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Global Theoretical Investigation of Diffusion Driven Instability for Three Coupled Equations of a Reaction Diffusion System

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Abstract: In this paper, we investigate the mechanism called DDI (Diffusion Driven Instability) for a full three dimensional matrix of diffusion coefficients. We apply a linear approach in the neighborhood of an arbitrary equilibrium point using the Routh-Hurwitz stability criterion and we study the existence of at least one eigenvalue with positive real part of the matrix $A(k)$. Our main result is the proof of sufficient and necessary condition for the Turing instability. The research is extended to a reaction-diffusion system for three species.

Keywords: *reaction-diffusion system; Turing instability; cross diffusion; predator-prey.*

Mathematics Subject Classification (2020): 35K57, 35B36, 35B32, 92D25, 92D40, 37N25, 70K42.

1 Introduction

Back in the 1950s, Alan Turing published a paper under the title “The Chemical Basis of Morphogenesis”. Turing demonstrated that under certain circumstances, chemicals can react and diffuse in a way that results in solutions that do not have concentration equilibrium. To study the process of morphogenesis, he took into account two coupled reaction-diffusion systems. Mathematically, Turing’s idea was as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f(u,v), \quad t>0 \quad x \in \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + g(u,v), \quad t>0 \quad x \in \Omega, \end{cases} \tag{E}$$

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and its corresponding kinetic equations was

$$\begin{cases} \dfrac{du}{dt} = f(u,v), \quad t > 0, \\ \dfrac{dv}{dt} = g(u,v), \quad t > 0. \end{cases} \tag{K}$$

The principal idea is that if there is no diffusion (regarding K), u and v converge to a linear stable uniform steady state, then the uniform stable steady state can be unstable due to the presence of diffusion and some other conditions (taking into account E) in certain instances. Turing realized that diffusions are the primary cause of the Turing instability in reaction-diffusion systems, which is referred to as Diffusion-Driven Instability. Qian and Murray [5] considered the case where the matrix of diffusion coefficients D is diagonal. Jia and Wang [6] considered the case

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} - d_1(t)\,\Delta u_1(x,t) = u_1(x,t)\big[r_1(t) - a_{11}(t)\,u_1(x,t-\tau_1) - a_{12}(t)\,u_2(x,t)\big], \\ \dfrac{\partial u_2(x,t)}{\partial t} - d_2(t)\,\Delta u_2(x,t) = u_2(x,t)\big[-r_2(t) - a_{22}(t)\,u_2(x,t-\tau_2) + a_{21}(t)\,u_1(x,t-\tau_1)\big]. \end{cases}$$

Wu et al. [8] investigated the fractional-order predator-prey reaction-diffusion model

$$\begin{cases} \dfrac{\partial^\eta u}{\partial t^\eta} = d_1\Delta u + ru\Big(1 - \dfrac{u}{k}\Big) - \dfrac{qu^2v}{u^2+a}, \\ \dfrac{\partial^\eta v}{\partial t^\eta} = d_2\Delta v - cv + \dfrac{pu^2v}{u^2+a}. \end{cases}$$

Didiharyono et al. [1] dealt with the kinetic equations

$$\begin{cases} \dfrac{dB}{dt} = r_1B\Big(1 - \dfrac{B}{K}\Big) - \dfrac{\alpha NB}{(1+\eta B)(1+\mu N)} - \beta MB, \\ \dfrac{dN}{dt} = \dfrac{\delta\alpha NB}{(1+\eta B)(1+\mu N)} - r_2N + \theta M, \\ \dfrac{dM}{dt} = \vartheta MB - r_3M + \sigma N. \end{cases}$$

Also, in [7], Agus et al. considered the kinetic equations such that

$$\begin{cases} \dfrac{dx}{dt} = x\Big[(R - R_1x) - \dfrac{cx^2z}{c_1+x^2} - uy\Big] + \sigma_2 y, \\ \dfrac{dy}{dt} = y\big[(S - S_1y) - vx\big] + \sigma_1 x, \\ \dfrac{dz}{dt} = z\Big(\dfrac{g_1x^2}{c_1+x^2} - e - q_3z\Big). \end{cases}$$

Das K.P. and Said K. [2] handled

$$\begin{cases} \partial_t u - D_1\Delta u = u(1-u) - \dfrac{a_1uv}{1+b_1u}, \\ \partial_t v - D_2\Delta v = \dfrac{a_1uv}{1+b_1u} - \dfrac{a_2vw}{1+b_2v} - d_1v - dv^2, \qquad \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_t r - D_3\Delta r = \dfrac{a_2vw}{1+b_2v} - d_2w. \end{cases}$$

In this paper, we handle a more general situations, where D is a full matrix, and f, g and h are any continuously differentiable reaction terms. We focus only on a class of reaction-diffusion systems that fulfill the following hypotheses:

$$\operatorname{tr} A < 0, \quad \det A < 0, \quad \det A - \operatorname{tr} A \cdot \operatorname{tr} \operatorname{com} A > 0,$$

where A is the Jacobian matrix of linearization, and $\operatorname{com} A$ is the adjugate matrix of A.

2 Linearization and Stability of the Jacobian Matrix

Consider the reaction-diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_{11}\Delta u + d_{12}\Delta v + d_{13}\Delta w + f(u,v,w), \\ \dfrac{\partial v}{\partial t} = d_{21}\Delta u + d_{22}\Delta v + d_{23}\Delta w + g(u,v,w), \\ \dfrac{\partial w}{\partial t} = d_{31}\Delta u + d_{32}\Delta v + d_{33}\Delta w + h(u,v,w), \end{cases} \tag{1}$$

where f, g and h are nonlinear reaction terms and d_{ij}, $1 \leq i, j \leq 3$, represent diffusion coefficients of u, v and w, respectively. As discussed by Murray in 2003, see [3], system (1) is used to model predator-prey dynamics by blending local interactions (prey reproduce, predators eat prey) with space movement, resulting in patterns that demonstrate the evolution of predator-prey populations in time. A linear approach analysis of (1) near the steady state of its corresponding nonlinear ordinary differential equation yields

$$\frac{\partial y}{\partial t} = \left(A - k^2 D\right) y,$$

where

$$y = \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{w} \end{pmatrix} \quad A = \begin{pmatrix} f_u & f_v & f_w \\ g_u & g_v & g_w \\ h_u & h_v & h_w \end{pmatrix} \quad D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

and k is the wave number of the Fourier Transform $\psi(x,t) = \int \widetilde{\psi}(k,t) e^{-ikx}\, dx$. The coefficients d_{ij} satisfy the conditions

$$\begin{cases} d_{11} > 0, \\ 4d_{11}d_{22} - (d_{12}+d_{21})^2 > 0, \\ d_{33} - \dfrac{(d_{13}+d_{31})^2}{d_{11}} - \dfrac{\left(2d_{11}(d_{23}+d_{32}) - (d_{12}+d_{21})(d_{13}+d_{31})\right)^2}{4d_{11}\left(4d_{11}d_{22} - (d_{12}+d_{21})^2\right)} > 0, \end{cases} \tag{P}$$

which reflects the parabolicity of the system (1) and implies at the same time that the matrix D is positive definite.

Definition 2.1 Diffusion-Driven Instability is the process of making conditions on A, k and D such that, with the presence of the diffusions $\left(d_{ij} \neq 0\right)$, the matrix $A(k) = A - k^2 D$ has at least one eigenvalue with a positive real part.

Remark 2.1 We assume that without the diffusion coefficients, i.e., $\big(d_{ij} = 0\big)$, the steady state is stable.

In what follows, we give some more details concerning the previous remark. We set:

$$A = \begin{pmatrix} f_u\big(U^*\big) & f_v\big(U^*\big) & f_w\big(U^*\big) \\ g_u\big(U^*\big) & g_v\big(U^*\big) & g_w\big(U^*\big) \\ h_u\big(U^*\big) & h_v\big(U^*\big) & h_w\big(U^*\big) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $U^* = \big(u^*, v^*, w^*\big)$ denotes any equilibria of (1). The characteristic polynomial of A is given by

$$P(\lambda) = -\lambda^3 + \big(\mathrm{tr}A\big)\lambda^2 - \big(\mathrm{tr}\,\mathrm{com}\,A\big)\lambda + \big(\det A\big). \tag{2}$$

To determine if the matrix A is stable, we check that all its eigenvalues have a negative real part. For this purpose, we set

$$a = -\mathrm{tr}A, \quad b = \mathrm{tr}\,\mathrm{com}\,A, \quad c = -\det A,$$

and then P takes the form

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0.$$

In order to get the roots of (2), i.e., all eigenvalues of A, with negative real parts, we apply the Routh-Hurwitz stability criterion. Hence,

$$a > 0, \quad c > 0, \quad \text{and} \quad ab - c > 0.$$

More precisely,

$$\mathrm{tr}A < 0, \tag{i}$$

$$\det A < 0, \tag{ii}$$

$$\det A - \mathrm{tr}A \cdot \mathrm{tr}\,\mathrm{com}\,A > 0. \tag{iii}$$

3 Explicit Conditions

For our main result, before we give the expression of the characteristic polynomial of the matrix $A\,(k)$, we first identify some tools. We set the submatrices

$$M_1 = \begin{pmatrix} a_{11} & a_{12} \\ d_{21} & d_{22} \end{pmatrix} \quad M_2 = \begin{pmatrix} a_{11} & a_{13} \\ d_{31} & d_{33} \end{pmatrix} \quad M_3 = \begin{pmatrix} a_{22} & a_{21} \\ d_{12} & d_{11} \end{pmatrix} \quad M_4 = \begin{pmatrix} a_{33} & a_{31} \\ d_{13} & d_{11} \end{pmatrix}$$

$$M_5 = \begin{pmatrix} a_{33} & a_{23} \\ d_{32} & d_{22} \end{pmatrix} \quad M_6 = \begin{pmatrix} a_{22} & a_{32} \\ d_{23} & d_{33} \end{pmatrix} \quad M_7 = \begin{pmatrix} a_{11} & d_{12} & d_{13} \\ a_{21} & d_{22} & d_{23} \\ a_{31} & d_{32} & d_{33} \end{pmatrix}$$

$$M_8 = \begin{pmatrix} d_{11} & a_{12} & a_{13} \\ d_{21} & a_{22} & a_{23} \\ d_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We also define two constants σ_1 and σ_2 as follows:

$$\begin{aligned}\sigma_1 = &-d_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) + d_{21}(a_{13}d_{32} + a_{32}d_{13} - a_{12}d_{33} - a_{33}d_{12}) \\ &- d_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}),\end{aligned}$$

$$\begin{aligned}\sigma_2 = &\, a_{11}(a_{23}d_{32} + a_{32}d_{23} - a_{22}d_{33} - a_{33}d_{22}) + a_{21}(a_{12}d_{33} + a_{33}d_{12} - a_{13}d_{32} - a_{32}d_{13}) \\ &+ a_{31}(a_{13}d_{22} + a_{22}d_{13} - a_{12}d_{23} - a_{23}d_{12}),\end{aligned}$$

then we have the following lemma.

Lemma 3.1 *The characteristic polynomial of $A(k)$ is*

$$\tilde{P}(\lambda) = -\lambda^3 + [\,\mathrm{tr}A - k^2\mathrm{tr}D\,]\,\lambda^2 + \Big[-\mathrm{tr}(\mathrm{com}D)\,k^4 + \Big(\sum_{i=1}^{6}\det M_i\Big)k^2 - \mathrm{tr}(\mathrm{com}A)\Big]\lambda$$
$$-\ (\det D)\,k^6 + (\sigma_1 + \det M_7)\,k^4 + (\sigma_2 - \det M_8)\,k^2 + \det A.$$

Proof. It suffices to apply the definition and use the above auxiliary tools.
We observe that $\tilde{P} = 0$ takes the form

$$\lambda^3 + \varphi_1\left(k^2\right)\lambda^2 + \varphi_2\left(k^2\right)\lambda + \varphi_3\left(k^2\right) = 0,$$

where

$$\begin{aligned}
\varphi_1\left(k^2\right) &= \mathrm{tr}(D)\,k^2 - \mathrm{tr}(A),\\
\varphi_2\left(k^2\right) &= \mathrm{tr}\big(\mathrm{com}(D)\big)\,\left(k^2\right)^2 - \left(\sum_{i=1}^{6}\det M_i\right)k^2 + \mathrm{tr}\big(\mathrm{com}(A)\big),\\
\varphi_3\left(k^2\right) &= (\det D)\left(k^2\right)^3 - (\sigma_1 + \det M_7)\left(k^2\right)^2 + (\det M_8 - \sigma_2)\,k^2 - \det A.
\end{aligned}$$

We notice that the coefficients φ_1, φ_2, and φ_3 are polynomial functions in k^2. Now, we apply the Routh-Hurwitz stability criterion, see F. R. Gantmacher [2],

$$\varphi_1\left(k^2\right) > 0 \quad \textbf{and} \quad \varphi_3\left(k^2\right) > 0 \quad \textbf{and} \quad \left(\varphi_1\varphi_2 - \varphi_3\right)\left(k^2\right) > 0.$$

This means that all eigenvalues of $A(k)$ have a negative real part. In order to achieve our goal "Diffusion Driven Instability", we violate at least one of the above conditions, so

$$\begin{gathered}
A(k) \text{ has at least one eigenvalue with positive real part}\\
\Leftrightarrow\\
\varphi_1\left(k^2\right) \le 0 \quad \textbf{or} \quad \varphi_3\left(k^2\right) \le 0 \quad \textbf{or} \quad \left(\varphi_1\varphi_2 - \varphi_3\right)\left(k^2\right) \le 0.
\end{gathered}$$

Remark 3.1 Since the functions we are dealing with, φ_1, φ_3, and $(\varphi_1\varphi_2 - \varphi_3)$, are even, we limit our study over $\mathbb{R}^+ = [0, +\infty)$.

At this point, we are now ready to establish the explicit conditions on k^2.

3.1 First condition

We have $\varphi_1\left(k^2\right) = \mathrm{tr}(D)\,k^2 - \mathrm{tr}(A)$, and since $\mathrm{tr}(D) > 0$, thanks to $d_{ii} > 0,\ i = 1, 2, 3$, and using the assumption $\mathrm{tr}(A) < 0$ (i), we get that $\forall\, k^2 \ge 0, \varphi_1\left(k^2\right) > 0$. This means we get nothing as a result.

Remark 3.2 The Routh-Hurwitz criterion implies that $\varphi_1\left(k^2\right) \le 0$, or $\varphi_3\left(k^2\right) \le 0$, or $(\varphi_1\varphi_2 - \varphi_3)\left(k^2\right) \le 0$, for some values of k^2, and because $\varphi_1\left(k^2\right) \le 0$ is impossible to to fulfil, the only option left is to deal with $\varphi_3\left(k^2\right) \le 0$ or $(\varphi_1\varphi_2 - \varphi_3)\left(k^2\right) \le 0$. Since we have the logical coordinator "**or",** it is much easier to handle $\varphi_3\left(k^2\right) \le 0$ instead of $(\varphi_1\varphi_2 - \varphi_3)\left(k^2\right) \le 0$. In other words, it suffices to focus on finding the values of k^2 such that $\varphi_3\left(k^2\right) \le 0$.

3.2 Second condition

The derivative of φ_3 with respect to k^2 is given by

$$\frac{\partial}{\partial k^2}\,\varphi_3\big(k^2\big) = 3\,\det D\,\big(k^2\big)^2 - 2\,\big(\sigma_1 + \det M_7\big)\,k^2 + \det M_8 - \sigma_2. \tag{3}$$

It appears that the derivative of φ_3 is a second degree polynomial in k^2. Set

$$\tilde{A} = 3\,\det D, \quad \tilde{B} = -2\,\big(\sigma_1 + \det M_7\big), \quad \tilde{C} = \det M_8 - \sigma_2.$$

The discriminant $\tilde{\Delta}$ of (3) is

$$\tilde{\Delta} = 4\Big[\,\big(\sigma_1 + \det M_7\big)^2 - 3\,\det D\,\big(\det M_8 - \sigma_2\big)\Big].$$

We denote by $\tilde{k}_i^2$ the roots of φ_3, and by k_i^2 the roots of $\dfrac{\partial}{\partial k^2}\,\varphi_3$. Following the signs of $\det D$ and $\tilde{\Delta}$, we give two propositions.

Proposition 3.1 *If the following conditions hold:*

$$\det D > 0, \quad \tilde{\Delta} > 0, \quad \tilde{B} < 0 \textit{ and } \tilde{C} > 0, \tag{4}$$

then $A\,(k)$ has at least one eigenvalue with positive real part. Furthermore,

$$\forall\, k^2 \in [\tilde{k}_1^2,\, \tilde{k}_2^2], \quad \varphi_3\,\big(k^2\big) \le 0.$$

Proof. Since $\tilde{\Delta} > 0$, there exist two distinct roots, obviously with $k_1^2 < k_2^2$:

$$k_1^2 = \frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\,\tilde{A}}, \quad k_2^2 = \frac{-\tilde{B} + \sqrt{\tilde{\Delta}}}{2\,\tilde{A}}.$$

We must ensure that both roots are positive; otherwise, the analysis cannot proceed. So, the following conditions are required:

$$\tilde{B} < 0 \quad \text{and} \quad \tilde{C} > 0, \tag{5}$$

equivalently,

$$-2\,(\sigma_1 + \det M_7) < 0, \quad \det M_8 - \sigma_2 > 0.$$

Under the hypotheses given by (4), the behavior of φ_3 is determined as follows.

- The value $\varphi_3\big(k_1^2\big)$ must be positive:

$$\big(\det D\big)\left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}}\right)^3 - \big(\sigma_1 + \det M_7\big)\left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}}\right)^2$$
$$+ \big(\det M_8 - \sigma_2\big)\left(\frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\tilde{A}}\right) - \det A > 0.$$

- Regarding $\varphi_3\big(k_2^2\big)$, there are two cases:

– If $\varphi_3\big(k_2^2\big) > 0$, then

$$\forall\, k^2 \geq 0, \quad \varphi_3\big(k^2\big) \geq 0.$$

– If $\varphi_3\big(k_2^2\big) < 0$, then there exist $\tilde{k}_1^2 \in [\,k_1^2,\, k_2^2]$ and $\tilde{k}_2^2 > k_2^2$ such that

$$\varphi_3\big(\tilde{k}_1^2\big) = \varphi_3\big(\tilde{k}_2^2\big) = 0.$$

Thus,

$$\forall\, k^2 \in [\,\tilde{k}_1^2,\, \tilde{k}_2^2], \quad \varphi_3\big(k^2\big) \leq 0.$$

Proposition 3.2 *If the following assumptions hold:*

$$\det D < 0 \quad \textit{and} \quad \tilde{\Delta} < 0, \tag{6}$$

then $A(k)$ has at least one eigenvalue with positive real part. Furthermore,

$$\forall\, k^2 \in [\,\tilde{k}_3^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Moreover, if

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3\big(k_3^2\big) > 0 \quad \textit{and} \quad \varphi_3\big(k_4^2\big) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall\, k^2 \in [\,\tilde{k}_4^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

If

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3\big(k_3^2\big) < 0 \quad \textit{and} \quad \varphi_3\big(k_4^2\big) < 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall\, k^2 \in [\,\tilde{k}_5^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Finally, if

$$\det D < 0, \quad \tilde{\Delta} > 0, \quad \varphi_3\big(k_3^2\big) < 0 \quad \textit{and} \quad \varphi_3\big(k_4^2\big) > 0,$$

then $A(k)$ has at least one eigenvalue with positive real part, and

$$\forall\, k^2 \in [\,\tilde{k}_6^2,\, \tilde{k}_7^2] \cup [\,\tilde{k}_8^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Proof. We begin with the trivial case: if $\tilde{\Delta} < 0$, then there are no real roots of (3). The behavior of φ_3 under the hypotheses given by (6), ensures the existence of some $\tilde{k}_3^2 > 0$ such that $\varphi_3(\tilde{k}_3^2) = 0$. Hence,

$$\forall\, k^2 \in [\,\tilde{k}_3^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

Now, consider the case $\tilde{\Delta} > 0$. Then there exist two distinct real roots of (3), given by

$$k_3^2 = \frac{-\tilde{B} + \sqrt{\tilde{\Delta}}}{2\,\tilde{A}}, \quad k_4^2 = \frac{-\tilde{B} - \sqrt{\tilde{\Delta}}}{2\,\tilde{A}}, \quad \text{with} \quad k_3^2 < k_4^2.$$

Again, the roots must be positive and we require

$$\tilde{B} > 0 \quad \text{and} \quad \tilde{C} < 0. \tag{7}$$

Depending on the sign of $\varphi_3(k_3^2)$ and $\varphi_3(k_4^2)$, we have the following cases:
- If $\varphi_3(k_3^2) > 0$ and $\varphi_3(k_4^2) > 0$, then there exists $\tilde{k}_4^2 > k_4^2$ such that $\varphi_3(\tilde{k}_4^2) = 0$. Thus,

$$\forall\, k^2 \in [\,\tilde{k}_4^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

- If $\varphi_3(k_3^2) < 0$ and $\varphi_3(k_4^2) < 0$, then there exists $\tilde{k}_5^2 < k_3^2$ such that $\varphi_3(\tilde{k}_5^2) = 0$. Therefore,

$$\forall\, k^2 \in [\,\tilde{k}_5^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

- If $\varphi_3(k_3^2) < 0$ and $\varphi_3(k_4^2) > 0$, then there exist $\tilde{k}_6^2 \in (0, k_3^2)$, $\tilde{k}_7^2 \in (k_3^2, k_4^2)$, and $\tilde{k}_8^2 > k_4^2$ such that

$$\varphi_3(\tilde{k}_6^2) = \varphi_3(\tilde{k}_7^2) = \varphi_3(\tilde{k}_8^2) = 0.$$

In this case, we get

$$\forall\, k^2 \in [\,\tilde{k}_6^2, \tilde{k}_7^2] \cup [\,\tilde{k}_8^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

In addition, we discuss the sub-cases where $\det D = 0$. In this case, the function φ_3 takes the form

$$\varphi_3(k^2) = -(\sigma_1 + \det M_7)\,(k^2)^2 + (\det M_8 - \sigma_2)\,k^2 - \det A.$$

Suppose that $\sigma_1 + \det M_7 \neq 0$. Then φ_3 can be rewritten as

$$\varphi_3(k^2) = \tilde{A}_1\,(k^2)^2 + \tilde{B}_1\,k^2 + \tilde{C}_1, \tag{8}$$

where

$$\tilde{A}_1 = -(\sigma_1 + \det M_7), \quad \tilde{B}_1 = \det M_8 - \sigma_2, \quad \tilde{C}_1 = -\det A.$$

The discriminant of (8) is

$$\tilde{\Delta}_1 = \tilde{B}_1^2 - 4\,\tilde{A}_1\,\tilde{C}_1.$$

If $\tilde{\Delta}_1 > 0$ and $\tilde{A}_1 > 0$ (equivalently, $\sigma_1 + \det M_7 < 0$), then there exist two real roots

$$\tilde{k}_9^2 = \frac{-\tilde{B}_1 - \sqrt{\tilde{\Delta}_1}}{2\,\tilde{A}_1}, \quad \tilde{k}_{10}^2 = \frac{-\tilde{B}_1 + \sqrt{\tilde{\Delta}_1}}{2\,\tilde{A}_1}, \quad \text{with} \quad \tilde{k}_9^2 < \tilde{k}_{10}^2.$$

To ensure that both roots are positive, we require

$$\tilde{B}_1 < 0 \quad \text{and} \quad \tilde{C}_1 > 0. \tag{9}$$

That is,

$$\det M_8 - \sigma_2 < 0 \quad \text{and} \quad -\det A > 0.$$

In this case, we have

$$\forall\, k^2 \in [\,\tilde{k}_9^2, \tilde{k}_{10}^2], \quad \varphi_3(k^2) \leq 0.$$

Furthermore, if $\sigma_1 + \det M_7 = 0$, then φ_3 reduces to

$$\varphi_3(k^2) = (\det M_8 - \sigma_2)\,k^2 - \det A.$$

If $\det M_8 - \sigma_2 < 0$, then there exists $\tilde{k}_{11}^2 > 0$ such that $\varphi_3(\tilde{k}_{11}^2) = 0$. Hence,

$$\forall\, k^2 \in [\,\tilde{k}_{11}^2, +\infty), \quad \varphi_3(k^2) \leq 0.$$

4 Application

Consider the following reaction-diffusion system:

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_u \Delta u = \alpha u - \beta uv + \delta v & \text{in } Q_t, \\ \dfrac{\partial v}{\partial t} - d_v \Delta v - d_{vw} \Delta w = \gamma uv - \mu v + \eta w & \text{in } Q_t, \\ \dfrac{\partial w}{\partial t} - d_w \Delta w = \tau w - \xi uw & \text{in } Q_t, \end{cases} \tag{10}$$

where $Q_t = \{t > 0\} \times \Omega$, Ω is an open bounded subset of $\mathbf{R}^n$.

This ecological system models the interactions of predator-prey dynamics, and the solutions $u(x,t), v(x,t)$ and $w(x,t)$ represent the concentrations of three interacting biological species. Now, we give the biological interpretation of constants.

Constant	Signification
α	Growth rate of species u
β	Interaction strength between u and v
γ	Production rate of v via interaction with u
μ	Natural decay rate of v
δ	Effect of v on u
η	Coupling term from w to v
τ	Growth rate of species w
ξ	Interaction term between v and w
d_{vw}	Cross-diffusion coefficient

Equating the left-hand side of (10) to 0, we get the equilibrium points:

- $E_0(0,0,0)$ exists always;
- $E_m\Big(\dfrac{\mu}{\gamma}, \dfrac{\alpha\mu}{\beta\mu - \delta\gamma}, 0\Big)$ exists under the conditions $\gamma \neq 0$ and $\beta\mu - \delta\gamma \neq 0$,
- $E_*\Big(\dfrac{\tau}{\xi}, \dfrac{\alpha\tau}{\beta\tau - \delta\xi}, \dfrac{\alpha\tau(\mu\xi - \gamma\tau)}{\eta\xi(\beta\tau - \delta\xi)}\Big)$ exists under the conditions $\xi \neq 0$, $\beta\tau - \delta\xi \neq 0$, and $\eta\xi(\beta\tau - \delta\xi) \neq 0$.

According to (P), the diffusion coefficients must satisfy

$$d_u > 0, \quad d_w > 0, \quad 4d_v d_w - d_{vw}^2 > 0,$$

so can they reflect the parabolicity of the system.

4.1 Turing instability

To obtain the Turing instability, we need find the values of k^2, for which the matrix $A(k^2) = A - k^2 D$ can be unstable; in other words, to define conditions on k^2 under which that the matrix $A(k)$ has at least one eigenvalue with positive real part. The discriminant D of the matrix is equal to $d_u d_v d_w > 0$, we apply only Proposition 3.1..

The Jacobian matrix of linearization in the neighborhood of the first equilibrium point E_0 is given by

$$A_0 = \begin{pmatrix} \alpha & \delta & 0 \\ 0 & -\mu & \eta \\ 0 & 0 & \tau \end{pmatrix}.$$

Note that this matrix is stable under the conditions (i), (ii) and (iii), respectively, hence

$$\begin{array}{l} \alpha - \mu + \tau < 0, \\ -\alpha\mu\tau < 0, \\ \alpha\mu\tau + \big(\alpha - \mu + \tau\big)\big(\alpha\tau - \mu\tau - \alpha\mu\big) < 0. \end{array}$$

To obtain the desired result, as we have mentioned earlier, the matrix $A_0\big(k^2\big) = A_0 - k^2 D$ must have an eigenvalue with positive real part.

By Proposition 3.1, if

$$\begin{array}{l} \tilde{A}_0 = 3d_u d_v d_w > 0, \\ \tilde{B}_0 = 2\mu d_u d_w - 2\tau d_u d_v - 2\alpha d_v d_w < 0, \\ \tilde{C}_0 = \alpha\tau d_v - \mu\tau d_u - \alpha\mu d_w > 0, \\ \tilde{\Delta}_0 = \tilde{B}_0^2 - 4\tilde{A}_0\tilde{C}_0 > 0, \\ \varphi_3\Big(\frac{-\tilde{B}_0 - \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}\Big) > 0 \text{ and } \varphi_3\Big(\frac{-\tilde{B}_0 + \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}\Big) < 0, \end{array}$$

it results in

$$\forall\, k^2 \in \left[\frac{-\tilde{B}_0 - \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}, \frac{-\tilde{B}_0 + \sqrt{\tilde{\Delta}_0}}{2\tilde{A}_0}\right] \quad \varphi_3\big(k^2\big) \leq 0,$$

where

$$\begin{array}{rl} \varphi_3(k^2) = & \big(\det D\big)(k^2)^3 - \big(\sigma_1 + \det M_7\big)(k^2)^2 \\ & +\big(\det M_8 - \sigma_2\big)k^2 - \det A_0, \end{array}$$

and we get the Turing instability Move now to the second equilibrium point E_m, its associated matrix of linearization is given by

$$A_m = \begin{pmatrix} \alpha\Big(1 - \frac{\beta\mu}{\beta\mu - \gamma\delta}\Big) & \delta - \frac{\beta\mu}{\gamma} & 0 \\ \frac{\alpha\gamma\mu}{\beta\mu - \gamma\delta} & 0 & \eta \\ 0 & 0 & \tau - \frac{\xi\mu}{\gamma} \end{pmatrix}.$$

A_m is stable under the following conditions:

$$\begin{array}{l} \tau - \frac{\xi\mu}{\gamma} - \frac{\alpha\gamma\delta}{\beta\mu - \gamma\delta} < 0, \\ \frac{\alpha\mu}{\delta}\big(\xi\mu - \gamma\tau\big) < 0, \\ \frac{\alpha\mu}{\delta}\big(\xi\mu - \gamma\tau\big) - \Big(\tau - \frac{\xi\mu}{\gamma} - \frac{\alpha\gamma\delta}{\beta\mu - \gamma\delta}\Big)\Big(\alpha + \alpha\frac{\xi\mu - \gamma\tau}{\beta\mu - \gamma\delta}\Big) > 0. \end{array}$$

Following the same reasoning, we apply Proposition 3.1 under the following assumptions:

$$\begin{aligned}&\tilde{A}_m = 3d_u d_v d_w > 0,\\ &\tilde{B}_m = 2\tfrac{\xi\mu-\gamma\tau}{\gamma} d_u d_v + 2\tfrac{\alpha\gamma\delta}{\beta\mu-\gamma\delta} d_v d_w < 0,\\ &\tilde{C}_m = \alpha\mu d_w - \tfrac{\alpha(\gamma\tau-\xi\mu)}{\beta\mu-\gamma\delta} d_v > 0,\\ &\tilde{\Delta}_m = \tilde{B}_m^2 - 4\tilde{A}_m\tilde{C}_m > 0,\\ &\varphi_3\Big(\tfrac{-\tilde{B}_m-\sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}\Big) > 0 \quad \text{and} \quad \varphi_3\Big(\tfrac{-\tilde{B}_m+\sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}\Big) < 0.\end{aligned}$$

It results in

$$\forall\, k^2 \in \left[\frac{-\tilde{B}_m - \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}, \frac{-\tilde{B}_m + \sqrt{\tilde{\Delta}_m}}{2\tilde{A}_m}\right] \quad \varphi_3\big(k^2\big) \le 0,$$

where

$$\begin{aligned}\varphi_3(k^2) = \ & \big(\det D\big)(k^2)^3 - \big(\sigma_1 + \det M_7\big)(k^2)^2\\ & + \big(\det M_8 - \sigma_2\big)k^2 - \det A_m,\end{aligned}$$

and we get the Turing instability. Finally, the matrix

$$A_* = \begin{pmatrix} \alpha\Big(1 - \frac{\beta\tau}{\beta\tau-\delta\xi}\Big) & \delta - \frac{\beta\tau}{\xi} & 0\\ \frac{\alpha\gamma\tau}{\beta\tau-\delta\xi} & \frac{\gamma\tau}{\xi} - \mu & \eta\\ \frac{\alpha\tau(\gamma\tau-\mu\xi)}{\eta(\beta\tau-\delta\xi)} & 0 & 0 \end{pmatrix}$$

is the Jacobian matrix of linearization around the final equilibrium point E_*, with the following conditions of stability:

$$\begin{aligned}&\frac{\gamma\tau}{\xi} - \mu - \frac{\alpha\delta\xi}{\beta\tau-\delta\xi} < 0,\\ &\frac{\alpha\tau}{\xi}(\mu\xi - \gamma\tau) < 0,\\ &\frac{\alpha\tau}{\xi}(\mu\xi - \gamma\tau) - \alpha\Big(\frac{\gamma\tau}{\xi} - \mu - \frac{\alpha\delta\xi}{\beta\tau-\delta\xi}\Big)\Big(\frac{\gamma\tau}{\xi} - \delta\cdot\frac{\gamma\tau-\mu\xi}{\beta\tau-\delta\xi}\Big) > 0.\end{aligned}$$

We apply Proposition 3.1 under the following assumptions:

$$\begin{aligned}&\tilde{A}_* = 3d_u d_v d_w > 0,\\ &\tilde{B}_* = 2\tfrac{\mu\xi-\gamma\tau}{\xi} d_u d_w + 2\tfrac{\alpha\delta\xi}{\beta\tau-\delta\xi} d_v d_w < 0,\\ &\tilde{C}_* = \Big(\tfrac{\alpha\delta(\mu\xi-\gamma\tau)}{\beta\tau-\delta\xi} - \tfrac{\gamma\tau}{\xi}\Big) d_w + \tfrac{\alpha\tau(\gamma\tau-\mu\xi)}{\xi} d_{vw} > 0,\\ &\tilde{\Delta}_* = \tilde{B}_*^2 - 4\tilde{A}_*\tilde{C}_* > 0,\\ &\varphi_3\Big(\tfrac{-\tilde{B}_*-\sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}\Big) > 0 \quad \text{and} \quad \varphi_3\Big(\tfrac{-\tilde{B}_*+\sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}\Big) < 0,\end{aligned}$$

this leads to

$$\forall\, k^2 \in \left[\frac{-\tilde{B}_* - \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}, \frac{-\tilde{B}_* + \sqrt{\tilde{\Delta}_*}}{2\tilde{A}_*}\right] \quad \varphi_3(k^2) \le 0,$$

where

$$\varphi_3(k^2) = \big(\det D\big)(k^2)^3 - \big(\sigma_1 + \det M_7\big)(k^2)^2 + \big(\det M_8 - \sigma_2\big)k^2 - \det A_*.$$

and we get the Turing instability.

5 Final Remarks

Corollary 5.1 *For $k = 0$, we recover the characteristic polynomial of the matrix A given by* (2).

Remark 5.1 We strongly recommend that the roots k_i^2 of $\frac{\partial}{\partial k^2}\varphi_3$ are positive. Otherwise, the analysis cannot proceed. We emphasize that conditions (5), (7) and (9) are necessary and sufficient.

Remark 5.2 The conditions

$$\text{sign}\left(\varphi_3(k_i^2)\right) = -\ \text{sign}\left(\varphi_3(k_{i+1}^2)\right)$$

for all $i \geq 1$ ensure the existence of $\tilde{k}_i^2$ for all $i \geq 1$.

Remark 5.3 The following cases do not ensure the existence of an eigenvalue with positive real part and lead to a contradiction.

- **Regarding φ_3:** If $\det D > 0$ and $\tilde{\Delta} < 0$, then $\varphi_3(k^2) > 0$ for all $k^2 \geq 0$, so no instability occurs.
- **Special case for φ_3 when it is reduced to**

$$\varphi_3 = -(\sigma_1 + \det M_7)(k^2)^2 + (\det M_8 - \sigma_2)k^2 - \det A.$$

If $\tilde{\Delta}_1 < 0$ and $(\sigma_1 + \det M_7) < 0$, then $\varphi_3(k^2) > 0$ for all $k^2 \geq 0$. If ($\tilde{\Delta}_1 < 0$ or $\tilde{\Delta}_1 > 0$) and $(\sigma_1 + \det M_7) > 0$, this contradicts hypothesis (ii).

6 Conclusion

This paper presents a detailed overview of a three-species reaction-diffusion system incorporating normal diffusion, cross-diffusion, and nonlinear interspecies interactions. It elucidates the role and significance of each model parameter, including growth rates, interaction strengths, decay rates and cross-diffusion coefficients. The system exhibits a wide range of complex dynamics such as the Turing pattern formation, oscillations, spatiotemporal chaos, and traveling wave phenomena. Such models are fundamental for advancing the understanding of chemical reactions, biological population dynamics, and natural pattern formation processes. The deep connection between reaction-diffusion systems and non-linear dynamical systems is highlighted by these phenomena, the equilibrium and stability of the system are determined by its local reaction kinetics, the driving force behind symmetry breaking and pattern selection is diffusion and crossdiffusion terms. The analysis of the Turing instability in this context reveals the process of small perturbations growing and evolving into rich spatiotemporal structures that bridge local dynamics and large-scale spatial organization.

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