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Solving Two–Dimensional Lane–Emden System Equations by MDTM

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Abstract: In this paper, we introduce and solve the nonlinear forms of two-dimensional Lane–Emden system equations. Using the properties of a Modified Differential Transform Method, we obtain exact analytical solutions for these equations without resorting to linearization, discretization, or perturbation, while requiring minimal computation.

Keywords: *two–dimensional Lane–Emden system equations; reduced differential transform method; modified differential transform method; initial value problems.*

Mathematics Subject Classification (2020): 35C10, 65L05, 35J15, 35J47, 70F15, 35J75.

1 Introduction of Lane-Emden System Equations

The linear and nonlinear two–dimensional Lane–Emden type equations were first introduced by Wazwaz, Rach and Duan in [1], as follows:

$$u_{xx} + \frac{\propto}{x} u_x + u_{yy} + \frac{\beta}{y} u_y + g(x,y) f(u) = 0, \tag{1}$$

$$x > 0, \qquad y > 0, \qquad \propto > 0, \qquad \beta > 0,$$

$$u(x,0) = h(x), \quad u_y(x,0) = 0, \quad u(0,y) = h(y), \quad u_x(0,y) = 0, \tag{2}$$

where $g(x,y) f(u)$ is a linear or nonlinear term.

In [2], N. Teyar introduced the linear and nonlinear two–dimensional Lane–Emden system equations

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$$\begin{cases} u_{xx} + \frac{\propto}{x}u_x + u_{yy} + \frac{\beta}{y}u_y + f\left(x,y,v\right) = 0, \\ \\ v_{xx} + \frac{\gamma}{x}v_x + v_{yy} + \frac{\theta}{y}v_y + g\left(x,y,u\right) = 0, \end{cases} \tag{3}$$

$x > 0, \qquad y > 0, \qquad \propto > 0, \qquad \beta > 0, \qquad \gamma > 0, \qquad \theta > 0,$

$$\begin{cases} u\left(x,0\right) = h\left(x\right), & u_y\left(x,0\right) = 0, \\ v\left(x,0\right) = k\left(x\right), & v_y\left(x,0\right) = 0, \\ u\left(0,y\right) = h\left(y\right), & u_x\left(0,y\right) = 0, \\ v\left(0,y\right) = k\left(y\right), & v_x\left(0,y\right) = 0. \end{cases} \tag{4}$$

This type of systems can model configurations such as double stars in gravitational interaction or gaseous structures in hydrostatic equilibrium under the influence of several components. Because of the spherical symmetry of the solutions of the Lane–Emden equation, the second line of conditions (4) is not mentioned in general. These equations generalize the Lane-Emden equation to a coupled two-field system in two dimensions, which could describe: 1- Anisotropic self-gravitating fluids (e.g., accretion disks, gas clouds); 2- Two interacting astrophysical components (e.g., a coupled gas-star system); 3- Modified gravity effects (e.g., alternative gravity models for stellar structures).

In this paper, we apply modified differential transform methods for solving this kind of elliptic system problems, with singularities in both x and y, for obtaining exact solutions, by using a new product and quotient properties of the MDTM proved in [2], which implies a minimum of computation. We will obtain exact analytic solutions for these system equation. Liliane Maia, Gabrielle Nornberg and Filomena Pacella in [3], introduce a dynamical system approach for the second-order Lane–Emden type problems by defining some new variables that allow us to transform the radial fully nonlinear Lane-Emden equations into a quadratic dynamical system. For the one-dimension Lane-Emden equation, the original formal conservation of specific entropy along streamlines was given by a PDE in function of t (time) and r (radius).

2 Definition and Properties of Modified Differential Transform Method (MDTM)

We introduce the basic definitions of the modified differential transform method as follows.

Definition 2.1 The modified differential transform of $u(x, y)$ with respect to the variable y at y_0 is defined as

$$U(x,h) = \frac{1}{h!}\left(\frac{\partial^h}{\partial x^h}u(x,y)\right)_{y=y_0}, \qquad k \in \mathbb{N}, \tag{5}$$

where $u(x, y)$ is the original function and $\mathrm{U}(x, h)$ is the transformed function.

Definition 2.2 *The modified inverse differential transform $U(x, h)$ of $u(x, y)$ is defined as*

$$\mathrm{u(x,y)} = \sum_{\mathrm{h}=0}^{\infty} \mathrm{U(x,h)(y-y_0)^h}. \tag{6}$$

Original functions	Transformed functions
$w(x,y)=\alpha u(x,y)\pm\beta v(x,y)$	$W(x,h)=\alpha U(x,h)+\beta V(x,h)$
$w(x,y)=x^m y^n$	$W(x,h)=x^m\delta(h-n)$
$w(x,y)=x^m y^n u(x,y)$	$W(x,h)=x^m U(x,h-n)$
$w(x,y)=u(x,y)\,v(x,y)$	$W(x,h)=\sum_{s=0}^{k}U(x,s)\,V(x,h-s)$
$w(x,y)=[u(x,y)]^3$	$W(x,h)=\sum_{r=0}^{h}\sum_{s=0}^{r}U(x,h-r)\,U(x,s)\,U(x,r-s)$
$w(x,y)=\frac{\partial u(x,y)}{\partial x}$	$W(x,h)=\frac{\partial U(x,h)}{\partial x}$
$w(x,y)=\frac{\partial^2 u(x,y)}{\partial x^2}$	$W(x,h)=\frac{\partial^2 U(x,h)}{\partial x^2}$
$w(x,y)=\frac{\partial u(x,y)}{\partial y}$	$W(x,h)=(h+1)\,U(x,h+1)$
$w(x,y)=\frac{\partial^2 u(x,y)}{\partial y^2}$	$W(x,h)=(h+1)(h+2)\,U(x,h+2)$
$w(x,y)=e^{au(x,y)}$	$W(x,h)=\begin{cases} e^{aU(x,0)},\ h=0 \\ a\sum_{s=0}^{h-1}\frac{s+1}{h}U(x,s+1)\,W(x,h-s-1),\ h\geq 1\end{cases}$

Table 1: Fundamental properties of the MDTM.

Then combining equations (5) *and*(6), *we write*

$$u(x,y)=\sum_{h=0}^{\infty}\frac{1}{h!}\left(\frac{\partial^h}{\partial x^h}u(x,y)\right)_{y=y_0}(y-y_0)^h. \tag{7}$$

When (x,y_0) *is taken as* $(x,0)$, *then* (7) *can be expressed as*

$$u(x,y)=\sum_{h=0}^{\infty}U(x,h)y^h.$$

Some important properties of the MDTM used in this paper, are listed in Table 1.

3 Theorems and Corollaries

Theorem 3.1 *[2] If* $w(x,y)=\frac{u(x,y)}{v(x,y)}$ *and* $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\frac{U(x,0)}{V(x,0)}, & \text{if } h=0,\\ \frac{U(x,h)-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)}, & \text{if } h\geq 1.\end{cases}$$

Corollary 3.1 *[2] If* $w(x,y)=\frac{x^m y^n}{v(x,y)}$ *and* $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\frac{x^m\delta(n)}{V(x,0)} & \text{if } h=0,\\ \frac{x^m\delta(h-n)-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)} & \text{if } h\geq 1.\end{cases}$$

Corollary 3.2 *[2] If* $w(x,y)=\frac{1}{v(x,y)}$ *and* $V(x,0)\neq 0$, *then the modified differential transform version is*

$$W(x,h)=\begin{cases}\frac{1}{V(x,0)} & \text{if } h=0,\\ \frac{-\sum_{i=0}^{h-1}W(x,i)V(x,h-i)}{V(x,0)} & \text{if } h\geq 1.\end{cases}$$

Theorem 3.2 *[2] If* $w(x,y) = \frac{u(x,y)}{x^m y^n}$ *and* $(x,y) \neq (0,0)$*, then the modified differential transform version of* $w(x,y)$ *is*

$$W(x,h) = \frac{U(x,h+n)}{x^m}.$$

Corollary 3.3 *[2] If* $w(x,y) = \frac{u(x,y)}{x}, x \neq 0$*, then the modified differential transform version is*

$$W(x,h) = \frac{U(x,h)}{x}.$$

Corollary 3.4 *[2] If* $w(x,y) = \frac{u(x,y)}{y}, y \neq 0$*, then the modified differential transform version is*

$$W(x,h) = U(x,h+1).$$

4 Main Result and Description of Method

The classical Lane-Emden equation describes the structure of a self-gravitating, spherically symmetric polytropic gas cloud:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) + \theta^n = 0,$$

where $\theta(r)$ represents a dimensionless density profile, and n is the polytropic index. We will solve the following system of two equations:

$$\begin{cases} u_{xx} + \frac{\propto}{x}u_x + u_{yy} + \frac{\beta}{y}u_y + f\left(x,y,v\right) = 0, \\ v_{xx} + \frac{\gamma}{x}v_x + v_{yy} + \frac{\theta}{y}v_y + g\left(x,y,u\right) = 0, \end{cases} \tag{8}$$

$$x > 0, \quad y > 0, \quad \propto > 0, \quad \beta > 0, \quad \gamma > 0, \quad \theta > 0,$$

$$u\left(x,0\right) = h\left(x\right), \quad u_y\left(x,0\right) = 0, \quad v\left(x,0\right) = k\left(x\right), \quad v_y\left(x,0\right) = 0. \tag{9}$$

In Table 2, we compare the physical meaning of some variables in the classical Lane-Emden equation and the variables in our system of equations.

The following theorem constitutes the main result of this paper.

Theorem 4.1 *The Lane-Emden system of equations given by* (8)*, subject to the initial conditions* (9)*, can be effectively solved using the Modified Differential Transform Method (MDTM), yielding exact symmetric solutions.*

Proof: From Table 1 and Corollaries 3.3, 3.4, the modified differential transform version of (8) is

$$\begin{cases} \frac{\partial^2 U(x,h)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,h)}{\partial x} + \left(h+\beta+1\right)\left(h+2\right)U\left(x,h+2\right) + F\left(x,h\right) = 0, \\ \frac{\partial^2 V(x,h)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,h)}{\partial x} + \left(h+\theta+1\right)\left(h+2\right)V\left(x,h+2\right) + G\left(x,h\right) = 0, \end{cases}$$

Feature	**Classical Lane-Emden**	**Our System**
Dimension	$1D$ (radial r)	$2D$ (x, y)
Symmetry	Spherical	Possibly cylindrical or more general
Nonlinear source term	θ^n	$f(x, y, v), g(x, y, u)$ (self-interacting fields or relativistic corrections)
Number of equations	Single equation	Coupled system
Physical meaning	Polytropic stars	Two-component fluid, modified gravity, or plasma model

Table 2: Comparison of Classical Lane-Emden system and our System.

where $F\left(x, h\right)$ and $G\left(x, h\right)$ are, respectively, the modified differential transform version of $f\left(x, y, v\right)$ and $g\left(x, y, v\right)$. Also, the modified MDTM versions of initial conditions (9) are

$$U\left(x, 0\right) = H(x), \qquad U\left(x, 1\right) = 0, \qquad V\left(x, 0\right) = K(x), \qquad V\left(x, 1\right) = 0.$$

Then

$$\begin{cases} U\left(x, h+2\right) = \frac{-1}{(h+\beta+1\ )(h+2)}\left[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,h)}{\partial x} + F\left(x, h\right)\right] \\ V\left(x, h+2\right) = \frac{-1}{(h+\theta+1\ )(h+2)}\left[\frac{\partial^2 V(x,h)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,h)}{\partial x} + G\left(x, h\right)\right]. \end{cases}$$

For $h = 0$:

$$\begin{cases} U\left(x, 2\right) = \frac{-1}{2(\beta+1\ )}\left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,0)}{\partial x} + F\left(x, 0\right)\right] \\ V\left(x, 2\right) = \frac{-1}{2(\theta+1\ )}\left[\frac{\partial^2 V(x,0)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,0)}{\partial x} + G\left(x, 0\right)\right]. \end{cases}$$

For $h = 1$:

$$\begin{cases} U\left(x, 3\right) = \frac{-1}{3(\beta+2\ )}\left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,1)}{\partial x} + F\left(x, 1\right)\right] \\ \\ V\left(x, 3\right) = \frac{-1}{3(\theta+2\ )}\left[\frac{\partial^2 V(x,1)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,1)}{\partial x} + G\left(x, 1\right)\right]. \end{cases}$$

For $h = 2$:

$$\begin{cases} U\left(x, 4\right) = \frac{-1}{4(\beta+3\ )}\left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,2)}{\partial x} + F\left(x, 2\right)\right] \\ \\ V\left(x, 4\right) = \frac{-1}{4(\theta+3\ )}\left[\frac{\partial^2 V(x,2)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,2)}{\partial x} + G\left(x, 2\right)\right]. \end{cases}$$

For $h = 3$:

$$\begin{cases} U\left(x, 5\right) = \frac{-1}{5(\beta+4\ )}\left[\frac{\partial^2 U(x,3)}{\partial x^2} + \frac{\propto}{x}\frac{\partial U(x,3)}{\partial x} + F\left(x, 3\right)\right] \\ \\ V\left(x, 5\right) = \frac{-1}{5(\theta+4\ )}\left[\frac{\partial^2 V(x,3)}{\partial x^2} + \frac{\gamma}{x}\frac{\partial V(x,3)}{\partial x} + G\left(x, 3\right)\right]. \end{cases}$$

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After computing the first terms $U(x,h)$, we obtain the series solution

$$u(x,y) = \sum_{h=0}^{\infty} U(x,h)y^h.$$

5 Applications

Example 1. Consider the system of equations

$$\begin{cases} u_{xx} + \frac{2}{x}u_x + u_{yy} + \frac{3}{y}u_y + 6e^{-2v} = 0, \\ v_{xx} + \frac{3}{x}v_x + v_{yy} + \frac{4}{y}v_y - 14u = 0, \end{cases} \tag{10}$$

$$u(x,0) = \frac{1}{x^2}, \qquad u_y(x,0) = 0, \qquad v(x,0) = 2lnx, \qquad v_y(x,0) = 0 \tag{11}$$

with exact solutions

$$u(x,y) = \frac{1}{x^2+y^2}, \quad v(x,y) = \ln(x^2+y^2).$$

The terms $\frac{2}{x}u_x$ and $\frac{3}{y}u_y$ suggest a formulation in cylindrical or spherical coordinates, commonly used for modeling a stellar structure or gravitational equilibrium in astrophysical settings.

The term $6e^{-2v}$ suggests a nonlinear source term, which could represent an energy density or a self-interaction potential, as in plasma physics or cosmology.

The coupling between u and v in the equations suggests an interaction between two physical fields, possibly related to density-pressure relations in a self-gravitating system. From Table 1 and Corollaries 3.3, 3.4, the modified differential transform version of (10) is

$$\begin{cases} \frac{\partial^2 U(x,h)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,h)}{\partial x} + (h+4)(h+2)U(x,h+2) + 6F(x,h) = 0, \\ \frac{\partial^2 V(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,h)}{\partial x} + (h+5)(h+2)V(x,h+2) - 14U(x,h) = 0, \end{cases}$$

where $F(x,h)$ is the modified differential transform version of e^{-2v} (Table 1), such as

$$F(x,h) = \begin{cases} e^{-2V(x,0)} = \frac{1}{x^4}, & h = 0, \\ -2\sum_{s=0}^{h-1} \frac{s+1}{h} V(x,s+1)F(x,h-s-1), & h \geq 1. \end{cases}$$

Also, the modified MDTM versions of initial conditions (11) are

$$U(x,0) = \frac{1}{x^2}, \qquad U(x,1) = 0, \qquad V(x,0) = 2lnx, \qquad V(x,1) = 0.$$

Then

$$\begin{cases} U(x,h+2) = \frac{-1}{(h+4)(h+2)}\left[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,h)}{\partial x} + 6F(x,h)\right], \\ V(x,h+2) = \frac{-1}{(h+5)(h+2)}\left[\frac{\partial^2 V(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,h)}{\partial x} - 14U(x,h)\right]. \end{cases}$$

For $h=0$: $\begin{cases} U(x,2) = \frac{-1}{8}\left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,0)}{\partial x} + 6F(x,0)\right], \\ \quad = \frac{-1}{8}\left[\frac{6}{x^4} + \frac{2}{x}.\frac{-2}{x^3} + \frac{6}{x^4}\right] = \frac{-1}{x^4}, \\ V(x,2) = \frac{-1}{10}\left[\frac{\partial^2 V(x,0)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,0)}{\partial x} - 14U(x,0)\right], \\ \quad = \frac{-1}{10}\left[\frac{-2}{x^2} + \frac{3}{x}.\frac{2}{x} - \frac{14}{x^2}\right] = \frac{10}{10x^2} = \frac{1}{x^2}. \end{cases}$

For $h=1$: $(F(x,1) = -2V(x,1)\,F(x,0) = 0)$

$$\begin{cases} U(x,3) = \frac{-1}{15}\left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,1)}{\partial x} - V(x,1)\,F(x,0)\right] = 0, \\ V(x,3) = \frac{-1}{18}\left[\frac{\partial^2 V(x,1)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,1)}{\partial x} - 14U(x,1)\right] = 0. \end{cases}$$

For $h=2$:

$$\begin{cases} U(x,4) = \frac{-1}{24}\left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,2)}{\partial x} + 6F(x,2)\right], \\ \quad = \frac{-1}{24}\left[\frac{-20}{x^6} + \frac{2}{x}.\frac{4}{x^5} - \frac{12}{x^6}\right] = \frac{1}{x^6}, \\ V(x,4) = \frac{-1}{28}\left[\frac{\partial^2 V(x,2)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,2)}{\partial x} - 14U(x,2)\right], \\ \quad = \frac{-1}{28}\left[\frac{6}{x^4} + \frac{3}{x}.\frac{-2}{x^3} + \frac{14}{x^4}\right] = \frac{-14}{28x^4} = \frac{-1}{2x^4}. \end{cases}$$

For $h=3$:

$$\begin{cases} U(x,5) = \frac{-1}{35}\left[\frac{\partial^2 U(x,3)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,3)}{\partial x} + 6F(x,3)\right] = 0, \\ V(x,5) = \frac{-1}{40}\left[\frac{\partial^2 V(x,3)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,3)}{\partial x} - 14U(x,3)\right] = 0. \end{cases}$$

For $h=4$:

$$\begin{cases} U(x,6) = \frac{-1}{48}\left[\frac{\partial^2 U(x,4)}{\partial x^2} + \frac{2}{x}\frac{\partial U(x,4)}{\partial x} + 6F(x,4)\right] = \frac{1}{x^8}, \\ V(x,6) = \frac{-1}{54}\left[\frac{\partial^2 V(x,4)}{\partial x^2} + \frac{3}{x}\frac{\partial V(x,4)}{\partial x} - 14U(x,4)\right] = \frac{1}{3x^6}. \end{cases}$$

Then by substituting the quantities $U(x,h)$, $V(x,h)$ in (7), we get the series solutions

$$u(x,y) = \frac{1}{x^2}\left(1 - \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^4 - \left(\frac{y}{x}\right)^6 \cdots\right)$$

$$v(x,y) = 2lnx + \left(\frac{y}{x}\right)^2 - \frac{1}{2}\left(\frac{y}{x}\right)^4 + \frac{1}{3}\left(\frac{y}{x}\right)^6 \cdots$$

And the exact solutions are

$$u(x,y) = \frac{1}{x^2+y^2}, \quad v(x,y) = \ln\left(x^2+y^2\right)$$

on the region $x \geq y$.

By applying the symmetric conditions of (11)
$u(0,y) = \frac{1}{y^2}, u_x(0,y) = 0, v(0,y) = 2lny, v_x(0,y) = 0,$
we obtain the series solutions

$u(x,y)=\frac{1}{y^2}\left(1-\left(\frac{x}{y}\right)^2+\left(\frac{x}{y}\right)^4-\left(\frac{x}{y}\right)^6\cdots\right)$
$v(x,y)=2lny+\left(\frac{x}{y}\right)^2-\frac{1}{2}\left(\frac{x}{y}\right)^4+\frac{1}{3}\left(\frac{x}{y}\right)^6\cdots$
which converge to the same exact solutions on the region $x\le y$.

Example 2. Consider the system of equations

$$\begin{cases} u_{xx}+\frac{1}{x}u_x+u_{yy}+\frac{1}{y}u_y-24v^{-1}=0, \\ v_{xx}+\frac{2}{x}v_x+v_{yy}+\frac{3}{y}v_y+6u^{-1}=0, \end{cases} \tag{12}$$

$$u(x,0)=x^4, \quad u_y(x,0)=0, \quad v(x,0)=\frac{1}{x^2}, \quad v_y(x,0)=0 \tag{13}$$

with exact solutions

$$u(x,y)=\left(x^2+y^2\right)^2, \quad v(x,y)=\frac{1}{x^2+y^2}.$$

This system could model two interdependent physical quantities evolving under nonlinear interactions and spatial symmetries possibly within a self-gravitating system, plasma configuration, or cosmological model. The presence of the terms $+6u^{-1}$, $-24v^{-1}$ in the equations indicates a nonlinear coupling between two functions u and v. Physically, this might correspond to the interactions between two fields such as pressure and density, or temperature and concentration in self-regulating systems like stars or gases under gravity.

From Table 1 and Corollaries 3.3, 3.4, the modified differential transform version of (12) is

$$\begin{cases} \frac{\partial^2 U(x,h)}{\partial x^2}+\frac{1}{x}\frac{\partial U(x,h)}{\partial x}+(h+2)^2U(x,h+2)-24F(x,h)=0, \\ \frac{\partial^2 V(x,h)}{\partial x^2}+\frac{2}{x}\frac{\partial V(x,h)}{\partial x}+(h+4)(h+2)V(x,h+2)+6G(x,h)=0. \end{cases},$$

where $F(x,h)$ and $G(x,h)$ are the modified differential transform versions of v^{-1}, u^{-1}, respectively, such as (from Corollary 3.2)

$$F(x,h)=\begin{cases} \frac{1}{V(x,0)}=x^2, & h=0, \\ \frac{-\sum_{i=0}^{h-1}F(x,i)V(x,h-i)}{V(x,0)}, & h\ge 1. \end{cases}$$

$$G(x,h)=\begin{cases} \frac{1}{U(x,0)}=\frac{1}{x^4}, & h=0, \\ \frac{-\sum_{i=0}^{h-1}G(x,i)U(x,h-i)}{U(x,0)}, & h\ge 1. \end{cases}$$

Also, the MDTM versions of initial conditions (13) are

$$U(x,0)=x^4, \quad U(x,1)=0, \quad V(x,0)=\frac{1}{x^2}, \quad V(x,1)=0.$$

Then

$$\begin{cases} U(x,h+2)=\frac{-1}{(h+2)^2}\left[\frac{\partial^2 U(x,h)}{\partial x^2}+\frac{1}{x}\frac{\partial U(x,h)}{\partial x}-24F(x,h)\right], \\ V(x,h+2)=\frac{-1}{(h+4)(h+2)}\left[\frac{\partial^2 V(x,h)}{\partial x^2}+\frac{2}{x}\frac{\partial V(x,h)}{\partial x}+6G(x,h)\right]. \end{cases}$$

For $h=0$:
$$\begin{cases} U(x,2) = \frac{-1}{4}\left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{1}{x}\frac{\partial U(x,0)}{\partial x} - 24F(x,0)\right], \\ \quad = \frac{-1}{4}\left[12x^2 + \frac{1}{x}.4x^3 - 24x^2\right] = 2x^2, \\ V(x,2) = \frac{-1}{8}\left[\frac{\partial^2 V(x,0)}{\partial x^2} + \frac{2}{x}\frac{\partial V(x,0)}{\partial x} + 6G(x,0)\right], \\ \quad = \frac{-1}{8}\left[\frac{6}{x^4} + \frac{2}{x}.\frac{-2}{x^3} + 6\frac{1}{x^4}\right] = \frac{-8}{8x^4} = \frac{-1}{x^4}. \end{cases}$$

For $h=1$: $(F(x,1)=0, \quad G(x,1)=0)$

$$\begin{cases} U(x,3) = \frac{-1}{9}\left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{1}{x}\frac{\partial U(x,1)}{\partial x} - 24F(x,1)\right] = 0, \\ V(x,3) = \frac{-1}{15}\left[\frac{\partial^2 V(x,1)}{\partial x^2} + \frac{2}{x}\frac{\partial V(x,1)}{\partial x} + 6G(x,1)\right] = 0. \end{cases}$$

For $h=2$:

$$\begin{cases} U(x,4) = \frac{-1}{16}\left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{1}{x}\frac{\partial U(x,2)}{\partial x} - 24F(x,2)\right], \\ \quad = \frac{-1}{16}\left[4 + \frac{1}{x}.4x - 24\right] = 1, \\ V(x,4) = \frac{-1}{24}\left[\frac{\partial^2 V(x,2)}{\partial x^2} + \frac{2}{x}\frac{\partial V(x,2)}{\partial x} + 6G(x,2)\right], \\ \quad = \frac{-1}{24}\left[\frac{-20}{x^6} + \frac{2}{x}.\frac{4}{x^5} - \frac{12}{x^6}\right] = \frac{24}{24x^6} = \frac{1}{x^6}. \end{cases}$$

For $h=3$:

$$\begin{cases} U(x,5) = \frac{-1}{25}\left[\frac{\partial^2 U(x,3)}{\partial x^2} + \frac{1}{x}\frac{\partial U(x,3)}{\partial x} - 24F(x,3)\right] = 0, \\ V(x,5) = \frac{-1}{35}\left[\frac{\partial^2 V(x,3)}{\partial x^2} + \frac{2}{x}\frac{\partial V(x,3)}{\partial x} + 6G(x,3)\right] = 0. \end{cases}$$

For $h=4$:

$$\begin{cases} U(x,6) = \frac{-1}{36}\left[\frac{\partial^2 U(x,4)}{\partial x^2} + \frac{1}{x}\frac{\partial U(x,4)}{\partial x} - 24F(x,4)\right] = 0, \\ V(x,6) = \frac{-1}{48}\left[\frac{\partial^2 V(x,4)}{\partial x^2} + \frac{2}{x}\frac{\partial V(x,4)}{\partial x} + 6G(x,4)\right] = \frac{-1}{x^8}. \end{cases}$$

Then by substituting the quantities $U(x,h)$ in (7), we get the exact solution

$$u(x,y) = x^4 + 2x^2y^2 + y^4 = \left(x^2+y^2\right)^2$$

and the series solution

$$v(x,y) = \frac{1}{x^2}\left(1 - \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^4 - \left(\frac{y}{x}\right)^6 \cdots\right).$$

The exact solution is

$$v(x,y) = \frac{1}{x^2+y^2}$$

on the region $x \geq y$.

By appling the symmetric conditions of (13)

$u(0,y) = y^4, \quad u_x(0,y) = 0, \quad v(0,y) = \frac{1}{y^2}, \quad v_x(0,y) = 0,$

we obtain the series solutions

$v(x,y) = \frac{1}{y^2}\left(1 - \left(\frac{x}{y}\right)^2 + \left(\frac{x}{y}\right)^4 - \left(\frac{x}{y}\right)^6 \cdots\right)$
which converge to the same exact solutions on the region $x \le y$.

Example 3. Consider the system of equations

$$\begin{cases} u_{xx} + \frac{3}{x}u_x + u_{yy} + \frac{4}{y}u_y - \frac{14}{\sqrt{v}} = 0, \\ v_{xx} + \frac{1}{x}v_x + v_{yy} + \frac{1}{y}v_y - 24e^u = 0, \end{cases} \tag{14}$$

$$u(x,0) = 2lnx, \quad u_y(x,0) = 0, \quad v(x,0) = x^4, \quad v_y(x,0) = 0 \tag{15}$$

with exact solutions

$$u(x,y) = \ln(x^2+y^2), \quad v(x,y) = (x^2+y^2)^2.$$

From Table 1 and Corollaries 3.3, 3.4, the modified differential transform version of (14) is

$$\begin{cases} \frac{\partial^2 U(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,h)}{\partial x} + (h+5)(h+2)U(x,h+2) - 14F(x,h) = 0, \\ \frac{\partial^2 V(x,h)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,h)}{\partial x} + (h+2)^2 V(x,h+2) - 24G(x,h) = 0, \end{cases}$$

where $F(x,h)$, $G(x,h)$ are the modified differential transform versions of

$$\frac{1}{\sqrt{v}} \text{ and } e^u, \text{ respectively, such as}$$

$$G(x,h) = \begin{cases} e^{U(x,0)} = x^2, & h = 0, \\ \sum_{s=0}^{h-1} \frac{s+1}{h} U(x,s+1)\, G(x,h-s-1), & h \ge 1. \end{cases}$$

Also, the MDTM versions of initial conditions (15) are

$$U(x,0) = 2lnx, \qquad U(x,1) = 0, \qquad V(x,0) = x^4, \qquad V(x,1) = 0.$$

Then

$$\begin{cases} U(x,h+2) = \frac{-1}{(h+5)(h+2)}\left[\frac{\partial^2 U(x,h)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,h)}{\partial x} - 14F(x,h)\right], \\ V(x,h+2) = \frac{-1}{(h+2)^2}\left[\frac{\partial^2 V(x,h)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,h)}{\partial x} - 24G(x,h)\right]. \end{cases}$$

For $h = 0$, we have

$$F(x,0) = \frac{1}{0!}\left(\frac{\partial^0}{\partial y^0}\left(\frac{1}{\sqrt{v(x,y)}}\right)\right)_{y=0} = \frac{1}{\sqrt{V(x,0)}} = \frac{1}{x^2}.$$

So,
$$\begin{cases} U(x,2) = \frac{-1}{10}\left[\frac{\partial^2 U(x,0)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,0)}{\partial x} - 14F(x,0)\right], \\ \qquad = \frac{-1}{10}\left[\frac{-2}{x^2} + \frac{3}{x}.\frac{2}{x} - \frac{14}{x^2}\right] = \frac{1}{x^2}, \\ V(x,2) = \frac{-1}{4}\left[\frac{\partial^2 V(x,0)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,0)}{\partial x} - 24G(x,0)\right], \\ \quad = \frac{-1}{4}\left[12x^2 + \frac{1}{x}.4x^3 - 24x^2\right] = \frac{8x^2}{4} = 2x^2. \end{cases}$$

For $h=1$, we have

$$\begin{aligned} F(x,1) &= \frac{1}{1!}\left(\frac{\partial}{\partial y}\left(\frac{1}{\sqrt{v(x,y)}}\right)\right)_{y=0} = \frac{-1}{2}\left(\frac{\partial v(x,y)}{\partial y}.\frac{1}{v(x,y)\sqrt{v(x,y)}}\right)_{y=0} \\ &= \frac{-1}{2}.V(x,1).\frac{1}{V(x,0)\sqrt{V(x,0)}} = 0\ , \end{aligned}$$

$$G(x,1) = U(x,1)\,G(x,0) = 0.$$

So,

$$\begin{cases} U(x,3) = \frac{-1}{18}\left[\frac{\partial^2 U(x,1)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,1)}{\partial x} - 14F(x,1)\right] = 0, \\ V(x,3) = \frac{-1}{9}\left[\frac{\partial^2 V(x,1)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,1)}{\partial x} - 24G(x,1)\right] = 0. \end{cases}$$

For $h=2$, we have

$$\begin{aligned} F(x,2) &= \frac{1}{2!}\left(\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{v(x,y)}}\right)\right)_{y=0} = \frac{-1}{4}\left(\frac{\partial}{\partial y}\left(\frac{\partial v(x,y)}{\partial y}.(v(x,y))^{-\frac{3}{2}}\right)\right)_{y=0} \\ &= \frac{-1}{4}\left(\frac{\partial^2 v(x,y)}{\partial y^2}.(v(x,y))^{-\frac{3}{2}} - \frac{3}{2}.\ (v(x,y))^{-\frac{5}{2}}.\frac{\partial v(x,y)}{\partial y}\right)_{y=0} \end{aligned}$$

$$\begin{aligned} &= \frac{-1}{2}.V(x,2)\,(V(x,0))^{-\frac{3}{2}} + \frac{3}{8}.(V(x,0))^{-\frac{5}{2}}.V(x,1) \\ &= \frac{-1}{2}.V(x,2)\,(V(x,0))^{-\frac{3}{2}} = \frac{-1}{2}.\frac{2x^2}{x^6} \\ &= -\frac{1}{x^4}\ . \end{aligned}$$

$$G(x,2) = \sum_{s=0}^{1}\frac{s+1}{2}U(x,s+1)\,G(x,1-s) = \frac{1}{2}U(x,1)\,G(x,1) + U(x,2)\,G(x,0)$$

$$= \frac{1}{x^2}.x^2 = 1.$$

So,

$$\begin{cases} U(x,4) = \frac{-1}{28}\left[\frac{\partial^2 U(x,2)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,2)}{\partial x} - 14F(x,2)\right], \\ \qquad = \frac{-1}{28}\left[\frac{6}{x^4} + \frac{3}{x}.\frac{-2}{x^3} + \frac{14}{x^4}\right], \\ \qquad = \frac{-1}{2x^4}, \\ V(x,4) = \frac{-1}{16}\left[\frac{\partial^2 V(x,2)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,2)}{\partial x} - 24G(x,2)\right], \\ \qquad = \frac{-1}{16}\left[4 + \frac{1}{x}.4x - 24\right] = 1. \end{cases}$$

For $h=3$:

$$\begin{cases} U(x,5) = \frac{-1}{25}\left[\frac{\partial^2 U(x,3)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,3)}{\partial x} - 14F(x,3)\right] = 0, \\ V(x,5) = \frac{-1}{35}\left[\frac{\partial^2 V(x,3)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,3)}{\partial x} - 24G(x,3)\right] = 0. \end{cases}$$

For $h = 4$:

$$\begin{cases} U(x,6) = \frac{-1}{36}\left[\frac{\partial^2 U(x,4)}{\partial x^2} + \frac{3}{x}\frac{\partial U(x,4)}{\partial x} - 14F(x,4)\right] = \frac{1}{3x^6}, \\ V(x,6) = \frac{-1}{48}\left[\frac{\partial^2 V(x,4)}{\partial x^2} + \frac{1}{x}\frac{\partial V(x,4)}{\partial x} - 24G(x,4)\right] = 0. \end{cases}$$

Then by substituting the quantities $U(x,h)$ in (7), we get the exact solution

$$v(x,y) = x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2$$

and the series solution

$$u(x,y) = 2lnx + \left(\frac{y}{x}\right)^2 - \frac{1}{2}\left(\frac{y}{x}\right)^4 + \frac{1}{3}\left(\frac{y}{x}\right)^6 \cdots$$

the exact solution is

$$u(x,y) = \ln(x^2 + y^2)$$

on the region $x \geq y$.

By applying the symmetric conditions of (15)

$u(0,y) = 2lny, \quad u_x(0,y) = 0, \quad v(0,y) = y^4, \quad v_x(0,y) = 0,$

we obtain the series solutions

$u(x,y) = 2lny + \left(\frac{x}{y}\right)^2 - \frac{1}{2}\left(\frac{x}{y}\right)^4 + \frac{1}{3}\left(\frac{x}{y}\right)^6 \cdots$

which converge to the same exact solutions on the region $x \leq y$.

6 Conclusion

In this study, we are the first to introduce the two-dimensional Lane–Emden system equations and to derive exact analytical solutions without resorting to linearization, discretization, or perturbation techniques. By successfully applying the Modified Differential Transformation Method (MDTM) to these nonlinear forms, we obtain highly accurate solutions with significantly reduced computational effort. This approach leverages novel product and quotient properties specific to various differential transform methods, which we previously established in another paper.

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