Nonlinear Dynamics and Systems Theory, **25** (6) (2025) 608–620

Adaptive Controller with Fixed-Time Convergence for Combination Synchronization of Multiple Master and Slave Chaotic Systems

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Received: December 24, 2024; Revised: November 20, 2025

Abstract: This paper mainly studies the fixed-time combination synchronization between different multiple chaotic systems using adaptive control. Asymptotic combination synchronization has previously been considered, but the fixed-time combination synchronisation of multiple chaotic systems with unknown parameters is the first of its kind. The fixed time control and adaptive control algorithms are successfully included to realize the combination synchronization between different multiple chaotic systems. The advantages of the proposed scheme include the possibility of realizing synchronization between almost all different chaotic systems in a short fixed time. According to the Lyapunov theory and fixed time laws, the unknown parameters are estimated and the settling time is calculated. Numerical simulation results are presented to prove the effectiveness of the proposed scheme.

Keywords: *chaotic systems; Lyapunov stability; fixed time stability; synchronization; adaptive control.*

Mathematics Subject Classification (2020): 93D40, 93D15, 34H05, 37C75.

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1 Introduction

Chaotic synchronization is one of the most important branches of chaos theory due to its real applications in numerous fields including secure communications [1], optical communication [2], biological systems [3], finance [4], neural networks [5], cryptography [6, 7], and lasers [8].

Various control techniques have been proposed to achieve chaos synchronization, namely active control [9, 10], sliding mode [11], event-triggered strategy [12], backstepping [13], adaptive control [14, 15].

In recent years, more attention has been paid to the study of combination synchronization. This method was initially proposed by Luo [16], when two master systems were synchronized with one slave system. Subsequently, Ayub et al. [17] proposed dual combination-combination synchronization (DCCS) between four master and four slave systems.

The DCCS scheme is a promising technique to improve both privacy and security in communication networks, to encode and decode information at the hardware level.

In the literature, there is also a number of results on combination synchronization, see [18, 19].

On the other hand, note that all the above papers [18,19] concern only the asymptotic chaos synchronization, in which the master and slave systems synchronize with the infinite settling time.

Later, some papers studied combination synchronization in finite-time, see [20], but this method has some limitations, in particular, the settling time of synchronisation depends on the initial conditions.

This is a shortcoming because the application of this method in reality is impractical, especially if the initial conditions are not known in advance, in addition, the design of the synchronization scheme applied in these studies relates to the synchronization between two identical chaotic systems with known parameters. This is also a deficiency since the parameters of chaotic systems may be unknown, moreover, generalized synchronization is worth studying more than identical synchronization.

So, achieving chaos combination synchronization in determined time independent of any initial value in the practical engineering application is a desirable objective.

Researchers have recently been interested in studying synchronization in fixed time, see [21, 22], and so far, there are few results in the literature on this subject.

Compared to the current results, there are no published papers in which the adaptive fixed-time combination synchronization of multiple master and slave chaotic systems has been studied. This has prompted us to undertake this work to address some of the aforementioned deficiencies.

Inspired by the discussion above, this paper discussed the combination synchronization of different multiple chaotic systems with unknown parameters in a fixed time.

The main contribution of this paper is to show how a basic technique can be used to design an appropriate controller to accomplish fixed time combination synchronization and it is summarized as follows.

- Based on the fixed time stability theory and adaptive method, an effective control is designed to solve the fixed time combination synchronization of different multiple chaotic systems.

- A new theorem is proposed to demonstrate that the presented algorithm is an appropriate choice to achieve the combination synchronization of chaotic systems with unknown parameters.

- The settling time of the fixed time combination synchronization is bounded for any initial condition states.

The features of the proposed scheme that made it innovative and very important are:

- The scheme can be applied to almost all chaotic and even hyperchaotic dynamic systems.
- The possibility of applying it in case all master and slave systems are different and customizing it in case they are identical.
- The combination synchronization is achievable in a short fixed time.

The remainder of this paper is organizes as follows.

Some necessary definitions and lemmas are presented in Section 2. Section 3 contains the proposed scheme that is designed to obtain the desired fixed time combination synchronization. Some numerical simulations are presented in Section 4. The conclusion of the work is given in the last section.

2 Definitions and Lemmas

This section presents some of the definitions and lemmas related to this work.

Consider the nonlinear autonomous system

$$\begin{cases} \frac{dX(t)}{dt} = F(X(t)), \\ X(0) = X_0, \end{cases} \tag{1}$$

where $X \in \mathbb{R}^n$. Here, the results are presented under the assumption that the origin is an equilibrium point of the system (1).

Definition 2.1 The origin of (1) is said to be **fixed-time stable** if it is globally finite-time stable and the settling-time function $T^*(X_0)$ is bounded, i.e., $\exists T^*_{Max} > 0$: $T^*(X_0) < T^*_{Max}$, $\forall\ X_0 \in \mathbb{R}^n$.

Lemma 2.1 *[23] If there exists a continuous radically unbounded function $V : \mathbb{R}^n \longrightarrow \mathbb{R}^+ \cup \{0\}$ such that*

- $V(x) = 0 \Leftrightarrow x = 0$,
- *any solution $X(t)$ of (1) satisfies the inequality*

$$\begin{array}{l} \dot{V}(X(t)) + \beta_1 V^{\alpha_1}(X(t)) + \beta_2 V^{\alpha_2}(X(t)) \ \leq 0 \\ \textit{for } \beta_1,\ \beta_2,\ \alpha_1,\ \alpha_2 > 0,\ \alpha_1 < 1, \quad \alpha_2 > 1, \end{array}$$

then the origin of (1) is fixed-time stable and the settling-time function is estimated by

$$T^*(X_0) \leq \frac{1}{\beta_1 (1 - \alpha_1)} + \frac{1}{\beta_2 (\alpha_2 - 1)}, \forall X_0 \in \mathbb{R}^n.$$

Lemma 2.2 *[24] For any V_i, $(i = \overline{1,n})$, $0 < a < 1$, $b > 1$,*

$$\begin{cases} \sum_{i=1}^{n} (V_i)^a \geq \left[\sum_{i=1}^{n} (V_i) \right]^a, \\ \sum_{i=1}^{n} (V_i)^b \geq n^{1-b} \left[\sum_{i=1}^{n} (V_i) \right]^b. \end{cases}$$

3 Formulation of Fixed Time Combination Synchronization Scheme

In this section, we propose a new control law to achieve the adaptive fixed time combination synchronization of multiple master and slave systems. Under the proposed control laws, errors converge to zero in a bounded time. Consider the first two master chaotic systems given as

$$\begin{cases} \dot{x} = f_1(x) + A(x)\eta, \\ \dot{y} = f_2(y) + B(y)\xi, \end{cases} \tag{2}$$

where $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$ are the state vectors, $f_1, f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two nonlinear functions, $A(x) \in \mathbb{R}^{n\times r}$, $B(y) \in \mathbb{R}^{n\times s}$ are matrix functions, $\eta \in \mathbb{R}^r$, $\xi \in \mathbb{R}^s$ are unknown parameter vectors.

The second two master chaotic systems are described as follows:

$$\begin{cases} \dot{\bar{x}} = \bar{f}_1(\bar{x}) + \bar{A}(\bar{x})\bar{\eta}, \\ \dot{\bar{y}} = \bar{f}_2(\bar{y}) + \bar{B}(\bar{y})\bar{\xi}, \end{cases} \tag{3}$$

where $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)^T$, $\bar{y} = (\bar{y}_1, \bar{y}_2, ..., \bar{y}_n)^T$ are the state vectors, $\bar{f}_1, \bar{f}_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two nonlinear functions, $\bar{A}(\bar{x}) \in \mathbb{R}^{n\times\bar{r}}$, $\bar{B}(\bar{y}) \in \mathbb{R}^{n\times\bar{s}}$ are matrix functions, $\bar{\eta} \in \mathbb{R}^{\bar{r}}$, $\bar{\xi} \in \mathbb{R}^{\bar{s}}$ are unknown parameter vectors.

The combination of the state vectors of two primary (2) and two secondary (3) master systems is given, respectively, by

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = (x_1, ..., x_n, y_1, y_2, ..., y_n)^T, \bar{X} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = (\bar{x}_1, ..., \bar{x}_n, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n)^T.$$

The combination of nonlinear functions of two primary and two secondary master systems is represented, respectively, by

$$f(X) = \begin{pmatrix} f_1(x) \\ f_2(y) \end{pmatrix} = [f_{11}(x), f_{12}(x), ..., f_{1n}(x), f_{21}(y), f_{22}(y), ..., f_{2n}(y)]^T, \tag{4}$$

$$\bar{f}(\bar{X}) = \begin{pmatrix} \bar{f}_1(\bar{x}) \\ \bar{f}_2(\bar{y}) \end{pmatrix} = \left[\bar{f}_{11}(\bar{x}), \bar{f}_{12}(\bar{x}), ..., \bar{f}_{1n}(\bar{x}), \bar{f}_{21}(\bar{y}), \bar{f}_{22}(\bar{y}), ..., \bar{f}_{2n}(\bar{y})\right]^T. \tag{5}$$

Consider the first two chaotic slave systems given below:

$$\begin{cases} \dot{z} = g_1(z) + C(z)\rho + u_1, \\ \dot{w} = g_2(w) + D(w)\theta + u_2, \end{cases} \tag{6}$$

where $z = (z_1, z_2, ..., z_n)^T$, $w = (w_1, w_2, ..., w_n)^T$ are the state vectors, $g_1, g_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two nonlinear functions, $C(z) \in \mathbb{R}^{n\times q}$, $D(w) \in \mathbb{R}^{n\times l}$ are matrix functions,

$\rho \in \mathbb{R}^q$, $\theta \in \mathbb{R}^l$ are unknown parameter vectors, $u_1 = (u_{11},\ u_{12}, ...,\ u_{1n})$, and $u_2 = (u_{21},\ u_{22}, ...,\ u_{2n})$ represent the controllers which are to be designed.

Suppose the second two slave chaotic systems are given as follows:

$$\begin{cases} \dot{\bar{z}} = \bar{g}_1(\bar{z}) + \bar{C}(\bar{z})\,\bar{\rho} + \bar{u}_1, \\ \dot{\bar{w}} = \bar{g}_2(\bar{w}) + \bar{D}(\bar{w})\,\bar{\theta} + \bar{u}_2, \end{cases} \tag{7}$$

where $\bar{z} = (\bar{z}_1, \bar{z}_2, ..., \bar{z}_n)^T$, $\bar{w} = (\bar{w}_1, \bar{w}_2, ..., \bar{w}_n)^T$ are the state vectors, $\bar{g}_1, \bar{g}_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are two nonlinear functions, $\bar{C}(\bar{z}) \in \mathbb{R}^{n\times\bar{q}}$, $\bar{D}(\bar{w}) \in \mathbb{R}^{n\times\bar{l}}$ are matrix functions, $\bar{\rho} \in \mathbb{R}^{\bar{q}}$, $\bar{\theta} \in \mathbb{R}^{\bar{l}}$ are unknown parameter vectors, $\bar{u}_1, \bar{u}_2 \in \mathbb{R}^n$ represent the controllers of the second slave systems, where $\bar{u}_1 = (\bar{u}_{11},\ \bar{u}_{12}, ...,\ \bar{u}_{1n})$, $\bar{u}_2 = (\bar{u}_{21},\ \bar{u}_{22}, ...,\ \bar{u}_{2n})$.

The combination of the state vectors of two primary (6) and two secondary (7) slave systems is given, respectively, by

$Y = \binom{z}{w} = (z_1,\ z_2, ...,\ z_n,\ w_1, ...,\ w_n)^T$, $\bar{Y} = \binom{\bar{z}}{\bar{w}} = (\bar{z}_1,\ \bar{z}_2, ...,\ \bar{z}_n,\ \bar{w}_1, ...,\ \bar{w}_n)^T$.

The combination of nonlinear functions of two primary and two secondary slave systems is represented, respectively, by

$$g(Y) = \begin{pmatrix} g_1(z) \\ g_2(w) \end{pmatrix} = [g_{11}(z),\ g_{12}(z), ...,\ g_{1n}(z),\ g_{21}(w),\ g_{22}(w), ...,\ g_{2n}(w)]^T, \tag{8}$$

$$\bar{g}(\bar{Y}) = \begin{pmatrix} \bar{g}_1(\bar{z}) \\ \bar{g}_2(\bar{w}) \end{pmatrix} = [\bar{g}_{11}(\bar{z}),\ \bar{g}_{12}(\bar{z}), ...,\ \bar{g}_{1n}(\bar{z}),\ \bar{g}_{21}(\bar{w}),\ \bar{g}_{22}(\bar{w}), ...,\ \bar{g}_{2n}(\bar{w})]^T. \tag{9}$$

Remark 3.1 The matrix functions are combined as shown below:

$\psi(X) = \begin{pmatrix} A(x) & 0 \\ 0 & B(y) \end{pmatrix}$, $\bar{\psi}(\bar{X}) = \begin{pmatrix} \bar{A}(\bar{x}) & 0 \\ 0 & \bar{B}(\bar{y}) \end{pmatrix}$, $\phi(Y) = \begin{pmatrix} C(z) & 0 \\ 0 & D(w) \end{pmatrix}$, $\bar{\phi}(\bar{Y}) = \begin{pmatrix} \bar{C}(\bar{z}) & 0 \\ 0 & \bar{D}(\bar{w}) \end{pmatrix}$.

The combination synchronization error between the systems (2), (3) and the systems (6), (7) is defined by

$$e = Y + \bar{Y} - \left(X + \bar{X}\right),$$

where $e = \binom{e_1}{e_2} = \binom{Y-X}{\bar{Y}-\bar{X}}$, $e_1 = (e_{11}, e_{12}, ..., e_{1n})^T$, and $e_2 = (e_{21}, e_{22}, ..., e_{2n})^T$.

Then we obtain the error dynamical system

$$\dot{e} = \dot{Y} + \dot{\bar{Y}} - \left(\dot{X} + \dot{\bar{X}}\right). \tag{10}$$

Theorem 3.1 *Let T^* be given by*

$$T^* = \frac{1}{2^{\frac{\mu+1}{2}}\left(\frac{1-\mu}{2}\right)k} + \frac{1}{2^{\frac{\lambda+1}{2}}(Q)^{\frac{1-\lambda}{2}}\left(\frac{\lambda-1}{2}\right)k}, \tag{11}$$

where $Q = 2n + m + \bar{m} + p + \bar{p}$.

Adaptive combination synchronization between systems (2), (3) and systems (6), (7) in fixed time (11) can be obtained if the following conditions are met.

1. Control laws are designed as

$$\begin{aligned} u(t)+\bar{u}(t) &= -\left[g(Y)+\bar{g}\left(\bar{Y}\right)+\phi(Y)\tilde{\gamma}+\bar{\phi}\left(\bar{Y}\right)\tilde{\bar{\gamma}}\right] \\ &+\left(f(X)+\bar{f}\left(\bar{X}\right)+\psi(X)\tilde{\delta}+\bar{\psi}\left(\bar{X}\right)\tilde{\bar{\delta}}\right)-k\left(|e|^{\mu}+|e|^{\lambda}\right)sgn(e)\,. \end{aligned}$$

2. Parameter update laws are given by

$$\begin{cases} \dot{\tilde{\delta}}=-\left[\psi(X)\right]^{T}e+k\left(\left|\delta-\tilde{\delta}\right|^{\mu}+\left|\delta-\tilde{\delta}\right|^{\lambda}\right)sgn\left(\delta-\tilde{\delta}\right), \\ \dot{\tilde{\bar{\delta}}}=-\left[\bar{\psi}\left(\bar{X}\right)\right]^{T}e+k\left(\left|\bar{\delta}-\tilde{\bar{\delta}}\right|^{\mu}+\left|\delta-\tilde{\bar{\delta}}\right|^{\lambda}\right)sgn\left(\bar{\delta}-\tilde{\bar{\delta}}\right), \\ \dot{\tilde{\gamma}}=\left[\phi(Y)\right]^{T}e+k\left(\left|\gamma-\tilde{\gamma}\right|^{\mu}+\left|\gamma-\tilde{\gamma}\right|^{\lambda}\right)sgn\left(\gamma-\tilde{\gamma}\right), \\ \dot{\tilde{\bar{\gamma}}}=\left[\bar{\phi}\left(\bar{Y}\right)\right]^{T}e+k\left(\left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\mu}+\left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\lambda}\right)sgn\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right), \end{cases} \tag{12}$$

where $k>0$, $0<\mu<1$, $\lambda>1$.

Proof. Substituting (12) into (10) yields the error dynamics

$$\begin{aligned} \dot{e} &= \phi(Y)\left(\gamma-\tilde{\gamma}\right)+\bar{\phi}\left(\bar{Y}\right)\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)-\left(\psi(X)\left(\delta-\tilde{\delta}\right)+\bar{\psi}\left(\bar{X}\right)\left(\bar{\delta}-\tilde{\bar{\delta}}\right)\right) \\ &-k\left(|e|^{\mu}+|e|^{\lambda}\right)sgn(e)\,. \end{aligned} \tag{13}$$

Take the Lyapunov function candidate as

$$V=\frac{1}{2}\left[\begin{array}{c} e^{T}e+\left(\delta-\tilde{\delta}\right)^{T}\left(\delta-\tilde{\delta}\right)+\left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left(\bar{\delta}-\tilde{\bar{\delta}}\right)+\left(\gamma-\tilde{\gamma}\right)^{T}\left(\gamma-\tilde{\gamma}\right)+ \\ +\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right) \end{array}\right]. \tag{14}$$

Compute the derivative of Lyapunov function

$$\dot{V}=\frac{1}{2}\left[\begin{array}{c} \dot{e}^{T}e+\left(\delta-\tilde{\delta}\right)^{T}\left(-\dot{\tilde{\delta}}\right)+\left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left(-\dot{\tilde{\bar{\delta}}}\right)+\left(\gamma-\tilde{\gamma}\right)^{T}\left(-\dot{\tilde{\gamma}}\right) \\ +\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left(-\dot{\tilde{\bar{\gamma}}}\right) \end{array}\right].$$

Substituting expressions (12) and (13), we get

$$\begin{aligned}
\dot{V} &= \begin{bmatrix} \phi(Y)(\gamma-\tilde{\gamma}) + \bar{\phi}(\bar{Y})\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right) - \left(\psi(X)\left(\delta-\tilde{\delta}\right) + \bar{\psi}(\bar{X})\left(\bar{\delta}-\tilde{\bar{\delta}}\right)\right) \\ -k\left(|e|^{\mu}+|e|^{\lambda}\right) sgn(e) \end{bmatrix}^{T} e \\
&\quad + \left(\delta-\tilde{\delta}\right)^{T}\left[[\psi(X)]^{T} e - k\left(\left|\delta-\tilde{\delta}\right|^{\mu} + \left|\delta-\tilde{\delta}\right|^{\lambda}\right) sgn\left(\delta-\tilde{\delta}\right)\right] \\
&\quad + \left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left[\left[\bar{\psi}(\bar{X})\right]^{T} e - k\left(\left|\bar{\delta}-\tilde{\bar{\delta}}\right|^{\mu} + \left|\delta-\tilde{\bar{\delta}}\right|^{\lambda}\right) sgn\left(\bar{\delta}-\tilde{\bar{\delta}}\right)\right] \\
&\quad + (\gamma-\tilde{\gamma})^{T}\left[-[\phi(Y)]^{T} e - k\left(|\gamma-\tilde{\gamma}|^{\mu} + |\gamma-\tilde{\gamma}|^{\lambda}\right) sgn(\gamma-\tilde{\gamma})\right]
\end{aligned}$$

$$\begin{aligned}
&\quad + \left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left[-\left[\bar{\phi}(\bar{Y})\right]^{T} e - k\left(\left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\mu} + \left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\lambda}\right) sgn\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)\right] \\
&= (\gamma-\tilde{\gamma})^{T}[\phi(Y)]^{T} e + \left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left[\bar{\phi}(\bar{Y})\right]^{T} e - \left(\delta-\tilde{\delta}\right)^{T}[\psi(X)]^{T} e \\
&\quad - \left(\bar{\delta}-\tilde{\bar{\delta}}\right)\left[\bar{\psi}(\bar{X})\right]^{T} e - k(sgn(e))^{T}\left(|e|^{\mu}+|e|^{\lambda}\right)^{T} e \\
&\quad + \left(\delta-\tilde{\delta}\right)^{T}[\psi(X)]^{T} e - k\left(\delta-\tilde{\delta}\right)^{T}\left(\left|\delta-\tilde{\delta}\right|^{\mu} + \left|\delta-\tilde{\delta}\right|^{\lambda}\right) sgn\left(\delta-\tilde{\delta}\right) \\
&\quad + \left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left[\bar{\psi}(\bar{X})\right]^{T} e - k\left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left(\left|\bar{\delta}-\tilde{\bar{\delta}}\right|^{\mu} + \left|\delta-\tilde{\bar{\delta}}\right|^{\lambda}\right) sgn\left(\bar{\delta}-\tilde{\bar{\delta}}\right) - \\
&\quad - (\gamma-\tilde{\gamma})^{T}[\phi(Y)]^{T} e - k(\gamma-\tilde{\gamma})^{T}\left(|\gamma-\tilde{\gamma}|^{\mu} + |\gamma-\tilde{\gamma}|^{\lambda}\right) sgn(\gamma-\tilde{\gamma}) - \\
&\quad - \left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left[\bar{\phi}(\bar{Y})\right]^{T} e - k\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left(\left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\mu} + \left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\lambda}\right) sgn\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= -k(sgn(e))^{T}\left(|e|^{\mu}+|e|^{\lambda}\right)^{T} e - k\left(\delta-\tilde{\delta}\right)^{T}\left(\left|\delta-\tilde{\delta}\right|^{\mu} + \left|\delta-\tilde{\delta}\right|^{\lambda}\right) sgn\left(\delta-\tilde{\delta}\right) \\
&\quad -k\left(\bar{\delta}-\tilde{\bar{\delta}}\right)^{T}\left(\left|\bar{\delta}-\tilde{\bar{\delta}}\right|^{\mu} + \left|\delta-\tilde{\bar{\delta}}\right|^{\lambda}\right) sgn\left(\bar{\delta}-\tilde{\bar{\delta}}\right) \\
&\quad -k(\gamma-\tilde{\gamma})^{T}\left(|\gamma-\tilde{\gamma}|^{\mu} + |\gamma-\tilde{\gamma}|^{\lambda}\right) sgn(\gamma-\tilde{\gamma}) \\
&\quad -k\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)^{T}\left(\left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\mu} + \left|\bar{\gamma}-\tilde{\bar{\gamma}}\right|^{\lambda}\right) sgn\left(\bar{\gamma}-\tilde{\bar{\gamma}}\right)
\end{aligned}$$

$$\begin{aligned}
\dot{V} &= -k\sum_{i=1}^{2n}\left(|e_i|^{\mu+1} + |e_i|^{\lambda+1}\right) - k\sum_{i=1}^{m}\left(\left|\delta_i-\tilde{\delta}_i\right|^{\mu+1} + \left|\delta_i-\tilde{\delta}_i\right|^{\lambda+1}\right) \\
&\quad -k\sum_{i=1}^{\bar{m}}\left(\left|\bar{\delta}_i-\tilde{\bar{\delta}}_i\right|^{\mu+1} + \left|\delta_i-\tilde{\bar{\delta}}_i\right|^{\lambda+1}\right) - k\sum_{i=1}^{p}\left(|\gamma_i-\tilde{\gamma}_i|^{\mu+1} + |\gamma_i-\tilde{\gamma}_i|^{\lambda+1}\right) \\
&\quad -k\sum_{i=1}^{\bar{p}}\left(\left|\bar{\gamma}_i-\tilde{\bar{\gamma}}_i\right|^{\mu+1} + \left|\bar{\gamma}_i-\tilde{\bar{\gamma}}_i\right|^{\lambda+1}\right),
\end{aligned}$$

then

$$\dot{V} = -k\left[\begin{array}{c}\sum_{i=1}^{2n}\left(\left(e_i^2\right)^{\frac{\mu+1}{2}}\right)+\sum_{i=1}^{m}\left(\left(\delta_i-\tilde{\delta}_i\right)^2\right)^{\frac{\mu+1}{2}}+\sum_{i=1}^{\bar{m}}\left(\left(\bar{\delta}_i-\tilde{\bar{\delta}}_i\right)^2\right)^{\frac{\mu+1}{2}}\\ +\sum_{i=1}^{p}\left(\left(\gamma_i-\tilde{\gamma}_i\right)^2\right)^{\frac{\mu+1}{2}}+\sum_{i=1}^{\bar{p}}\left(\left(\bar{\gamma}_i-\tilde{\bar{\gamma}}_i\right)^2\right)^{\frac{\mu+1}{2}}\end{array}\right] -k\left[\begin{array}{c}\sum_{i=1}^{2n}\left(\left(e_i^2\right)^{\frac{\lambda+1}{2}}\right)+\sum_{i=1}^{m}\left(\left(\delta_i-\tilde{\delta}_i\right)^2\right)^{\frac{\lambda+1}{2}}+\sum_{i=1}^{\bar{m}}\left(\left(\bar{\delta}_i-\tilde{\bar{\delta}}_i\right)^2\right)^{\frac{\lambda+1}{2}}\\ +\sum_{i=1}^{p}\left(\left(\gamma_i-\tilde{\gamma}_i\right)^2\right)^{\frac{\lambda+1}{2}}+\sum_{i=1}^{\bar{p}}\left(\left(\bar{\gamma}_i-\tilde{\bar{\gamma}}_i\right)^2\right)^{\frac{\lambda+1}{2}}\end{array}\right].$$

According to the equation (14),

$$V=\frac{1}{2}\left[\sum_{i=1}^{2n}(e_i)^2+\sum_{i=1}^{m}\left(\delta_i-\tilde{\delta}_i\right)^2+\sum_{i=1}^{\bar{m}}\left(\bar{\delta}_i-\tilde{\bar{\delta}}_i\right)^2+\sum_{i=1}^{p}(\gamma_i-\tilde{\gamma}_i)^2+\sum_{i=1}^{\bar{p}}\left(\bar{\gamma}_i-\tilde{\bar{\gamma}}_i\right)^2\right].$$

According to Lemma (2.2),

$$\dot{V}\le -2^{\frac{\mu+1}{2}}k\left(\sum_{i=1}^{Q}V_i\right)^{\frac{\mu+1}{2}}-2^{\frac{\lambda+1}{2}}k\ (Q)^{\frac{1-\lambda}{2}}\left(\sum_{i=1}^{Q}V_i\right)^{\frac{\lambda+1}{2}}.$$

This means

$$\dot{V}\le -2^{\frac{\mu+1}{2}}k\ V^{\frac{\mu+1}{2}}-2^{\frac{\lambda+1}{2}}k\ (Q)^{\frac{1-\lambda}{2}}\ V^{\frac{\lambda+1}{2}}.$$

According to Lemma (2.1), we can conclude that the origin e_0 of system (10) is fixed time stable under the proposed controller (12), this means that the adaptive combination synchronization is achieved between the systems (2), (3) and the systems (6), (7) in fixed time convergence, and the settling-time function $T^*(e_0)$ is bounded by (11), and also, $e(t)=0$ for $t\ge T^*_{Max}$. This completes the proof.

4 Numerical Simulations

In this section, four different master systems and four different response systems are used to verify the effectiveness of the proposed scheme.

First two master systems [25], [26] are given, respectively, by

$$\begin{cases}\dot{x}_1=-\eta_1x_1+\eta_2x_2x_3,\\ \dot{x}_2=\eta_3x_2-\eta_4x_1x_3,\\ \dot{x}_3=-\eta_5x_3+\eta_6x_1x_2,\end{cases}\tag{15}$$

$$\begin{cases}\dot{y}_1=\xi_1(y_2-y_1),\\ \dot{y}_2=-y_1y_3+\xi_2y_2,\\ \dot{y}_3=y_1y_2-\xi_3y_3,\end{cases}\tag{16}$$

where η_i, $(i=\overline{1,6})$ and ξ_i, $(i=\overline{1,3})$ are unknown parameters, according to [25] and [26], the systems (15) and (16) are chaotic when the parameters take the values ($\eta_1=4$, $\eta_2=3$, $\eta_3=1$, $\eta_4=7$, $\eta_5=1$, $\eta_6=2$) and ($\xi_1=36$, $\xi_2=28$, $\xi_3=3$).

The second two master systems defined in [27] are given as follows:

$$\begin{cases} \dot{\bar{x}}_1 = \bar{x}_2\bar{x}_3, \\ \dot{\bar{x}}_2 = \bar{x}_1 - \bar{x}_2, \\ \dot{\bar{x}}_3 = \bar{\eta}_1 \left|\bar{x}_1\right| - \bar{\eta}_2 x_1^2, \end{cases} \tag{17}$$

$$\begin{cases} \dot{\bar{y}}_1 = \bar{\xi}_1(\bar{y}_2 - \bar{y}_1), \\ \dot{\bar{y}}_2 = -\bar{y}_1\bar{y}_3 + \bar{\xi}_2\bar{y}_2, \\ \dot{\bar{y}}_3 = \exp(\bar{y}_1\bar{y}_2) - \bar{\xi}_3\bar{y}_3, \end{cases} \tag{18}$$

where $\bar{\eta}_i$, $(i = 1,2)$ and $\bar{\xi}_i$, $(i = \overline{1,3})$ are unknown parameters. The systems (17) and (18) are chaotic when the parameters are given by $(\bar{\eta}_1 = 5,\ \bar{\eta}_2 = 2)$ and $(\bar{\xi}_1 = 33,\ \bar{\xi}_2 = 19.5,\ \bar{\xi}_3 = 8)$.

The two first slave systems [28], [29] are given as follows:

$$\begin{cases} \dot{z}_1 = \rho_1(z_2 - z_1) + u_{11}, \\ \dot{z}_2 = \rho_2 z_1 - \rho_3 z_1 z_2 + u_{12}, \\ \dot{z}_3 = -\rho_4 z_3 + \rho_5 z_1^2 + u_{13}, \end{cases} \tag{19}$$

$$\begin{cases} \dot{w}_1 = \theta_1(w_2 - w_1) + u_{21}, \\ \dot{w}_2 = w_1 w_3 + \theta_2 w_2 + u_{22}, \\ \dot{w}_3 = -w_1^2 - \theta_3 w_3 + u_{23}, \end{cases} \tag{20}$$

where ρ_i, $(i = \overline{1,5})$ and θ_i, $(i = \overline{1,3})$ are unknown parameters, according to [28] and [29], the systems (19) and (20) are chaotic when the parameters take the values $(\rho_1 = 10,\ \rho_2 = 40,\ \rho_3 = 1,\ \rho_4 = 2.5,\ \rho_5 = 4)$ and $(\theta_1 = 30,\ \theta_2 = 15,\ \theta_3 = 11)$.

Second slave systems [30], [31] are defined as follows:

$$\begin{cases} \dot{\bar{z}}_1 = \bar{z}_2 + \bar{u}_{11}, \\ \dot{\bar{z}}_2 = \bar{z}_3 + \bar{\rho}_1\bar{z}_2 + \bar{u}_{12}, \\ \dot{\bar{z}}_3 = \bar{\rho}_2\bar{z}_1 + \bar{\rho}_3\bar{z}_2 + \bar{\rho}_4\bar{z}_3 + \bar{\rho}_5\bar{z}_1^2 + \bar{u}_{13}, \ , \end{cases} \tag{21}$$

$$\begin{cases} \dot{\bar{w}}_1 = \bar{\theta}_1(\bar{w}_2 - \bar{w}_1 + \bar{w}_2\bar{w}_3) + \bar{u}_{21}, \\ \dot{\bar{w}}_2 = -\bar{w}_1\bar{w}_3 + \bar{\theta}_2\bar{w}_2 + \bar{u}_{22}, \\ \dot{\bar{w}}_3 = \bar{w}_1\bar{w}_2 - \bar{\theta}_3\bar{w}_3 + \bar{u}_{23}, \end{cases} \tag{22}$$

where $\bar{\rho}_i$, $(i = \overline{1,5})$ and $\bar{\theta}_i$, $(i = \overline{1,3})$ are unknown parameters, according to [30] and [31], the systems (21) and (22) are chaotic when the parameters are $(\bar{\rho}_1 = -0.5,\ \bar{\rho}_2 = -6,\ \bar{\rho}_3 = -2.85,\ \bar{\rho}_4 = -0.5,\ \bar{\rho}_5 = 3)$ and $(\bar{\theta}_1 = 35,\ \bar{\theta}_2 = 14,\ \bar{\theta}_3 = 5)$.

In the numerical simulation, we choose arbitrarily the initial state vectors as

$$\begin{aligned} X(0) &= (-1,\ -10,\ -4,\ -20,\ 10,\ -1)^T,\ \bar{X}(0) = (1,\ 1,\ 2,\ 1,\ 1,\ 10)^T, \\ Y(0) &= (10,\ 3,\ 1,\ 20,\ -20,\ 8)^T,\ \bar{Y}(0) = (-10,\ 10,\ -5,\ 3,\ 15,\ -25)^T. \end{aligned}$$

The initial values of unknown parameters are arbitrarily taken as

$$\begin{aligned} \delta &= \begin{pmatrix} \eta_i \left(i = \overline{1,6}\right) \\ \xi_i \left(i = \overline{1,3}\right) \end{pmatrix} = (10,\ -8,\ 3,\ 1,\ 7,\ -4,\ 1,\ 5,\ 10)^T, \\ \bar{\delta} &= \begin{pmatrix} \bar{\eta}_i \left(i = \overline{1,2}\right) \\ \bar{\xi}_i \left(i = \overline{1,3}\right) \end{pmatrix} = (-5,\ 8,\ -12,\ 3,\ 10)^T, \end{aligned}$$

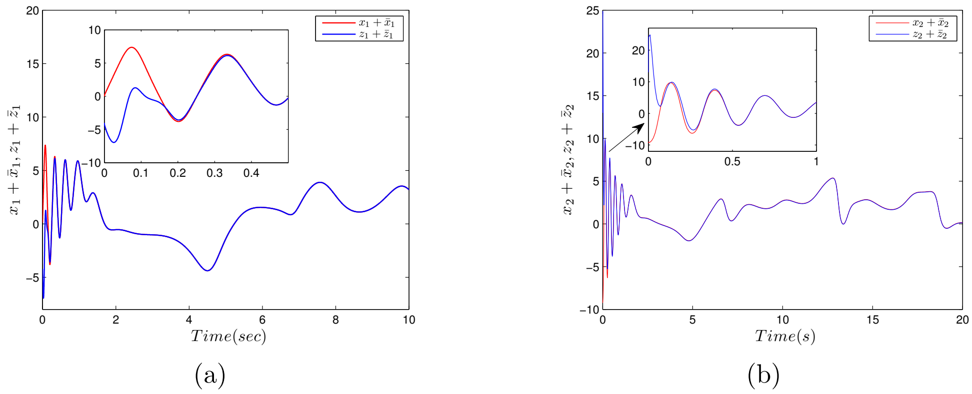


Figure 1: Synchronization for the state trajectories between **(a)**: $z_1 + \bar{z}_1$ and $x_1 + \bar{x}_1$, **(b)**: $z_2 + \bar{z}_2$ and $x_2 + \bar{x}_2$.

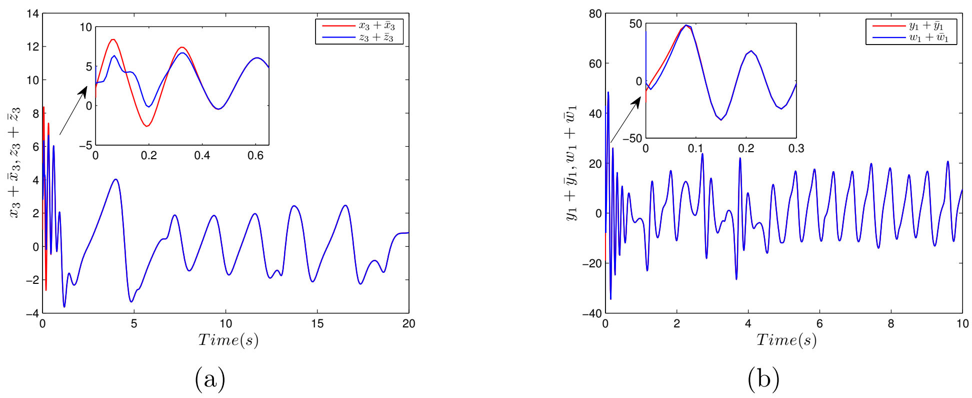


Figure 2: Synchronization for the state trajectories between **(a)**: $z_3 + \bar{z}_3$ and $x_3 + \bar{x}_3$, **(b)**: $w_1 + \bar{w}_1$ and $y_1 + \bar{y}_1$.

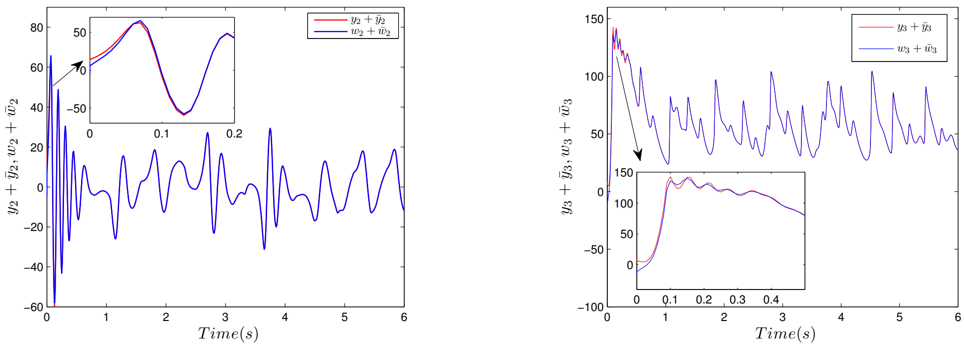
$$\gamma = \begin{pmatrix} \rho_i \left(i = \overline{1,5}\right) \\ \theta_i \left(i = \overline{1,3}\right) \end{pmatrix} = (5, \ -10, \ 45, \ 30, \ -8, \ -20, \ 40, \ 1)^T ,$$

$$\bar{\gamma} = \begin{pmatrix} \bar{\rho}_i \left(i = \overline{1,5}\right) \\ \bar{\theta}_i \left(i = \overline{1,3}\right) \end{pmatrix} = (-10, \ 1, \ 12, \ -20, \ 2, \ 16, \ 4, \ -6)^T .$$

The control gain is given by $k = 6$. Figures 1, 2 and 3 depict the combination synchronization between four master systems (2), (3) and four slave systems (6), (7).

5 Conclusion

The results obtained indicate that in this paper, we have sufficiently proved the combination synchronization of multiple chaotic systems with unknown parameters in a fixed time using an appropriate control algorithm, this algorithm is based on integrating an adaptive control strategy and a fixed time control. According to the Lyapunov theory



(a) (b)

Figure 3: Synchronization for the state trajectories between
(a): $w_2 + \bar{w}_2$ and $y_2 + \bar{y}_2$, **(b)**: $w_3 + \bar{w}_3$ and $y_3 + \bar{y}_3$.

and fixed time laws, the unknown parameters were estimated and the settling time was also determined.

The proposed scheme was applied to eight different chaotic systems with unknown parameters. Numerical simulation results were presented to demonstrate the effectiveness of theoretical analysis of the proposed scheme. The proposed scheme is an important extension of several existing schemes, which gives this work substantial merits.

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