Nonlinear Dynamics and Systems Theory, **25** (6) (2025) 669–678

A Numerical Approach for Solving Delay Volterra Integral Equations with a Spatial Variable and Mixed Kernels

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Received: January 6, 2025; Revised: October 25, 2025

Abstract: We propose a numerical method based on Taylor polynomials to construct a collocation solution for approximating the solution of delay Volterra integral equations (DVIEs) with a spatial variable. The method effectively handles both time delay and spatial dependence, which are essential in modeling nonlinear dynamic systems. A rigorous convergence analysis establishes that the method is accurate and stable, with an $O((h+k)^p)$ error bound. Numerical experiments confirm its efficiency and demonstrate its applicability to nonlinear dynamical problems governed by delay integral equations. The proposed approach provides a reliable and computationally efficient tool for solving DVIEs arising in nonlinear dynamics, setting a foundation for further extensions to higher-dimensional problems.

Keywords: *delay Volterra integral equation with spatial variable; collocation method; Taylor polynomials.*

Mathematics Subject Classification (2020): 45D05, 65R20, 70K99, 93C15.

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1 Introduction

In this paper, we analyze a numerical method for solving delay Volterra integral equations (DVIEs) with a spatial variable and a time delay $\tau > 0$, given by

$$u(t,x) = \begin{cases} g(t,x) + A(t,x)u(t-\tau,x) + \int_0^t K_1(t,x,s)u(s,x)\,ds \\ \quad + \int_0^{t-\tau} K_2(t,x,s)u(s,x)\,ds, & t \in [0,T],\ x \in [0,X], \\ \Phi(t,x), & t \in [-\tau,0], \end{cases} \tag{1}$$

where the functions g, Φ, A, K_1, and K_2 are given smooth, real-valued functions defined on $D := [0,T] \times [0,X] \subset \mathbb{R}^2$ and $S := \{(t,x,s) : 0 \le s \le t \le T,\ 0 \le x \le X\}$.

According to the classical theory of Volterra integral equations (VIEs), equation (1) admits a unique solution $u \in C(D)$ [6].

Delay Volterra integral equations (DVIEs) play a crucial role in modeling nonlinear dynamical systems where memory effects and delays significantly influence system behavior. Such systems arise in various disciplines, including population dynamics, epidemic models, viscoelastic materials, control systems, and financial modeling, where the future state of the system depends not only on its current state, but also on its past states. These delay effects often introduce nonlinearity and complexity, making analytical solutions difficult or impossible to obtain. Therefore, efficient and accurate numerical methods are essential for their analysis.

Several researchers have studied the numerical solutions of one-dimensional Volterra integral equations with time delays (see, for example, [1, 3, 5, 11, 12]). Ali et al. [1] proposed a spectral method for pantograph-type delay integral equations using the Legendre collocation method. Bellour and Bousselsal [3] constructed a collocation solution to approximate the solution of delay integral equations. Brunner and Yatsenko [11] investigated polynomial spline collocation methods for systems of VIEs with unknown delays. Zheng and Chen [12] developed spectral methods using Chebyshev bases and spectral collocation techniques to analyze VIEs with two types of delays.

While numerous studies have focused on two-dimensional Volterra integral equations [4, 7, 9, 10], fewer works have addressed delay Volterra integral equations with a spatial variable [13]. The primary challenge in solving such problems lies in handling both spatial dependence and time delay in the upper integral limit, distinguishing these equations from conventional two-dimensional Volterra integral equations.

To bridge this gap, we propose a Taylor collocation method, extending the approach in [3] to approximate the solution of equation (1). The main novelty of our work lies in:

1. The adaptation of the Taylor Collocation Method to DVIEs with spatial dependence, which is a relatively unexplored area in numerical analysis.

2. A computationally efficient framework, where the approximation coefficients are computed iteratively without requiring the solution of a system of algebraic equations.

3. A general numerical approach that can be extended to higher-dimensional delay integral equations encountered in physics, control theory, and engineering applications.

The paper is organized as follows. In Section 2, we partition the interval $[0,T] \times [0,X]$ into subintervals and approximate the solution of (1) in each subinterval using Taylor polynomials. Section 3 establishes the global convergence of the method. Section 4 presents numerical examples validating the method. Finally, the conclusion summarizes the main findings and potential future extensions.

2 Description of the Method

We assume, without loss of generality, that $T = r\tau$, where $r \in \mathbf{N}^*$. Let $\Pi_N^\epsilon = \{t_n^\epsilon = \epsilon\tau + nh,\ n = 0,1,...,N,\ \epsilon = 0,1,...,r\}$ and $\Pi_M = \{x_m = mk,\ m = 0,1,...,M\}$ denote uniform partitions of the intervals $[0,T]$ and $[0,X]$, respectively, with step sizes given by $h = \frac{\tau}{N}$ and $k = \frac{X}{M}$. These partitions define a grid for D:

$$\Pi_{N,M}^\epsilon = \Pi_N^\epsilon \times \Pi_M = \{(t_n^\epsilon, x_m) \mid 0 \le n \le N,\ 0 \le m \le M,\ \epsilon = 0,1,\dots,r\}.$$

Define the subintervals as follows:

$$\sigma_n^\epsilon = [t_n^\epsilon, t_{n+1}^\epsilon), \quad \sigma_{N-1}^\epsilon = [t_{N-1}^\epsilon, t_N^\epsilon], \quad n = 0,1,...,N-2, \quad \epsilon = 0,1,...,r-1.$$

$$\delta_m = [x_m, x_{m+1}), \quad m = 0,1,...,M-2, \quad \delta_{M-1} = [x_{M-1}, x_M].$$

The rectangular subdomains are given by

$$D_{n,m}^\epsilon := \sigma_n^\epsilon \times \delta_m, \quad \forall n = 0,1,...,N-1, \quad m = 0,1,...,M-1, \quad \epsilon = 0,1,...,r-1.$$

Moreover, let π_{p-1} denote the set of all real polynomials of degree at most $p-1$ in t and x. We define the real polynomial spline space of degree $p-1$ in t and x as follows:

$$S_{p-1}^{(-1)}(\Pi_{N,M}^\epsilon) = \{v \in D : v_{n,m}^\epsilon = v|_{D_{n,m}^\epsilon} \in \pi_{p-1}, \\ n = 0,...,N-1; \quad m = 0,1,...,M-1; \quad \epsilon = 0,1,...,r-1\}.$$

This represents the space of bivariate polynomial spline functions of degree at most $p-1$ in t and x. Its dimension is $rNMp^2$, which equals the total number of coefficients in the polynomials $v_{n,m}^\epsilon$, where $n = 0,...,N-1$, $m = 0,1,...,M-1$, and $\epsilon = 0,1,...,r-1$. To determine these coefficients, we apply the Taylor polynomial on each rectangle.

First, we approximate u in the rectangles $D_{0,m}^0$, where $m = 0,...,M-1$, using the polynomial

$$v_{0,m}^0(t,x) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \partial_t^{(i)} \partial_x^{(j)} u(0,x_m) t^i (x-x_m)^j. \tag{2}$$

Here, $\partial_t^{(i)}\partial_x^{(j)}u(0,x_m)$ represents the exact value of $\partial_t^{(i)}\partial_x^{(j)}u$ at the point $(0,x_m)$.

To determine $\partial_t^{(i)}\partial_x^{(j)}u(t,x)$, we differentiate equation (1) j times with respect to x and i times with respect to t, obtaining

$$\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}u(t,x) &= \partial_t^{(i)}\partial_x^{(j)}g(t,x) + \partial_t^{(i)}\partial_x^{(j)}\left(A(t,x)\Phi(t-\tau,x)\right) \\
&+ \sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q} \binom{j}{l}\binom{q}{\eta} \partial_t^{(q-\eta)} \left(\partial_t^{(i-1-q)}\partial_x^{(j-l)}K_1(t,x,s)\Big|_{s=t}\right) \partial_t^{(\eta)}\partial_x^{(l)}u(t,x) \\
&+ \int_0^t \sum_{l=0}^{j} \binom{j}{l} \partial_t^{(i)}\partial_x^{(j-l)}K_1(t,x,s)\, \partial_x^{(l)}u(s,x)ds \\
&- \partial_t^{(i)}\partial_x^{(j)}\left(\int_{t-\tau}^0 K_2(t,x,s)\Phi(s,x)\, ds\right).
\end{aligned}$$

Second, we approximate u in the rectangles $D^0_{n,m}$, where $n = 1, ..., N-1$ and $m = 0, ..., M-1$, using the polynomial

$$v^0_{n,m}(t,x) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \partial_t^{(i)} \partial_x^{(j)} \hat{v}^0_{n,m}(t^0_n, x_m)(t - t^0_n)^i (x - x_m)^j. \quad (3)$$

Here, $\hat{v}^0_{n,m}(t,x)$ is the exact solution of the integral equation

$$\hat{v}^0_{n,m}(t,x) = g(t,x) + A(t,x)\Phi(t-\tau,x) + \sum_{\eta=0}^{n-1} \int_{t^0_\eta}^{t^0_{\eta+1}} K_1(t,x,s) v^0_{\eta,m}(s,x) ds$$
$$+ \int_{t^0_n}^{t} K_1(t,x,s) \hat{v}^0_{n,m}(s,x) ds - \int_{t-\tau}^{0} K_2(t,x,s)\Phi(s,x) ds. \quad (4)$$

To compute $\partial_t^{(i)} \partial_x^{(j)} \hat{v}^0_{n,m}(t,x)$, we differentiate equation (4) j times with respect to x and i times with respect to t, obtaining

$$\partial_t^{(i)} \partial_x^{(j)} \hat{v}^0_{n,m}(t,x) = \partial_t^{(i)} \partial_x^{(j)} g(t,x) + \partial_t^{(i)} \partial_x^{(j)} \left(A(t,x)\Phi(t-\tau,x)\right)$$
$$+ \sum_{\eta=0}^{n-1} \int_{t^0_\eta}^{t^0_{\eta+1}} \sum_{l=0}^{j} \binom{j}{l} \partial_t^{(i)} \left(\partial_x^{(j-l)} K_1(t,x,s)\right) \partial_x^{(l)} v^0_{\eta,m}(s,x) ds$$
$$+ \sum_{q=0}^{i-1} \sum_{l=0}^{j} \sum_{\eta=0}^{q} \binom{j}{l} \binom{q}{\eta} \partial_t^{(q-\eta)} \left(\partial_t^{(i-1-q)} \partial_x^{(j-l)} K_1(t,x,s)\right)\Big|_{s=t} \partial_t^{(\eta)} \partial_x^{(l)} \hat{v}^0_{n,m}(t,x)$$
$$+ \int_{t^0_n}^{t} \sum_{l=0}^{j} \binom{j}{l} \partial_t^{(i)} \left(\partial_x^{(j-l)} K_1(t,x,s)\right) \partial_x^{(l)} \hat{v}^0_{n,m}(s,x) ds$$
$$- \partial_t^{(i)} \partial_x^{(j)} \left(\int_{t-\tau}^{0} K_2(t,x,s)\Phi(s,x)\, ds\right).$$

Finally, we approximate u in the rectangles $D^\epsilon_{n,m}$, where $n = 0, ..., N-1$, $\epsilon = 1, ..., r$, and $m = 0, 1, ..., M-1$, using the polynomial

$$v^\epsilon_{n,m}(t,x) = \sum_{j+i=0}^{p-1} \frac{1}{i!j!} \partial_t^{(i)} \partial_x^{(j)} \hat{v}^\epsilon_{n,m}(t^\epsilon_n, x_m)(t - t^\epsilon_n)^i (x - x_m)^j. \quad (5)$$

Here, $\hat{v}^\epsilon_{n,m}(t,x)$ is the exact solution of the integral equation

$$\hat{v}^\epsilon_{n,m}(t,x) = g(t,x) + A(t,x) v^{\epsilon-1}_{n,m}(t-\tau,x) + \sum_{e=0}^{\epsilon-1} \sum_{\eta=0}^{N-1} \int_{t^e_\eta}^{t^e_{\eta+1}} K_1(t,x,s) v^e_{\eta,m}(s,x) ds$$
$$+ \sum_{\eta=0}^{n-1} \int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}} K_1(t,x,s) v^\epsilon_{\eta,m}(s,x) ds + \int_{t^\epsilon_n}^{t} K_1(t,x,s) \hat{v}^\epsilon_{n,m}(s,x) ds$$
$$+ \sum_{e=0}^{\epsilon-2} \sum_{\eta=0}^{N-1} \int_{t^e_\eta}^{t^e_{\eta+1}} K_2(t,x,s) v^e_{\eta,m}(s,x) ds + \sum_{\eta=0}^{n-1} \int_{t^{\epsilon-1}_\eta}^{t^{\epsilon-1}_{\eta+1}} K_2(t,x,s) v^{\epsilon-1}_{\eta,m}(s,x) ds$$
$$+ \int_{t^{\epsilon-1}_n}^{t-\tau} K_2(t,x,s) v^{\epsilon-1}_{n,m}(s,x) ds. \quad (6)$$

To compute $\partial_t^{(i)}\partial_x^{(j)}\hat{v}^{\epsilon}_{n,m}(t,x)$, we differentiate equation (6) j times with respect to x and i times with respect to t, obtaining

$$
\begin{aligned}
\partial_t^{(i)}\partial_x^{(j)}\hat{v}^{\epsilon}_{n,m}(t,x) &= \partial_t^{(i)}\partial_x^{(j)}g(t,x)\\
&+\sum_{l=0}^{j}\sum_{\eta=0}^{i}\binom{j}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left(\partial_x^{(j-l)}A(t,x)\right)\partial_t^{(\eta)}\partial_x^{(l)}v^{\epsilon-1}_{n,m}(t-\tau,x)\\
&+\sum_{e=0}^{\epsilon-1}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}v^{e}_{\eta,m}(s,x)ds\\
&+\sum_{\eta=0}^{n-1}\int_{t^\epsilon_\eta}^{t^\epsilon_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}v^{\epsilon}_{\eta,m}(s,x)\,ds\\
&+\sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\right]\partial_t^{(\eta)}\partial_x^{(l)}\hat{v}^{\epsilon}_{n,m}(t,x)\\
&+\int_{t^\epsilon_n}^{t}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_1(t,x,s)\right)\partial_x^{(l)}\hat{v}^{\epsilon}_{n,m}(s,x)ds\\
&+\sum_{e=0}^{\epsilon-2}\sum_{\eta=0}^{N-1}\int_{t^e_\eta}^{t^e_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\partial_x^{(l)}v^{e}_{\eta,m}(s,x)ds\\
&+\sum_{\eta=0}^{n-1}\int_{t^{\epsilon-1}_\eta}^{t^{\epsilon-1}_{\eta+1}}\sum_{l=0}^{j}\binom{j}{l}\partial_t^{(i)}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\partial_x^{(l)}v^{\epsilon-1}_{\eta,m}(s,x)\,ds\\
&+\sum_{q=0}^{i-1}\sum_{l=0}^{j}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{s=t-\tau}\left(\partial_x^{(j-l)}K_2(t,x,s)\right)\right]\\
&\times\partial_t^{(\eta)}\partial_x^{(l)}v^{\epsilon-1}_{n,m}(t-\tau,x).
\end{aligned}
$$

3 Convergence Analysis

The following lemmas and theorems will be used in this section.

Theorem 3.1 (Taylor's Theorem for functions of two independent variables [8]) Let f be p times continuously differentiable on $D=[a,b]\times[c,d]$, and let $(x_0,y_0)\in D$. Then, for all $(x,y)\in D$, we have

$$
\begin{aligned}
f(x,y) &= \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_x^{(j)}f(x_0,y_0)(x-x_0)^i(y-y_0)^j\\
&\quad+\sum_{i+j=p}\frac{1}{i!j!}\partial_t^{(i)}\partial_x^{(j)}f(x_1,y_1)(x-x_0)^i(y-y_0)^j,
\end{aligned}
$$

where

$$
\begin{cases}
x_1=\theta x+(1-\theta)x_0, & x_1\in[a,b],\\
y_1=\theta y+(1-\theta)y_0, & y_1\in[c,d],
\end{cases}\qquad \theta\in(0,1).
$$

Lemma 3.1 (Discrete Gronwall-type inequality [6]) Let $\{k_j\}_{j=0}^{n}$ be a given non-negative sequence, and suppose that the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq p_0$ and $\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i$, for $n \geq 1$, with $p_0 \geq 0$. Then ε_n is bounded by

$$\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Lemma 3.2 (Bellman's inequality [2]) Let h and f be continuous and non-negative functions defined on $J = [\alpha, \beta]$, and let c be a non-negative constant. Then the inequality

$$h(t) \leq c + \int_{\alpha}^{t} f(s)h(s)ds, \quad t \in J,$$

implies that

$$h(t) \leq c \exp\left(\int_{\alpha}^{t} f(s)ds\right), \quad t \in J.$$

To establish the convergence of the proposed method, we require the following lemma. We work within the space $L^{\infty}(D)$ equipped with the norm

$$\|\varphi\|_{L^{\infty}(D)} = \inf\{C \in \mathbb{R} : |\varphi(t,x)| \leq C \quad \text{for a.e. } (t,x) \in D\} < \infty.$$

Lemma 3.3 Let g, Φ, A, K_1, and K_2 be p-times continuously differentiable on their respective domains. Then there exists a positive constant $\alpha(p)$ such that for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$, $\epsilon = 0, \ldots, r-1$, and $i + j = 0, \ldots, p$, we have

$$\left\| \frac{\partial^{i+j} \hat{v}_{n,m}^{\epsilon}}{\partial t^i \partial x^j} \right\|_{L^{\infty}(D_{n,m}^{\epsilon})} \leq \alpha(p).$$

Proof. The lemma can be proved by directly generalizing the procedures used in Claim 1 of Theorem 3.3 in [3]. The following theorem establishes the convergence of the proposed method.

Theorem 3.2 Suppose that g, Φ, A, K_1, and K_2 are p-times continuously differentiable on their respective domains. Then the equations $(2), (3), (5)$ uniquely define an approximation $v \in S_{p-1}^{(-1)}(\Pi_{N,M}^{\epsilon})$. Furthermore, the associated error function $E = u - v$ satisfies the following bound:

$$\|E\|_{L^{\infty}(D)} \leq C(h+k)^p,$$

where C is a finite constant independent of h and k.

Proof. Define the error function E on $D_{n,m}^{\epsilon}$ as

$$E_{n,m}^{\epsilon} = u - v_{n,m}^{\epsilon}, \quad \forall n \in \{0, \ldots, N-1\},\ m \in \{0, \ldots, M-1\},\ \epsilon \in \{0, \ldots, r-1\}.$$

Throughout this proof, the norm $\|\cdot\|$ refers to $\|\cdot\|_{L^{\infty}(D)}$.
Let $(t,x) \in D_{n,m}^{\epsilon}$. For $\epsilon = 0$, using Theorem 3.1 and equation (2), we write

$$|E_{0,m}^{0}(t,x)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \partial_t^{(i)} \partial_x^{(j)} u(0, x_m) \right\| h^i k^j.$$

According to Lemma 3.3, this implies

$$|E^0_{0,m}(t,x)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \frac{\alpha(p)}{p!}(h+k)^p.$$

Define $C_1(p) = \frac{\alpha(p)}{p!}$. Thus,

$$\|E^0_{0,m}\| \leq C_1(p)(h+k)^p.$$

For $\epsilon = 0$ and $n \in \{1, \dots, N-1\}$, using equation (4), we have

$$\begin{aligned} u(t,x) - \hat{v}^0_{n,m}(t,x) = &\sum_{\eta=0}^{n-1} \int_{t^0_\eta}^{t^0_{\eta+1}} K_1(t,x,s)(u(s,x) - v^0_{\eta,m}(s,x))\, ds \\ &+ \int_{t^0_n}^{t} K_1(t,x,s)(u(s,x) - \hat{v}^0_{n,m}(s,x))\, ds. \end{aligned}$$

Taking the norm, we get

$$\|u - \hat{v}^0_{n,m}\| \leq \sum_{\eta=0}^{n-1} hK \|E^0_{\eta,m}\| + K \int_{t^0_n}^{t} \|u - \hat{v}^0_{n,m}\|\, dt,$$

where $K = \max\{\|K_1\|, \|K_2\|\}$. By Lemma 3.2,

$$\|u - \hat{v}^0_{n,m}\| \leq \sum_{\eta=0}^{n-1} h d_1 \|E^0_{\eta,m}\|,$$

where $d_1 = K\exp(K\tau)$. On the other hand,

$$\|E^0_{n,m}\| \leq \|u - \hat{v}^0_{n,m}\| + \|\hat{v}^0_{n,m} - v^0_{n,m}\|.$$

Substituting the bounds and applying Theorem 3.1 and Lemma 3.1, we get

$$\|E^0_{n,m}\| \leq \frac{\alpha(p)}{p!}(h+k)^p \exp(\tau d_1).$$

Define $C_2(p) = \frac{\alpha(p)}{p!}\exp(\tau d_1)$.

For $\epsilon \in \{1, \dots, r-1\}$, using equation (6), we have

$$\|u - \hat{v}^\epsilon_{n,m}\| \leq \|A\| \sum_{e=0}^{\epsilon-1} \|E^e\| + Kh \sum_{\eta=0}^{n-1} \|E^\epsilon_{\eta,m}\| + K \int_{t^\epsilon_n}^{t} \|u - \hat{v}^\epsilon_{n,m}\|\, ds.$$

According to Lemma 3.2, this implies

$$\|u - \hat{v}^\epsilon_{n,m}\| \leq \widehat{A} \sum_{e=0}^{\epsilon-1} \|E^e\| + \widehat{K} h \sum_{\eta=0}^{n-1} \|E^\epsilon_{\eta,m}\|,$$

where $\widehat{A} = \|A\|\exp(K\tau)$ and $\widehat{K} = K\exp(K\tau)$.

On the other hand,

$$\|E^{\epsilon}_{n,m}\| \leq \widehat{A}\sum_{e=0}^{\epsilon-1}\|E^{e}\| + \widehat{K}h\sum_{\eta=0}^{n-1}\|E^{\epsilon}_{\eta,m}\| + \frac{\alpha(p)}{p!}(h+k)^{p}.$$

Applying Lemma 3.1, we have

$$\|E^{\epsilon}\| \leq \exp(\widehat{K}\tau)\frac{\alpha(p)}{p!}(h+k)^{p} + \exp(\widehat{K}\tau)\widehat{A}\sum_{e=0}^{\epsilon-1}\|E^{e}\|.$$

Using Lemma 3.1 again, we obtain

$$\|E^{\epsilon}\| \leq \underbrace{\exp(\widehat{K}\tau)\frac{\alpha(p)}{p!}\exp((r-1)\exp(\widehat{K}\tau)\widehat{A})}_{C_3(p)}(h+k)^{p}.$$

Finally, setting $C = \max\{C_1(p), C_2(p), C_3(p)\}$, we conclude

$$\|E^{\epsilon}_{n,m}\| \leq C(h+k)^{p}, \quad \forall n \in \{0,\dots,N-1\},\, m \in \{0,\dots,M-1\},\, \epsilon \in \{0,\dots,r-1\}. \ \Box$$

4 Numerical Examples

In this section, we validate the theoretical results derived in the previous section through numerical examples. The exact solutions for all examples are known in advance. For each example, we compute the error between the exact solution u and the Taylor collocation solution v.

Example 4.1 Consider the DVIEs with a time delay $\tau = \frac{1}{5}$ of the form

$$u(t,x) = \begin{cases} g(t,x) + (t+x)u(t-\frac{1}{5},x) + \int_0^t s\cos(xt)u(s,x)ds \\ + \int_0^{t-\frac{1}{5}} (2\sin(t+x)+s)u(s,x)ds, \quad t \in [0,1], \quad x \in [0,1], \\ \Phi(t,x), \quad t \in [-\frac{1}{5}, 0]. \end{cases}$$

The exact solution of this problem is given by $u(t,x) = t + \sin(x)$. The function $g(t,x)$ is then computed using the exact solution. Numerical results were obtained by applying the proposed Taylor collocation method for different numbers of collocation points. The absolute error values are reported in Table 1, showing that the absolute errors decrease as the number of collocation points increases.

Example 4.2 Consider the two-dimensional Volterra integral equation from [10]

$$u(x,t) = x^2(-1 + e^{-t} + x^2 + e^{t} - x^2e^{t}) + \int_0^t (x^2 + e^{-2s})u(x,s)ds$$

for $x, t \in [0,1]$, which has the exact solution $u(x,t) = x^2e^t$. The numerical results for $p-1 = 2,4,6$ and $h = k = 0.1$ obtained using the Taylor collocation method (TCM) are compared with the numerical results obtained using a method based on expanding the solution in terms of bivariate shifted Legendre polynomials [10]. The comparison is presented in Table 2.

x \ t	0	0.2	0.4	0.6	0.8	1
0	0	$1.32e-12$	$1.15e-11$	$2.36e-11$	$2.92e-10$	$8.23e-10$
0.2	$3.8e-14$	$6.54e-11$	$2.30e-11$	$1.50e-10$	$3.45e-10$	$9.21e-10$
0.4	$2.66e-11$	$7.81e-12$	$7.20e-11$	$4.01e-10$	$1.24e-09$	$1.18e-10$
0.6	$4.38e-11$	$3.29e-11$	$1.99e-10$	$1.48e-10$	$3.99e-10$	$2.79e-10$
0.8	$8.73e-11$	$3.45e-10$	$2.61e-10$	$2.59e-10$	$1.00e-09$	$5.41e-09$

Table 1: Numerical results for Example 4.1.

	Present method (TCM)			Method in Ref. [10]		
(x,t)	$p-1=2$	$p-1=4$	$p-1=6$	$M=2$	$M=4$	$M=6$
$(0,0)$	0	0	0	0	0	0
$(0.1,0.1)$	$3.6e-08$	$7.3e-12$	$1.4e-11$	$2.9e-05$	$2.0e-07$	$1.3e-09$
$(0.2,0.2)$	$3.0e-07$	$9.8e-11$	$1.3e-11$	$1.6e-04$	$4.7e-07$	$5.8e-09$
$(0.3,0.3)$	$1.0e-06$	$2.5e-10$	$3.4e-10$	$4.9e-04$	$9.5e-07$	$1.3e-08$
$(0.4,0.4)$	$2.8e-06$	$1.1e-09$	$3.2e-10$	$5.8e-04$	$2.4e-06$	$2.3e-08$
$(0.5,0.5)$	$6.3e-06$	$2.1e-09$	$1.0e-10$	$1.5e-05$	$3.4e-07$	$3.6e-08$
$(0.6,0.6)$	$1.3e-05$	$3.9e-09$	$3.0e-10$	$1.3e-03$	$4.9e-06$	$5.2e-08$
$(0.7,0.7)$	$2.5e-05$	$7.8e-09$	$2.8e-09$	$3.0e-03$	$5.6e-06$	$7.0e-08$
$(0.8,0.8)$	$4.8e-05$	$1.5e-08$	$1.5e-09$	$3.4e-03$	$6.9e-06$	$9.1e-08$
$(0.9,0.9)$	$9.1e-05$	$3.0e-08$	$1.3e-09$	$6.2e-04$	$1.7e-05$	$1.1e-07$
CPU time	$56.17\,sec$	$253.82\,sec$	$997.32\,sec$	/	/	/

Table 2: Comparison of absolute errors for Example 4.2.

5 Conclusion

In this paper, we proposed a new numerical method based on Taylor polynomials to construct a collocation solution for approximating the solution of delay Volterra integral equations (DVIEs) with a spatial variable. Unlike existing approaches, our method efficiently handles the time delay and spatial dependence in the integral equation, which is a key challenge in solving such problems.

A rigorous convergence analysis confirms that the proposed method is accurate and stable, with an $O((h+k)^p)$ error bound. Numerical examples validate its effectiveness, demonstrating that it provides high-precision approximations while maintaining computational efficiency.

The novelty of our approach lies in the extension of the Taylor Collocation Method to DVIEs with spatial dependence, an area that has received limited attention in previous studies. Additionally, our method is computationally efficient as it computes the approximation coefficients iteratively without requiring the solution of any system of algebraic equations. Moreover, it can be adapted to various delay integral equations encountered in applied sciences and engineering.

These contributions make our method a powerful and efficient alternative for solving DVIEs with spatial variables, laying the groundwork for further extensions to higher-dimensional problems.

References

[1] I. Ali, H. Brunner and T. Tang. Spectral methods for pantograph-type differential and integral equations with multiple delays. *Frontiers of Mathematics in China* **4** (2009) 49–61.

[2] R. Bellman. The stability of solutions of linear differential equations. *Duke Mathematical Journal* **10** (1943) 643–647.

[3] A. Bellour and M. Bousselsal. A Taylor collocation method for solving delay integral equations. *Numerical Algorithms* **65** (2014) 843–857.

[4] F. Birem, A. Boulmerka, H. Laib and C. Hennous. An algorithm for solving first-kind two-dimensional Volterra integral equations using collocation method. *Nonlinear Dynamics and Systems Theory* **23** (2023) 475–486.

[5] H. Bouzeraieb, H. Laib and A. Boulmerka. Numerical solution of neutral double delay Volterra integral equations using Taylor collocation method. *Nonlinear Dynamics and Systems Theory* **24** (2024) 236–245.

[6] H. Brunner. *Collocation Methods for Volterra Integral and Related Functional Differential Equations.* Cambridge University Press, Cambridge, 2004.

[7] H. Brunner. The numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation. *IMA Journal of Numerical Analysis* **9** (1989) 47–59.

[8] S. R. Ghorpade and B. V. Limaye. *A Course in Multivariable Calculus and Analysis.* Springer, New York, 2010.

[9] S. McKee, T. Tang and T. Diogo. An Euler-type method for two-dimensional Volterra integral equations of the first kind. *IMA Journal of Numerical Analysis* **20** (2000) 423–440.

[10] S. Nemati, P. M. Lima and Y. Ordokhani. Numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials. *Journal of Computational and Applied Mathematics* **242** (2013) 53–69.

[11] H. Brunner and Y. Yatsenko. Spline collocation methods for nonlinear Volterra integral equations with unknown delay. *Journal of Computational and Applied Mathematics* **71** (1996) 67–81.

[12] W. Zheng and Y. Chen. Numerical analysis for Volterra integral equation with two kinds of delay. *Acta Mathematica Scientia* **39** (2019) 607–617.

[13] L. Zhang, J. Huang and H. Li. Splitting extrapolation algorithms for solving linear delay Volterra integral equations with a spatial variable. *Applied Numerical Mathematics* **178** (2022) 372–385.