A is a closed operator. Thus, A is the Riesz spectral operator with real eigenvalues $-\lambda_n$ such that $\sup\{-\lambda_n : n \in \mathbb{N}_0\} = 0 < \infty$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Therefore, A is the infinitesimal generator of the C_0-semigroup $T(t)$ given by (11), Theorem 2.3.5 of [25]. Furthermore, since $\{\phi_n\}$ are orthonormal, for $v = \phi_n$, (11) gives

$$T(t)\phi_n = e^{-\lambda_n t}\phi_n \qquad \text{or} \qquad \|T(t)\| \leq 1, \quad t \geq 0,$$

i.e., $T(t)$ is a contraction.

Theorem 2.2 *The Fokker-Planck problem* (3) *is well-posed. Moreover, for any initial condition u_0, the solution $u(x,t)$ satisfies*

$$\lim_{t\to\infty} u(x,t) = \gamma_0 u_s(x),$$

where $\gamma_0 = \int_0^\ell u_0(x)\,dx$.

Proof. Since the operator A generates a C_0-semigroup in X, the Cauchy problem (8) is well-posed with a solution $v(x,t) = T(t)v_0(x)$. This implies the well-posedness of the Fokker-Planck problem (3) with the solution

$$u(x,t) = u_s(x)[T(t)u_s^{-1}u_0](x). \tag{12}$$

Moreover, the equations (11) and (12) give

$$u(x,t) = \gamma_0 u_s(x) + \sum_{n=1}^{\infty} e^{-\lambda_n t}\left[\int_0^\ell u_0(\tau)\phi_n(\tau)\,d\tau\right] u_s(x)\phi_n(x),$$

where $\gamma_0 = \int_0^\ell u_0(x)\,dx$. Since all $\lambda_n > 0$ for $n \geq 1$, this implies that $u(x,t)$ converges to $\gamma_0 u_s(x)$ as $t \to \infty$.

Theorem 2.3 *If $u_0(x) \geq 0$ and the ratio of the drift and diffusion $r(x) = \frac{f(x)}{D(x)}$ is nondecreasing in $(0,\ell)$, then the Fokker-Planck problem* (3) *has a uniquely nonnegative classical solution.*

Proof. The uniqueness follows the uniqueness of the solution of the Cauchy problems. To prove the nonnegativity of the solution, we set $u = \exp(\int_0^x \frac{f-2D'}{2D})v$ and substitute in equation (3), which leads to

$$v_t - Dv_{xx} + \frac{D}{2}\left[\frac{d}{dx}\left(\frac{f}{D}\right) + \frac{1}{2}\left(\frac{f}{D}\right)^2\right] v \equiv 0. \tag{13}$$

The nondecreasing of the ratio r implies that $r'(x) \geq 0$ for all $x \in (0,\ell)$. This imposes the coefficient of v in (13) is nonnegative. Consequently, by the maximum principle (Lemma 4.1, p. 19 of [26]), the solution v is nonnegative. The nonnegativity of u follows.

Corollary 2.1 *If f and D are constants, the solution of* (3) *is*

$$u(x,t) = \gamma_0 u_s(x) + \sum_{n=1}^{\infty} e^{\lambda_n t + \frac{f}{D}x}\left[\alpha_n \cos\frac{n\pi}{\ell}x + \beta_n \sin\frac{n\pi}{\ell}x\right], \tag{14}$$

where

$$\alpha_n = \frac{2N_0}{\ell}\int_0^\ell [u_0(\tau) - \gamma_0 u_s(\tau)] \exp\left(-\frac{f}{D}\tau\right) \cos\frac{n\pi}{\ell}\tau\, d\tau,$$
$$\beta_n = \frac{2N_0}{\ell}\int_0^\ell [u_0(\tau) - \gamma_0 u_s(\tau)] \exp\left(-\frac{f}{D}\tau\right) \sin\frac{n\pi}{\ell}\tau\, d\tau.$$

If f and D are constants, the stationary solution of (3) is $u_s(x) = N_0^{-1}\exp\left(\frac{2f}{D}x\right)$, where $N_0 = \int_0^\ell \exp\left(\frac{2f}{D}x\right) dx$ and the eigenvalues and eigenfunctions of A are given by

$$\lambda_0 = 0, \qquad \lambda_n = -\frac{1}{2}\left(\frac{n^2\pi^2 D}{\ell^2} + \frac{f^2}{D}\right),$$
$$\phi_0(x) = 1, \qquad \phi_n(x) = e^{-\frac{f}{D}x}\left[A_n \cos\frac{n\pi}{\ell}x + B_n \sin\frac{n\pi}{\ell}x\right], \quad n \geq 1,$$

respectively, where A_n and B_n are constants such that $\int_0^\ell \phi_n(x)\, dx = 1$. By solving the Fourier coefficients, the solution (12) gives the solution (14). Moreover, we verify that Theorems 2.2 and 2.3 are valid for this solution.

3 An Application to Production Function

In micro economics, a technical relationship between physical inputs (land, labour, capital) and physical outputs (quantity produced) is described by the production function. The relationship is not an economic relationship, but only studies the relationship of material inputs on one side and material outputs on the other side. The production function is denoted by $F = F(L(\tau), K(\tau))$, where $L(\tau)$ and $K(\tau)$ are the quantities of labour and capital at time τ in the time interval [0, T], respectively. Henceforth, we assume that the production function F is homogenous of degree one. Let $x(\tau) = K(\tau)/L(\tau)$ and $y = u(x(\tau), \tau)$, and we also assume that $x(\tau)$ is the solution to the following stochastic differential equation:

$$dx(\tau) = x(\tau)[a d\tau + b d\omega(\tau)], \tag{15}$$

where $\omega(\tau)$ is a standard Brownian motion, a and b are constants. Let $u(x, \tau)$ denote the value of the production function at any instant τ, $\tau \geq 0$. Using Ito's lemma [27], we have

$$du = \frac{\partial u}{\partial \tau} d\tau + \frac{\partial u}{\partial x} dx + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} dx^2. \tag{16}$$

Putting (15) and (16) together, we find that

$$du = \left[\frac{\partial u}{\partial \tau} + ax\frac{\partial u}{\partial x} + \frac{b^2}{2}x^2\frac{\partial^2 u}{\partial x^2}\right] d\tau + bx\frac{\partial u}{\partial x} d\omega(\tau). \tag{17}$$

We assume that the production function can be written as

$$u(x, \tau) = u_d(x, \tau) + \Delta x(\tau), \tag{18}$$

where Δ is unknown to be determined such that

$$du_d(x, \tau) = r u_d(x, \tau) d\tau$$

and r is a real positive constant. If $\Delta = \frac{\partial u}{\partial x}$, then the stochastic term vanishes, and we obtain

$$\frac{\partial u}{\partial \tau} + ax\frac{\partial u}{\partial x} + \frac{b^2}{2}x^2\frac{\partial^2 u}{\partial x^2} - ru = 0.$$

This is a linear second-order partial differential equation. Under the change of variable $t = \frac{b^2(\mathrm{T}-\tau)}{2}$ and equipped an initial value, it becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= x^2\frac{\partial^2 u}{\partial x^2} + \gamma x\frac{\partial u}{\partial x} - \gamma u, \\ u(x,0) &= u_0(x), \end{aligned} \tag{19}$$

where $\gamma := \frac{2r}{b^2}$.

Let $\mathcal{L}$ be a linear operator defined on the Hilbert space $X = L^2(a,b)$ by

$$(\mathcal{L}u)(x) := x^2 u^{''}(x) + \gamma x u'(x) - \gamma u(x) \tag{20}$$

on the domain

$$\mathcal{D}(\mathcal{L}) = \{u \in X : u, u' \text{ are absolute continuous and } u', u^{''} \in X\}.$$

The initial problem (19) can be written as a Cauchy problem in X

$$\begin{aligned} u_t &= \mathcal{L}u,\ t > 0, \\ u(0) &= u_0, \qquad u_0 \in X. \end{aligned} \tag{21}$$

The sufficient condition for well-posedness for the Cauchy problem (21) is that $\mathcal{L}$ is the infinitesimal generator of a C_0-semigroup in X. We note that the operator $\mathcal{L}$ is composed to be

$$\mathcal{L} = A - \gamma I,$$

where A is the backward Fokker-Planck operator given in (9) with $f(x) = \gamma x$ and $D(x) = 2x^2$. The main aim is to prove the operator $\mathcal{L}$ defined in (20) generates a C_0-semigroup. The form of the semigroup depends on the type of the spectrum of $\mathcal{L}$ or A, i.e., discrete or continuous spectrum.

3.1 The infinitesimal generator with discrete spectrum

We see that if the initial problem (19) works on the finite boundary values, then $\mathcal{L}$ has the discrete spectrum [28]. In this case, we assume the dynamics of production $u(x,t)$ is described by the Dirichlet boundary problem on $X = L^2(1,\ell)$, $\ell > 1$,

$$\begin{aligned} u_t &= \mathcal{L}u, \quad 1 < x < \ell,\ t > 0, \\ u(1,t) &= u(\ell,t) = 0,\ t > 0, \\ u(x,0) &= u_0(x), \quad 1 \le x \le \ell. \end{aligned} \tag{22}$$

In this subsection, we assume $\gamma \neq 1$. First, we consider the Cauchy problem generated by the backward Fokker-Planck operator A in X

$$\begin{aligned} u_t &= Au, \quad 1 < x < \ell,\ t > 0, \\ u(1,t) &= u(\ell,t) = 0,\ t > 0, \\ u(x,0) &= u_0(x), \quad 1 \le x \le \ell. \end{aligned} \tag{23}$$

In this case, the stationary solution of the forward Fokker-Planck equation is

$$u_s(x) = \frac{1}{2N} x^{\gamma-2}, \quad 1 \le x \le \ell, \tag{24}$$

where $N = (\ell^{\gamma-1} - 1)/2(\gamma - 1)$. We can find out that the eigenvalues and eigenfunctions of A are

$$\begin{aligned} \lambda_n &= -\left(\frac{n\pi}{\ln \ell}\right)^2 - \left(\frac{\gamma-1}{2}\right)^2, \\ \phi_n(x) &= 2x^{-(\gamma-1)/2}\sqrt{\frac{N}{\ln \ell}} \sin\left(\frac{n\pi}{\ln \ell} \ln x\right), \quad n \in \mathbb{N}, \end{aligned} \tag{25}$$

respectively. Therefore, the C_0-semigroup $T(t)$ generated by A in (11) is given by

$$(T(t)v)(x) = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle v, \phi_n \rangle_{u_s} \phi_n(x), \quad v \in L^2(1, \ell), \tag{26}$$

where λ_n and ϕ_n are the eigenvalues and eigenfunctions, respectively, given in (25) and the weighted inner product $\langle \cdot, \cdot \rangle_{u_s}$ in $L^2(1, \ell)$ with respect to u_s is given in (24).

By the theorem on the perturbation for the semigroups and the fact that $\mathcal{L} = A - \gamma I$, the operator $\mathcal{L}$ is the infinitesimal generator of the C_0-semigroup $S(t)$ given by

$$S(t) = e^{-\gamma t} T(t),$$

where $T(t)$ is given in (26). Theorem 3.6 of [29] implies that the dynamics of production inducted by $\mathcal{L}$ has a unique solution

$$u(x,t) = (S(t)u_0)(x) = \sum_{n=1}^{\infty} e^{(\lambda_n - \gamma)t} \langle u_0, \phi_n \rangle_{u_s} \phi_n(x).$$

Therefore, we have the following result.

Theorem 3.1 *The dynamics of production $u(x,t)$ described by the Cauchy problem* (22) *is well-posed.*

Remark 3.1 The forward Fokker-Planck problem corresponding to the operator A is

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}\left[\gamma x u\right] + \frac{\partial^2}{\partial x^2}\left[x^2 u\right], \quad 1 < x < \ell, \\ J(u(1)) &= J(u(\ell)) = 0, \\ u(x,0) &= u_0(x), \quad 1 \le x \le \ell, \end{aligned} \tag{27}$$

where

$$J(u(x)) = \gamma x u(x) - \frac{d}{dx}\left[x^2 u(x)\right], \quad 1 \le x \le \ell.$$

In particular, if we choose $\ell = e$ and $u_0(x) = \delta x^{\alpha}$, $\alpha \in (0,1)$, $\delta > 0$, Theorem 2.2 confirms that the solution of problem (27) is

$$u(x,t) = \gamma_0 u_s(x) + \sum_{n=1}^{\infty} \gamma_n e^{\lambda_n t} u_s(x) \phi_n(x), \quad 1 \le x \le e,$$

where u_s is given in (24) and

$$\begin{aligned}
\gamma_0 &= \frac{\delta(e^{\alpha-1}-1)}{\alpha-1}, \\
\gamma_n &= \frac{[1-(-1)^n e^{\nu}]\delta n\pi}{(\nu^2+n^2\pi^2)\sqrt{N}}, \qquad \nu = \frac{\gamma-1}{2}+\alpha, \\
\lambda_n &= -n^2\pi^2 - \left(\frac{\gamma-1}{2}\right)^2, \\
\phi_n(x) &= 2x^{-(\gamma-1)/2}\sqrt{N}\sin(n\pi\ln x).
\end{aligned}$$

Therefore, since $\lambda_n < 0$ for all $n \geq 1$, we have

$$\lim_{t\to\infty} u(x,t) = \gamma_0 u_s(x).$$

3.2 The infinitesimal generator with a continuous spectrum

Now, we consider the dynamics of production $u(x,t)$ of the Cauchy problem (19) in the Hilbert space $X = L^2(0,\infty)$,

$$\begin{aligned}
u_t &= \mathcal{L}u, \quad x \geq 0,\ t > 0, \\
u(x,0) &= u_0(x) = \begin{cases} \delta e^{\alpha x}, & 0 \leq x \leq \ell \\ 0, & x > \ell, \end{cases}
\end{aligned} \tag{28}$$

where $\alpha \in (0,1)$, $\delta > 0$, and $\ell > 0$. As before, we must prove that the operator $\mathcal{L}$ generates a C_0-semigroup. However, in this context, the spectrum of A in the form $\mathcal{L} = A - \gamma I$ is not discrete. Also, the corresponding forward Fokker-Planck equation does not have the stationary solution since the integral in (7) diverges. Indeed, the operator A has a continuous spectrum, so we can not apply the method used in the above section. For this purpose, first, we define the linear operator

$$(Vu)(x) := xu'(x) + \mu u(x), \qquad \mu = \tfrac{1}{2}(\gamma-1)$$

on the domain

$$\mathcal{D}(V) = \{u \in X : u \text{ is absolute continuous and } u' \in X\}.$$

We can check that

$$A = V^2 - \mu^2 I = (V-\mu I)(V+\mu I).$$

This implies that the spectrum of A depends on the spectrum of operators $(V-\mu I)$ and $(V+\mu I)$.

Lemma 3.1 *The operator A in* (20) *has a continuous spectrum in* $\mathbb{R}$.

Proof. We see that the resolvent operator $\mathcal{R}(\lambda, V+\mu I)$ exists only if the equation $(\lambda I - (V+\mu I))u = f$ has a unique solution. By the variation of constants formula, we find the resolvent operator given by $H : X \to X$,

$$(Hu)(x) := x^{\lambda-\gamma+1}\int_x^{\infty} \frac{u(s)}{s^{\lambda-\gamma+2}}ds, \quad u \in X, \quad \lambda > \gamma.$$

We need to show that the operator H is well-defined and bounded. We note that

$$\begin{aligned}|(Hu)(x)|^2 &\le x^{2(\lambda-\gamma+1)}\left(\int_x^\infty \frac{|u(s)|}{s^{\lambda-\gamma}}\frac{s^{\lambda-\gamma}}{s^{\lambda-\gamma+2}}\right)^2\\ &\le x^{2(\lambda-\gamma+1)}\int_x^\infty \frac{|u(s)|^2}{s^{2(\lambda-\gamma)}}ds\int_x^\infty \frac{s^{2(\lambda-\gamma)}}{s^{2(\lambda-\gamma)+4}}ds = \frac{1}{3}\int_x^\infty \frac{|u(s)|^2}{s^{2(\lambda-\gamma)}}x^{2(\lambda-\gamma)-1}ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\|Hu\|_2^2 &= \int_0^\infty |(Hu)(x)|^2dx\\ &\le \frac{1}{3}\int_0^\infty\int_x^\infty \frac{|u(s)|^2}{s^{2(\lambda-\gamma)}}x^{2(\lambda-\gamma)-1}dsdx\\ &= \frac{1}{3}\int_0^\infty\int_0^s \frac{|u(s)|^2}{s^{2(\lambda-\gamma)}}x^{2(\lambda-\gamma)-1}dxds = \frac{1}{6(\lambda-\gamma)}\|u\|_2^2.\end{aligned}$$

Thus H is well-defined and bounded when $\lambda > \gamma$. Moreover, we have $Hu \in \mathcal{D}(V)$ and

$$((V+\mu I)Hu)(x) = x(Hu)'(x) + 2\mu(Hu)(x) = -u(x) + \lambda(Hu)(x)$$

for all $u \in X$, and

$$(H(V+\mu I)u)(x) = (H(xu'))(x) + 2\mu(Hu)(x) = -u(x) + \lambda(Hu)(x)$$

for all $u \in \mathcal{D}(V)$. This proves that the resolvent operator $\mathcal{R}(\lambda, V+\mu I)$ exists and equals to H with $\rho(V+\mu I) = (\gamma, \infty)$.

Similarly, we can show that $\rho(V-\mu I) = (0,\infty)$. Therefore, the resolvent set $\rho(A) = (\tau, \infty)$, where $\tau = \max\{0, \gamma\}$. Thus, the spectrum $\sigma(A) = \mathbb{R} - \rho(A)$ is not discrete.

Since the spectrum $\sigma(A)$ is not discrete, we can not use the expansion of the eigenfunctions to solve the Cauchy problem (28). Using the concept of the fundamental solution for the diffusive equations, we can construct the Gaus-Weierstrass semigroup for the Cauchy problem (28), see [30].

Theorem 3.2 *The operator $\mathcal{L}$ is the infinitesimal generator of the C_0-semigroup $T(t)$ given by $T(0) = I$ and*

$$(T(t)u)(x) = \frac{e^{-\gamma t}}{\sqrt{4\pi t}}\int_{-\infty}^\infty \exp\left(-\frac{(\ln x + (\gamma-1)t - s)^2}{4t}\right)u(s)\,ds, \quad t > 0. \tag{29}$$

Moreover, the Cauchy problem (28) *is well-posed in X.*

Proof. Substitution $x = e^\xi$ reduces the operator A to be the differential operator with constant coefficients

$$A_0u := u_{\xi\xi} + (\gamma-1)u_\xi.$$

Following Example 1.8 of [30] and returning the substitution, we see that the operator A generates a C_0-semigroup $T_0(t)$ given by $T_0(0) = I$ and

$$(T_0(t)u)(x) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^\infty \exp\left(-\frac{(\ln x + (\gamma-1)t - s)^2}{4t}\right)u(s)\,ds, \quad t > 0. \tag{30}$$

Thus, the operator $\mathcal{L}$ is the infinitesimal generator of the C_0-semigroup $T(t)$ given by

$$T(t) = e^{-\gamma t} T_0(t), \quad t \geq 0.$$

Further, from Theorem 3.6 of [29], the Cauchy problem (28) is well-posed with the solution $u(x,t) = (T(t)u_0)(x)$, $u_0 \in X$. Indeed,

$$\begin{aligned} u(x,t) &= \frac{\delta e^{-\gamma t}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(\ln x + (\gamma-1)t - s)^2}{4t} + \alpha s \right) ds, \\ &= \frac{\delta x^\alpha e^{-(\alpha+(1-\alpha)\gamma)t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(z^2+2\alpha\sqrt{t}z)} dz, \qquad z = \frac{\ln x + (\gamma-1)t - s}{2\sqrt{t}} \\ &= \frac{\delta x^\alpha e^{-(1-\alpha)(\alpha+\gamma)t}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(z+\alpha\sqrt{t})^2} dz, \\ &= \delta x^\alpha e^{-(1-\alpha)(\alpha+\gamma)t}. \end{aligned}$$

Remark 3.2 (a) Theorem 3.2 gives an alternative way for solving the Fokker-Planck operator with the mixed eigenvalue spectrum of [31]. Furthermore, Theorem 3.2 can also be applied to solve the problem of the Fokker-Planck equation and Kramers' reaction rate theory [15].

(b) The FPE is also applicable to some advection-diffusion equations. As an example, we consider the advection-diffusion of the transport equation of open channels and river flows [32]

$$\frac{\partial C}{\partial t} = -U \frac{\partial C}{\partial x} + \frac{1}{A} \frac{\partial}{\partial x} \left(K A \frac{\partial C}{\partial x} \right). \tag{31}$$

Equation (31) describes the evolution of contaminant concentration $C(x,t)$ in a one-dimensional flow in a channel of cross-sectional area $A(x,t)$, with a mean flow velocity $U(x,t)$ and a diffusion coefficient $K(x,t)$. In cases where U and K depend only on x and A is a constant, equation (31) is the backward Kolmogorov equation with the drift coefficient $f(x) = \frac{\partial K}{\partial x} - U(x)$ and the diffusion coefficient $D(x) = 2K(x)$. Therefore, the methods used in the previous sections can be implemented to solve the equation.

(c) The FPE can also be used to solve the advection diffusion equation of larval dispersal alongshore [24]

$$\frac{\partial C}{\partial t} = -\frac{\partial}{\partial x} \left(U(x,t)C - K(x,t) \frac{\partial C}{\partial x} \right) + g(x,t), \tag{32}$$

where $C(x,t)$ is larval concentration at alongshore position x and time t, $U(x,t)$ is the mean advection velocity on a given time interval, $K(x,t)$ is the local dispersion coefficient and $g(x,t)$ is the 'reaction' term, which represents a source or sink. We see that equation (32) is the nonhomogenous perturbated backward Kolmogorov equation with the drift coefficient $f(x,t) = \frac{\partial K}{\partial x} - U$, the diffusion coefficient $D(x,t) = 2K(x,t)$ and the perturbation coefficient $-\frac{\partial U}{\partial x}$. Thus, the solution method for the FPE together with the Duhamel principle gives the solution of (32).

4 Conclusion

A semigroup approach is applicable for the Fokker-Planck equation with the drift and diffusion coefficients of a space variable. The stationary solution u_s of the FPE inducted a Hilbert space $X = L^2(a, b)$ with an inner product weighted by u_s. The corresponding backward Fokker-Planck operator A generates a C_0-semigroup $T(t)$ in X. The well-posedness of the Fokker-Planck equation follows the well-posedness of the Cauchy problem generated by A. The solution u of the FPE is asymptotically stable with respect to u_s as $t \to \infty$. As an application, the backward Fokker-Planck operator A confirms the well-posedness of product function equations.

Acknowledgements
The authors would like to thank the referees for valuable comments improving this paper and Sebelas Maret University for supporting and funding through Fundamental Research under Grant No.194.2/UN27.22/PT.01.03/2024.

References

[1] D. Liberzon and R.W. Brockett. Spectral analysis of Fokker-Planck and related operators arising from linear stochastic differential equations. *SIAM J. Control Optim.* **38** (5) (2000) 1453—1467.

[2] G.A. Pavliotis. *Stochastic Processes and Applications.* Springer Science Business Media, New York, 2014.

[3] T. Breiten, K. Kunisch and K. Pfeiffer. Control strategies for the Fokker-Planck equation. *ESAIM: Control Optim. Calc. Var.* **24** (2) (2018) 741-–763.

[4] J.M. Heninger, D. Lippolis and P. Cvitanović. Perturbation theory for the Fokker-Planck operator in chaos. *Commun. Nonlinear Sci. Numer. Simul.* **55** (2018) 16–28.

[5] J. Chandrasegaran and P. Sharma. A Short Review on Fokker-Planck equations, entropy production and entropy generation. *Bioinfo. Proteom. Img. Anal.* **4** (1) (2020) 21–24.

[6] U. Lucia. Entropy generation and the Fokker–Planck equation. *Physica A* **393** (2014) 256-–260.

[7] T. Tome. Entropy production in nonequilibrium systems described by a Fokker-Planck equation. *Braz. J. Phys.* **36** (4A) (2006) 1285–1289.

[8] F. Flandoli, M. Leocata and C. Ricci. The Navier–Stokes–Vlasov–Fokker–Planck system as a scaling limit of particles in a fluid. *J. Math. Fluid Mech.* (2021) 23-–40.

[9] A.G. Peeters and D. Strintzi. The Fokker-Planck equation, and its application in plasma physics. *Ann. Phys.* **17** (2-3) (2008) 142—157.

[10] L. Philipp and B.D. Shizgal. A pseudospectral solution of a bistable Fokker–Planck equation that models protein folding. *Physica A* **522** (2019) 158—166.

[11] G.W. Bluman. Similarity solutions of the one-dimensional Fokker-Planck equation. *Int. J. Non Linear Mech.* **6** (1971) 143–153.

[12] M.A. Peletier and D.R.M. Renger. Variational formulation of the Fokker-Planck equation with decay: a particle approach. *Commun. Contemp. Math.* **15** (5) (2013) 1350017.

[13] F. Guarnieri, W. Moon and J.S. Wettlaufer. Solution of the Fokker-Planck equation with a logarithmic potential and mixed eigenvalue spectrum. *J. Math. Phys.* **58** (093301) (2017) 1–22.

[14] R. Novak and X.R. Wang. On the Kramers–Fokker–Planck equation with decreasing potentials in dimension one. *J. Spectr. Theory* **10** (1) (2020) 1–32.

[15] Y. Zhan and B.D. Shizgal. Diffusion in a bistable system: The eigenvalue spectrum of the Fokker-Planck operator and Kramers' reaction rate theory. *Phys. Rev. E* **99** (042101) (2019) 1–12.

[16] M. Dehghan and M. Tatari. The use of He's variational iteration method for solving a Fokker–Planck equation. *Phys. Scr.* **74** (2006) 310-–316.

[17] S. Hesam, A.R. Nazemi and A. Haghbin. Analytical solution for the Fokker–Planck equation by differential transform method. *Sci. Iran. B* **19** (4) (2012) 1140—1145.

[18] M. Lakestani and M. Dehghan. Numerical solution of Fokker-Planck equation using the cubic B-spline scaling functions. *Numer. Methods Partial Differ. Equ.* **25** (2009) 418-–429.

[19] M. Tatari, M. Dehghan and M. Razzaghi. Application of the Adomian decomposition method for the Fokker–Planck equation. *Math. Comput. Model.* **45** (2007) 639-–650.

[20] A. Yildrim. Application of the homotopy perturbation method for the Fokker–Planck equation. *Int. J. Numer. Meth. Biomed. Eng.* **26** (2010) 1144-–1154.

[21] H. Risken. *The Fokker–Planck Equation Method of Solution and Applications*. Springer Verlag, Berlin, 1989.

[22] A. Friedman. *Partial Differential Equations of Parabolic Type*. Robert E. Krieger Publishing Company, Floria, 1983.

[23] C. Chilarescu and N. Vaneecloo. A Stochastic approach to the Cobb-Douglas production function. *Econ. Bull.* **3** (9) (2007) 1–9.

[24] J.L. Largier. Conssierations in estimating larval dispersial distances from oceanographic data. *Ecol. Appl.* **13** (1) (2003) S71-–S89.

[25] R. Curtain and H. Zwart. *An introdution to Infinite-Dimensional Linear System Theory*. Springer-Verlag, New York, 1995.

[26] C.V. Pao. *Nonlinear Parabolic and Elliptic Equations*. Plenum Press C.V, New York, 1992.

[27] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, Berlin, 1991.

[28] R.T. Lewis. The Discreteness of the spectrum of self-adjoint, even order, one-term, differential operators. *Proc. Am. Math. Soc.* **42** (2) (1974) 480–482.

[29] S. Sutrima, M. Mardiyana, R. Setiyowati and R. Respatiwulan. A new approach on the well-posedness of nonautonomous Cauhy problems: An application in population growth. *J. Nonlinear Funct. Anal.* **2021** (2021), Article ID 29.

[30] A.C. McBride. *Semigroups of Linear Operators: An Introduction*. John Wiley& Sons, Inc., New York, 1987

[31] M. Brics, J. Kaupuž and R. Mahnke. How to solve Fokker-Planck equation treating mixed eigenvalue spectrum?. *Condens. Matter Phys.* **16** (1) (2013) 1–13.

[32] B.H. Devkota and J. Imberger. Lagrangian modeling of advection-diffusion transport in open channel flow. *Water Resour. Res.* **45** (12406) (2009), doi:10.1029/2009WR008364.