Nonlinear Dynamics and Systems Theory, **25** (6) (2025) 643–655

Two-Parameter Quasi-Boundary Regularization for Backward Cauchy Problems

Nawel Moudjeb [1], Khaled Bessila [2], Ali Abdessemed [2] and Okba Zehrour [1*]

[1] *Dynamic Systems and Control Laboratory, Oum El Bouaghi University, Oum El Bouaghi, Algeria.*
[2] *Applied Mathematics and Modeling Laboratory, Constantine 1 University, Constantine, Algeria.*

Received: April 10, 2025 ; Revised: December 1, 2025

Abstract: We propose a two-parameter quasi-boundary regularization method for solving the ill-posed backward Cauchy problem. The Lambert W function is employed for the first time to derive enhanced stability bounds and refined convergence rates. Our approach perturbs the final data via two distinct parameters, providing better control of approximation errors. We prove the regularised problem's well-posedness and derive novel Hölder-Lambert stability estimates. Numerical experiments confirm that our method improves the accuracy of estimated errors, especially under high noise levels.

Keywords: *ill-posed problems; regularization; quasi-boundary value method; backward parabolic problem; stability analysis; Hölder–Lambert stability.*

Mathematics Subject Classification (2020): 35R20, 35R25, 35K90, 93C20, 93D20.

1 Introduction

Let H be a Hilbert space with the inner product $(.,.)$ and the norm $\|.\|$, and A be a self-adjoint operator on H. Assume that A admits an orthonormal eigenbasis $(\varphi_i)_{i\geq 1}$ in H, associated to the eigenvalues $(\lambda_i)_{i\geq 1}$ such that

$$0 < \lambda_1 < \lambda_2 < ...\text{et} \lim_{i\longrightarrow +\infty} \lambda_i = +\infty.$$

* Corresponding author: mailto:okbazehrour@yahoo.fr

The problem is to find a function $u : [0,T] \to H$, the solution of the final-value problem (FVP):

$$u'(t) + Au(t) = 0, \ 0 \le t \le T, \tag{1}$$

with a final condition

$$u(T) = f \tag{2}$$

for some predefined final value $f \in H$.

The FVP (1),(2) is an ill-posed problem. Even in the case of a unique solution that exists on $[0,T]$, the latter does not depend continuously on the final value of f. This type of problem has been considered by several authors using different regularisation methods, one of which is the quasi-reversibility method introduced by Lattes and Lions [1] and developed by Payne [2], Miller [3] and Showalter [4]. We replaced the operator A with a perturbation operator in this method. The other method of regularisation is the quasi-boundary value method introduced by Showalter [5] and developed in [6–10]. This method perturbs the final condition with one regularisation parameter. Moreover, in [6], the final condition (2) is replaced by

$$u(T) + \alpha u(0) = f,$$

and in [7], by

$$u(T) - \alpha u'(0) = f.$$

In other studies [11–13], the perturbation is applied directly to the final data instead of the condition, with either one or two regularization parameters. The origin of this method goes back to [11], by giving a specific value to f.

In this work, we propose a new version of the quasi-boundary regularization method based on a two-parameter perturbation, α and τ, of the final data f. The parameter α accounts for measurement errors, while τ reflects the regularity of the solution. Furthermore, we introduce, for the first time in this context, the Lambert W function as a central analytical tool such that equation (2) is replaced by

$$u(T) = f_{\alpha,\tau}, \tag{3}$$

where $\tau \ge 0$, $\alpha \in (0,1)$ and

$$f_{\alpha,\tau} = \sum_{i=1}^{\infty} \frac{b_i e^{-\lambda_i(T+\tau)}}{e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2} \varphi_i. \tag{4}$$

The backward Cauchy problem studied in this work naturally arises in nonlinear dynamical systems, especially in optimal control, inverse heat conduction, and system identification scenarios. Such problems typically involve reconstructing past states from final-time measurements, often subject to high noise levels. The ill-posedness of these problems directly impacts stability, accuracy, and predictive capability in nonlinear dynamical settings. Our proposed two-parameter quasi-boundary regularization method addresses these fundamental challenges by enhancing stability and convergence properties, making it particularly suitable for the reliable prediction and control of complex nonlinear systems.

We analyze the modified problem rigorously, proving the existence and uniqueness of the regularized solution. We then generate stability estimates that outperform the classical Holder results, introducing a new Holder-Lambert convergence rate. Furthermore, we provide error bounds for both exact and noisy data, and validate our theoretical results by comparing the numerical results in [13] with the remarkable accuracy and efficiency of our method, especially for small values of the noise level.

In Section 2, we introduce the Lambert W function, formulate the modified problem and present our theoretical results, including stability and convergence theorems. In Section 3, we provide numerical experiments that illustrate the effectiveness of the proposed method.

2 The Lambert W Function and the Approximate Problem

For historical notes and the upcoming Lambert W special function in the mathematical literature, we would like to refer the reader to the papers [14–17] and references there. To stay in the real case, it is the function usually denoted by $W(x)$ and defined as the inverse of the function (xe^x), it is generally used for solving transcendental algebraic equations. The function (xe^x) being one-to-one an $x \geq 1$, the principal Lambert function $W(x)$ is its reciprocal. We may find different analytic and differential properties of $W(x)$. In our context, we need its asymptotic expansion established in [15, 16], and given by

$$W(x) = \log(x) - \log(\log(x)) + \sum_{n\geq 1} \frac{(-1)^n}{(\log x)^n} a_n(x) \;; x \gg A, \tag{5}$$

where $a_n(x) = \sum_{m=1}^{n} (-1)^m s(n, n-m+1) \frac{(\log(\log(x)))^m}{m!}$, $s(n,k)$ are Stirling numbers of the first kind. Therefore, our results are much sharper and more exact through the use of (5) than any others known in the literature on the regularization of ill-posed problems, particularly backward in time heat problems.

We turn now to the study of the approximate problem $QBVP$ (1),(3). We first show that $QBVP$ (1),(3) is a well-posed problem. If $(\varphi_i)_{i\geq 1}$ is an orthonormal basis in H, then for all $f \in H$, we have

$$f = \sum_{i=1}^{\infty} b_i \varphi_i, \; b_i = (f, \varphi_i) \,, \forall i \geq 1. \tag{6}$$

If the problem FVP (1),(2) (respectively, $QBVP$ (1),(3)) admits a solution u (respectively, $u_{\alpha,\tau}$), then

$$u(t) = \sum_{i=1}^{\infty} b_i e^{\lambda_i (T-t)} \varphi_i \quad, \forall t \in [0, T], \tag{7}$$

and

$$u_{\alpha,\tau}(t) = \sum_{i=1}^{\infty} \frac{b_i e^{-\lambda_i (t+\tau)}}{e^{-\lambda_i (T+\tau)} + \alpha \lambda_i^2} \varphi_i \quad, \forall t \in [0, T]. \tag{8}$$

To start, for the upcoming results, we need the following lemma, which introduces the Lambert W special function.

Lemma 2.1 *For all* $t \in [0,T]$ *,* $\tau \geqslant 0$ *and* $\alpha \in (0,1)$*, we have*

$$\frac{e^{-\lambda(t+\tau)}}{e^{-\lambda(T+\tau)}+\alpha\lambda^2} \leqslant \frac{1}{\left(\alpha W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}} \left(\frac{(T+\tau)^2}{2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)}\right)^{\frac{T-t}{T+\tau}}. \tag{9}$$

Proof. Let $t \in [0,T]$, $\tau \geqslant 0$ and $\alpha \in (0,1)$. If we consider the function

$$h(\lambda) = \frac{1}{e^{-\lambda(T+\tau)}+\alpha\lambda^2},\ \lambda \in \mathbb{R}_+^*,$$

then $h(\lambda)$ attains its maximum at

$$\lambda_0 = \frac{1}{T+\tau} W\left(\frac{(T+\tau)^2}{2\alpha}\right), \tag{10}$$

hence

$$\frac{1}{e^{-\lambda(T+\tau)}+\alpha\lambda^2} \leqslant \frac{1}{\alpha W\left(\frac{(T+\tau)^2}{2\alpha}\right)} \frac{(T+\tau)^2}{\left(2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)}. \tag{11}$$

Consequently, the lemma follows from the equality

$$\frac{e^{-\lambda(t+\tau)}}{e^{-\lambda(T+\tau)}+\alpha\lambda^2} = \frac{e^{-\lambda(t+\tau)}}{\left(e^{-\lambda(T+\tau)}+\alpha\lambda^2\right)^{\frac{T-t}{T+\tau}} \left(e^{-\lambda(T+\tau)}+\alpha\lambda^2\right)^{\frac{t+\tau}{T+\tau}}}$$

and the estimate (11) above.

Theorem 2.1 *For all* $f \in H$*, the regularized problem QBVP (1),(3) has a unique solution* $u_{\alpha,\tau}$*. Moreover, the following estimate holds:*

$$\|u_{\alpha,\tau}(t)\| \leq \frac{1}{\left(\alpha W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}} \left(\frac{(T+\tau)^2}{2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)}\right)^{\frac{T-t}{T+\tau}} \|f\|,\ t \in [0,T], \tag{12}$$

where $\tau \geqslant 0$, $\alpha \in (0,1)$ *and* W *is the Lambert function.*

Proof. For $t \in [0,T]$, let us take

$$u_{\alpha,\tau(n)}(t) = \sum_{i=1}^{n} \frac{b_i e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)}+\alpha\lambda_i^2}\varphi_i.$$

Since $\left(u_{\alpha,\tau(n)}\right)_{n\geqslant 1}$ represents the sequence of partial sums of the series (8), it suffices to show that $u_{\alpha,\tau(n)} \in C^1([0,T],H)$ and $\left\|\lim_{n\longrightarrow+\infty} u_{\alpha,\tau(n)}(0)\right\| < +\infty$.
Set

$$v_{\alpha,\tau}(t) = -\sum_{i=1}^{+\infty} \frac{\lambda_i b_i e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)}+\alpha\lambda_i^2}\varphi_i.$$

So,

$$u'_{\alpha,\tau(n)}(t) - v_{\alpha,\tau}(t) = \sum_{i=n+1}^{+\infty} \frac{\lambda_i b_i e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2}\varphi_i.$$

Using the hypothesis $\lambda_i^2 \longrightarrow +\infty$ and the inequality

$$\frac{\lambda_i^2 b_i^2 e^{-2\lambda_i(t+\tau)}}{(e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2)^2} \leq \frac{b_i^2}{\alpha^2}\ ,\ \forall i \geq 1, \tag{13}$$

we get

$$\lim_{n\longrightarrow+\infty} \sup_{t\in[0,T]} \left\| u'_{\alpha,\tau(n)}(t) - v_{\alpha,\tau}(t) \right\| = 0.$$

Then the sequence $(u'_{\alpha,\tau(n)})_{n\geqslant 1}$ converges uniformly in t. Moreover, the use of the Weierstrass criterion insures that $u_{\alpha,\tau} \in C^1([0,T],H)$ and

$$u'_{\alpha,\tau}(t) = -\sum_{i=1}^{\infty} \frac{\lambda_i b_i e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2}\varphi_i; \forall t \in [0,T], \tau \geqslant 0. \tag{14}$$

Using (13), we have $u_{\alpha,\tau}(t) \in D(A)$ and

$$Au_{\alpha,\tau}(t) = \sum_{i=1}^{+\infty} \frac{\lambda_i b_i e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2}\varphi_i,\ \forall t \in [0,T],\ \tau \geqslant 0. \tag{15}$$

So, by virtue of (8), (14) and (15), we find that $u_{\alpha,\tau}$ is a classical solution of $QBVP$ (1),(3).
Let $t \in [0,T]$. From (8), we have

$$\|u_{\alpha,\tau}(t)\|^2 \leq \sum_{i=1}^{+\infty} b_i^2 \left(\frac{e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)} + \alpha\lambda_i^2} \right)^2, \tag{16}$$

and using Lemma 2.1, the estimate (12) is then deduced from (15).

Remark 2.1 We note in Theorem 2.1 above, we have a new stability order given by

$$\omega(\alpha,\tau,T) = \frac{1}{\left(\alpha W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}},\ \alpha > 0,\ \tau \geqslant 0,$$

where W^2 is the square of $W(x)$. This order does not exist in the literature concerning inverse problems backward in time. Our order of stability involves the Lambert W function, and improves significantly the existing Hölder order $\alpha^{\frac{t-\tau}{T+\tau}}$ known up to date. According to the asymptotic expansion of W given in (5), we largely have

$$\frac{1}{\alpha W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)} \ll \frac{1}{\alpha}, \alpha > 0,$$

for α small enough; $W\left(\frac{(T+\tau)^2}{2\alpha}\right)$ is approximatively

$$\log\left(\frac{(T+\tau)^2}{2\alpha}\right)-\log\left(\log\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)+O\left(\left(\frac{\log\left(\log\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)}{\log\left(\frac{(T+\tau)^2}{2\alpha}\right)}\right)^2\right),$$

the term $\frac{1}{W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)}$ restrains $\frac{1}{\alpha}$ from going to infinity rapidly; and consequently, stability is much more controlled, particularly when using numerical schemes for applications. Moreover, this mixed Hölder-Lambert type of stability is much more sharp and exact than the Hölder type, stability is much more weighted by the presence of the term $W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)$, particularly for applications and numerical computations. Finally, the choice of a second parameter τ in our approach gives rather good uniform results and regularity around the origin instant $t_0=0$. This improves many results where the stability is of order $\alpha^{-\frac{t}{T}}$, which is meaningless whenever t is in the neighborhood of the origin.

Theorem 2.2 *For every $f\in H$ and $\tau\geq 0$, $\|u_{\alpha,\tau}(T)-f\|\longrightarrow 0$ as $\alpha\to 0$. That is, $u_{\alpha,\tau}(T)$ converges to f in H.*

Proof. Let $f\in H$, $\tau\geq 0$, and $\varepsilon>0$. Choose some N for which $\sum_{i=N+1}^{+\infty} b_i^2<\frac{\varepsilon}{2}$.
From (4) and (8), we have

$$\|u_{\alpha,\tau}(T)-f\|^2\leq \alpha^2\sum_{i=1}^{N}\lambda_i^4 b_i^2 e^{2\lambda_i(T+\tau)}+\frac{\varepsilon}{2}. \tag{17}$$

If we take $\alpha^2<\varepsilon\left(2\sum_{i=1}^{N}\lambda_i^4 b_i^2 e^{2\lambda_i(T+\tau)}\right)^{-1}$, then we get

$$\lim_{\alpha\to 0}\|u_{\alpha,\tau}(T)-f\|=0,$$

as required.

Theorem 2.3 *For all $f\in H$ and $\tau\geq 0$, the FVP (1),(2) has a solution u given by (7) if and only if the sequence $(u_{\alpha,\tau}(0))$ converges in H. Furthermore, the sequence $(u_{\alpha,\tau})$ converges to u as α tends to zero, uniformly in t.*

Proof. Let $f\in H$ and $\tau\geq 0$. Assume that $\lim_{\alpha\to 0}u_{\alpha,\tau}(0)=u_0$ exists. Since $u_0\in H$, we have

$$u_0=\sum_{i=1}^{+\infty}u_{0i}\varphi_i.$$

The solution of the initial problem

$$\begin{cases} v'(t)+Av(t)=0 & ,\ 0\leq t\leq T,\\ v(0)=u_0, \end{cases}$$

is given by

$$v(t)=\sum_{i=1}^{+\infty}u_{0i}e^{-\lambda_i t}\varphi_i. \tag{18}$$

Let $t \in [0,T]$. From (8), we get

$$\|u_{\alpha,\tau}(t) - v(t)\| \leq \|u_{\alpha,\tau}(0) - u_0\|.$$

This means that $u_{\alpha,\tau}$ converges uniformly to v in H.
Moreover, $\lim_{\alpha\to 0} u_{\alpha,\tau}(T) = v(T)$. Using Theorem 2.2, we have $v(T) = f$ and v is a solution of the FVP (1),(2). Assuming that u is the solution to FVP (1),(2), given by (7), we prove that $(u_{\alpha,\tau}(0))$ converges in H. From (7) and (8), we have

$$\|u_{\alpha,\tau}(t) - u(t)\|^2 = \sum_{i=1}^{+\infty} \left(\frac{b_i e^{-\lambda_i \tau}}{e^{-\lambda_i (T+\tau)} + \alpha\lambda_i^2} - b_i e^{\lambda_i T} \right)^2 e^{-2\lambda_i t}$$

so that

$$\|u_{\alpha,\tau}(t) - u(t)\| \leq \|u_{\alpha,\tau}(0) - u(0)\|. \tag{19}$$

Since u is the solution of FVP (1)-(2), $u(0) \in H$ and $\sum_{i=1}^{+\infty} b_i^2 e^{2\lambda_i T} < +\infty$.
Let $\varepsilon > 0$ and choose some N for which $\sum_{i=N+1}^{+\infty} b_i^2 e^{2\lambda_i T} < \frac{\varepsilon}{2}$. From (7),(8), we have

$$\|u_{\alpha,\tau}(0) - u(0)\|^2 \leq \alpha^2 \sum_{i=1}^{N} e^{4\lambda_i T} \lambda_i^4 e^{2\lambda_i \tau} b_i^2 + \frac{\varepsilon}{2}. \tag{20}$$

So, when taking α such that $\alpha^2 < \varepsilon(2\sum_{i=1}^{N} e^{4\lambda_i T}\lambda_i^4 e^{2\lambda_i\tau} b_i^2)^{-1}$, the sequence $(u_{\alpha,\tau}(0))$ converges to $u(0)$ as α tends to zero. Furthermore, from (19),(20), $u_{\alpha,\tau}$ converges to u, which ends the proof.

The following theorem expresses the error estimate in the case of exact data.

Theorem 2.4 *Let $f \in H$, $\tau \geq 0$ and $\alpha \in (0,1)$. Suppose that FVP (1),(2) has a unique solution u such that the series $C_{T,\tau} = \sum_{i=1}^{+\infty} b_i^2 \lambda_i^4 e^{2\lambda_i (T+\tau)}$ is convergent. Then the following estimate holds for all $t \in [0,T]$:*

$$\|u(t) - u_{\alpha,\tau}(t)\| \leq C_{T,\tau} \frac{\alpha^{\frac{t+\tau}{T+\tau}}}{\left(W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}} \frac{(T+\tau)^{2\left(\frac{T-t}{T+\tau}\right)}}{\left(2 + W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}}, \tag{21}$$

where $u_{\alpha,\tau}$ is the solution of $QBVP$ (1),(3).

Proof. Let $\varepsilon > 0$. Suppose that the problem FVP (1),(2) has a unique solution u. Then u is given as

$$u(t) = \sum_{i=1}^{+\infty} b_i e^{\lambda_i (T-t)} \varphi_i.$$

Therefore

$$\|u(t) - u_{\alpha,\tau}(t)\|^2 = \sum_{i=1}^{+\infty} b_i^2 \alpha^2 \lambda_i^4 \left(\frac{e^{-\lambda_i (t-T)}}{e^{-\lambda_i (T+\tau)} + \alpha\lambda_i^2} \right)^2, \tag{22}$$

$$= \sum_{i=1}^{+\infty} b_i^2 \alpha^2 \lambda_i^4 e^{2\lambda_i (T+\tau)} \left(\frac{e^{-\lambda_i (t+\tau)}}{e^{-\lambda_i (T+\tau)} + \alpha\lambda_i^2} \right)^2. \tag{23}$$

Using (9), we get

$$\|u(t)-u_{\alpha,\tau}(t)\|^2 \leq \alpha^{2\left(\frac{t+\tau}{T+\tau}\right)}\left[\frac{(T+\tau)^2}{W\left(\frac{(T+\tau)^2}{2\alpha}\right)\left[2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right]}\right]^{2\left(\frac{T-t}{T+\tau}\right)}\sum_{i=1}^{+\infty} b_i^2\lambda_i^4 e^{2\lambda_i(T+\tau)}.$$

Based on the condition $\sum_{i=1}^{+\infty} b_i^2\lambda_i^4 e^{2\lambda_i(T+\tau)} < +\infty$, we find

$$\|u(t)-u_{\alpha,\tau}(t)\| \leq C_{T,\tau}\frac{\alpha^{\frac{t+\tau}{T+\tau}}}{\left(W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}}\frac{(T+\tau)^{2\left(\frac{T-t}{T+\tau}\right)}}{\left(2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)^{\frac{T-t}{T+\tau}}},$$

as required.

Remark 2.2 Thus, new mixed Hölder-Lambert type of convergence rate estimates improves very much the Hölder type $\alpha^{\frac{t+\tau}{T+\tau}}$ known in the litterature. Clearly, from Theorem 2.4, it is of the from

$$\omega_c(\alpha,T,\tau) = \frac{\alpha^{\frac{t+\tau}{T+\tau}}}{W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)^{\frac{T-t}{T+\tau}}},$$

where W^2 is the square of $W(x)$. From the asymptotic behaviour of the Lambert W function in (11), we have

$$\omega_c(\alpha,T,\tau) \ll \alpha^{\frac{t+\tau}{T+\tau}},$$

and $\omega_c(\alpha,T,\tau)$ tends to zero faster than $\alpha^{\frac{t+\tau}{T+\tau}}$ because of the new infinite term $W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)$ in the denominator. This Hölder-Lambert type is more strong and exact than any Hölder type or Hölder-logarithmic rate known, and involves better precision for numerical computation and applications.

Theorem 2.5 *Let u and $u^{\delta}_{\alpha,\tau}$ be solutions of the FVP (1),(2) and QBVP (1),(3), respectively, in the case where $f \equiv f_\delta$ such that $\|f-f_\delta\| < \delta$.*
Suppose that $\left\|\sum_{i=1}^{+\infty} b_i\lambda_i^2 e^{\lambda_i T}\varphi_i\right\| \leq K$. Taking into account the fact that K is a positive constant, we have

$$\|u(t)-u^{\delta}_{\alpha,\tau}(t)\| \leq \frac{C_{k,\delta,\alpha}}{W\left(\frac{(T+\tau)^2}{2\alpha}\right)}\cdot\frac{(T+\tau)^2}{\left(2+W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)}, \tag{24}$$

where $C_{k,\delta,\alpha} = (\frac{\delta}{\alpha}+K)$.

Proof. Let u and $u^{\delta}_{\alpha,\tau}$ be solutions of the FVP (1),(2) and QBVP (1),(3), respectively, in the case where $f \equiv f_\delta$ such that $\|f-f_\delta\| < \delta$.
Set

$$u^{\delta}_{\alpha,\tau}(t) = \sum_{i=1}^{+\infty}\frac{b_i^{\delta}e^{-\lambda_i(t+\tau)}}{e^{-\lambda_i(T+\tau)}+\alpha\lambda_i^2}\ \varphi_i\ , \quad \forall t\in[0,T], \tau \geqslant 0, \tag{25}$$

where

$$b_i^\delta = (f_\delta, \varphi_i) \ , \quad \forall i \geq 1. \tag{26}$$

Substitute (7) and (25) into the following inequality:

$$\|u(t) - u_{\alpha,\tau}^\delta(t)\| \leq \|u(t) - u_{\alpha,\tau}(t)\| + \|u_{\alpha,\tau}(t) - u_{\alpha,\tau}^\delta(t)\|.$$

From (11), we obtain

$$\|u(t) - u_{\alpha,\tau}(t)\| \leq \frac{(T+\tau)^2}{W\left(\frac{(T+\tau)^2}{2\alpha}\right)\left(2 + W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)} K \tag{27}$$

and

$$\|u_{\alpha,\tau}(t) - u_{\alpha,\tau}^\delta(t)\| \leq \frac{(T+\tau)^2\delta}{\alpha W\left(\frac{(T+\tau)^2}{2\alpha}\right)\left(2 + W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)}, \tag{28}$$

once again, the use of (27) and (28) gives us

$$\|u(t) - u_{\alpha,\tau}^\delta(t)\| \leq \frac{(T+\tau)^2 K}{W\left(\frac{(T+\tau)^2}{2\alpha}\right)\left(2 + W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)} + \frac{(T+\tau)^2\delta}{\alpha W\left(\frac{(T+\tau)^2}{2\alpha}\right)\left(2 + W\left(\frac{(T+\tau)^2}{2\alpha}\right)\right)},$$

from which, by taking $\alpha = \delta$, we obtain

$$\|u(t) - u_{\alpha,\tau}^\delta(t)\| \leq \frac{(T+\tau)^2}{W\left(\frac{(T+\tau)^2}{2\delta}\right)\left(2 + W\left(\frac{(T+\tau)^2}{2\delta}\right)\right)}(1+K).$$

In the above theorem, α and δ are kept proportional; and the rate of convergence is of order $\frac{C_{k,\delta,\alpha}}{W^2\left(\frac{(T+\tau)^2}{2\alpha}\right)}$, where W^2 is the square of $W(x)$.

3 Numerical Experiments

In this section, we present numerical experiments to evaluate the performance of our two-parameter modified quasi-boundary regularisation method for an inverse heat conduction problem in a cylindrical domain. We consider the same example as that studied in [13] to allow for a direct comparison of results.

We begin by recalling the following initial-boundary value problem, which has been previously studied in [13]:

$$(3.1)\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, 0 < r < r_0, 0 < t < T, \\ u(r_0, t) = 0, 0 \leq t \leq T, \\ u(r,t) \ \textit{bounded as} \ r \to 0 \ , \\ u(r,T) = f^{ex}(r), 0 \leq r \leq r_0. \end{cases}$$

Here, r is the radial coordinate, $f^{ex}(r) = e^T J_0(\frac{\mu_1}{r_0}r)$ represents the final temperature history of the cylinder. The objective is to reconstruct the temperature distribution $u(.,t)$ for $0 \leq t \leq T$, from noisy final-time data. The solution to problem (3.1) is given by

$$u^{ex}(r,t) = e^t e^{(\frac{\mu_1}{r_0})^2(T-t)} J_0(\frac{\mu_1}{r_0}r), \tag{29}$$

the function $J_0(x)$ is a Bessel function, where μ_n is its root. We will approximate (3.1) by the following regularised problem:

$$(3.2)\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, 0 < r < r_0, 0 < t < T, \\ u(r_0, t) = 0, 0 \leq t \leq T, \\ u(r,t) \; bounded \; as \; r \to 0 \;, \\ u(r,T) = f^{\epsilon}(r), 0 \leq r \leq r_0, \end{cases}$$

where

$$f^{\epsilon}(r) = e^T J_0(\frac{\mu_1}{r_0} r) + \sum_{p=1}^{p_0} \epsilon a_p J_0\left(\frac{\mu_p}{r_0} r\right),$$

where p_0 is a random natural number and a_p is a finite sequence of random normal numbers with mean 0 and variance A^2. It follows that the error in the measurement process is bounded by ϵ, $\| f^{\epsilon} - f^{ex} \| \leq \epsilon$. The regularized solution, which is obtained by (8) and corresponds to the data f^{ϵ}, is

$$u_{\epsilon,\tau}(r,t) = \frac{e^{1-(\frac{\mu_1}{r_0})^2(t+\tau)}}{e^{-(\frac{\mu_1}{r_0})^2(T+\tau)} + \epsilon(\frac{\mu_1}{r_0})^4} J_0(\frac{\mu_1}{r_0} r) + \sum_{i=1}^{p_0} a_p \frac{\epsilon e^{-(\frac{\mu_p}{r_0})^2(t+\tau)}}{e^{-(\frac{\mu_p}{r_0})^2(T+\tau)} + \epsilon(\frac{\mu_p}{r_0})^4} J_0(\frac{\mu_p}{r_0} r). \quad (30)$$

For each point of time, the relative error is obtained by the following relationship:

$$RE(\epsilon, t) = \frac{\| u^{app}(., t) - u^{ex}(., t) \|}{\| u^{ex}(., t) \|}.$$

Fix $T = 1, r_0 = 2, p_0 = 1000, A^2 = 100$.

Case 01:

Fix $\tau = 0.3$. We consider the discretizations $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, $\epsilon_3 = 10^{-3}$, respectively. The results obtained are presented in the following figures.

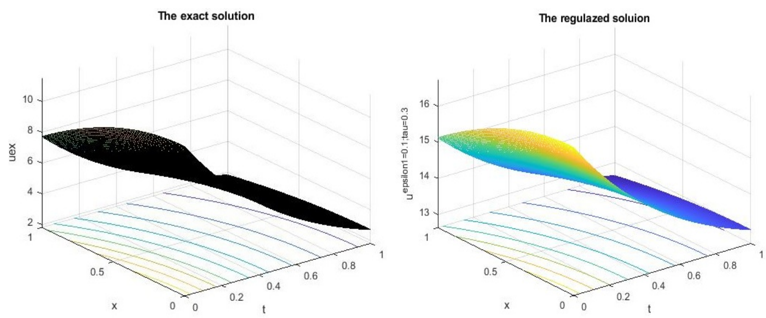

Figure 1: The exact solution and regularized solution with $\epsilon_1 = 10^{-1}$ and $\tau = 0.3$.

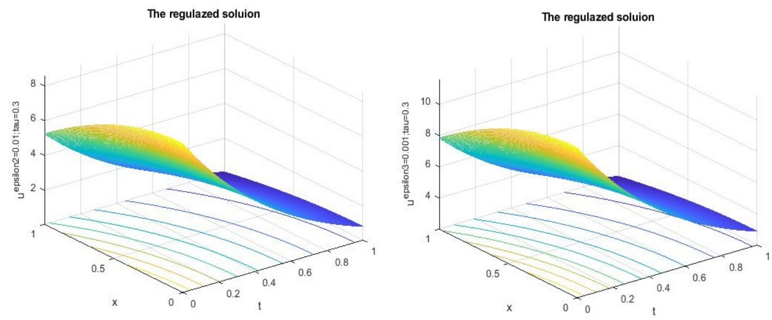

Figure 2: The approximate solution with $\tau = 0.3$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-3}$.

Case 02:
Fix $\epsilon = 10^{-3}$. We consider the discretizations $\tau_1 = 0$, $\tau_2 = 0.3$, respectively. The results obtained are presented in the following figures.

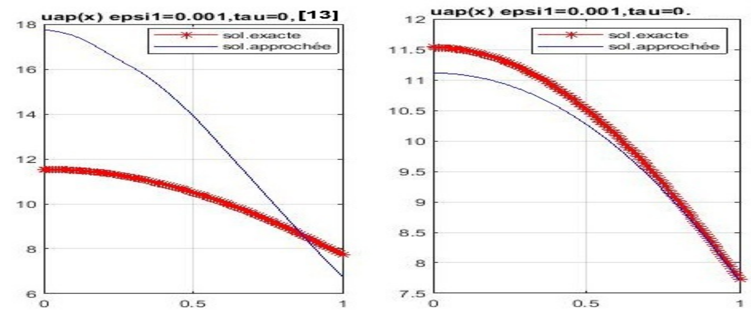

Figure 3: The comparison between the exact solution and the approximate solution in this paper and in [13] with $\epsilon = 10^{-3}$ for $\tau = 0$.

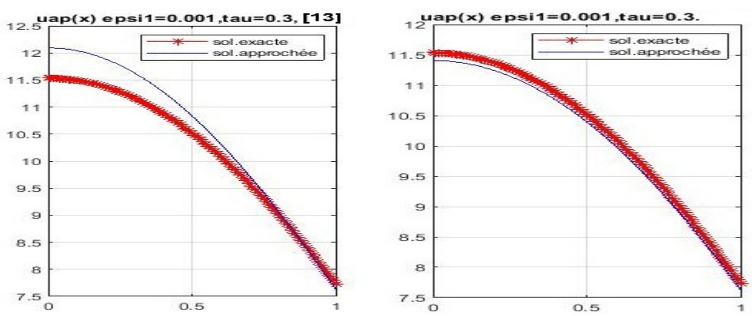

Figure 4: The comparison between the exact solution and the approximate solution in this paper and in [13] with $\epsilon = 10^{-3}$ for $\tau = 0.3$.

Furthermore, we provide comparison tables.

ε	$\|u_{\epsilon,\tau}(.,0) - u^{ex}(.,0)\|$	$RE(\varepsilon,0)$
$\varepsilon = 10^{-3}$	1.89912372122484	0.0183775205270803
$\varepsilon = 10^{-4}$	0.370482962729477	0.00358510516002786
$\varepsilon = 10^{-5}$	0.0568432599456435	0.000550063255385745
$\varepsilon = 10^{-6}$	0.0508581110298997	0.000492145913915346
$\varepsilon = 10^{-7}$	0.0111367060795689	0.00010776814711647

Table 1: Error and relative error of the proposed method for $\tau = 0.3$ and various values of ε.

In paper [13], we have the same table as that we created with the same values ε, but with results different from our results. We notice that whenever we reduce the values ε, we find that the relative error and absolute error in the two tables decrease, but in our work it decreases better, which makes the regular solution closer to the exact solution and better than that found in [13], and this shows that our method is more effective. We note that our method improved accuracy and so our results are more accurate than in [13]. We also reduced the errors in the data. For comparison, we reproduce below the table presented in paper [13] (see p. 11)

ε	$\|u_{\epsilon,\tau}(.,0) - u^{ex}(.,0)\|$	$RE(\varepsilon,0)$
$\varepsilon = 10^{-3}$	8.48702923827765	0.115626406608408
$\varepsilon = 10^{-4}$	5.84620043944197	0.079648028791562
$\varepsilon = 10^{-5}$	3.59734075211554	0.049009797519858
$\varepsilon = 10^{-6}$	0.723559578438893	0.009857700695156
$\varepsilon = 10^{-7}$	0.0550576642083171	0.000173396997915

Table 2: Error and relative error of the method by N.V. Hoa and T.Q. Khanh in [13] for $\tau = 0.3$ and various values of ε.

The numerical experiments confirm that our two-parameter regularisation method is accurate and stable. The solution improves as the noise level decreases, especially with $\tau = 0.3$. Our approach consistently achieves smaller errors than the method in [13]. These results validate the effectiveness of our method for inverse heat conduction problems with noisy data.

4 Conclusion

In this work, we proposed a two-parameter quasi-boundary regularisation method for solving the backward Cauchy problem commonly encountered in nonlinear dynamical systems. Our approach integrates the Lambert W function for the first time, significantly enhancing stability bounds and providing refined Hölder-Lambert-type convergence rates. Rigorous theoretical analysis established the well-posedness and stability of the regularised formulation. Numerical experiments demonstrated that our method achieves greater accuracy and robustness than existing techniques, particularly under high noise conditions. These results confirm our approach's effectiveness and practical applicability, especially for inverse problems involving noisy data in nonlinear dynamics and optimal control settings.

References

[1] R. T. Lattès and J. L. Lions. *Méthode de quasi-réversibilité et applications.* Dunod, Paris, 1967. (English translation by R. Bellman, Elsevier, New York, 1969.)

[2] L. E. Payne. Some general remarks on improperly posed problems for partial differential equations. In: *Lecture Notes in Mathematics* **316**. Springer, Berlin (1973), 1–30.

[3] K. Miller. Stabilized quasireversibility and other nearly best possible methods for non-well-posed problems. In: *Lecture Notes in Mathematics* **316**. Springer, Berlin (1973), 161–176.

[4] R. E. Showalter. The final value problem for evolution equations. *J. Math. Anal. Appl.* **47** (1974) 563–572.

[5] R. E. Showalter. *Cauchy Problem for Hyper-Parabolic Partial Differential Equations.* Elsevier, 1983.

[6] G. W. Clark and S. F. Oppenheimer. Quasireversibility methods for non-well-posed problems. *Electron. J. Diff. Eqns.* **8** (1994) 1–9.

[7] M. Denche and K. Bessila. A modified quasi-boundary value method for ill-posed problems. *J. Math. Anal. Appl.* **301** (2005) 419–426.

[8] K. Bessila. Regularization by a modified quasi-boundary value method of ill-posed differential-operator equations of first order. *J. Math. Anal. Appl.* **409** (1) (2014) 315–320.

[9] N. Teniou and S. Djezzar. An approximate solution for a class of ill-posed nonhomogeneous Cauchy problems. *Journal of Mathematics* **2021** (1) (2021).

[10] C. Wei and Y.-L. Liu. A quasi-boundary value regularization method for the spherically symmetric backward heat conduction problem. *Open Mathematics* **21** (2023) 20230171.

[11] D. D. Trong and N. H. Tuan. A nonhomogeneous backward heat problem: Regularization and error estimates. *Electron. J. Diff. Eqns.* **33** (2006) 1–14.

[12] M. Denche and A. Abdessemed. On regularization and error estimates for non-homogeneous backward Cauchy problem. *Arab. J. Math. Sci.* **18** (2) (2012) 149–164. DOI: 10.1016/j.ajmsc.2012.03.002.

[13] N. V. Hoa and T. Q. Khanh. Two-parameter regularization method for an axisymmetric inverse heat problem. *Bound. Value Probl.* **25** (2017).

[14] I. Mező. *The Lambert W Function, its Generalizations and Applications.* Chapman & Hall/CRC, New York, 2022.

[15] T. Dence. A brief look into the Lambert W function. *Appl. Math.* **4** (2013) 887–892.

[16] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey and D. E. Knuth. On the Lambert W function. *Adv. Comput. Math.* **5** (1996) 329–359.

[17] J. M. Borwein and S. B. Lindstrom. Meetings with Lambert W and other special functions in optimization and analysis. *Pure Appl. Funct. Anal.* **1** (2016).