Nonlinear Dynamics and Systems Theory, **25** (6) (2025) 679–690

Approximation of Invariant Solutions to the Nonlinear Filtration Equation by Modified Padé Approximants

S. Skurativskyi *, S. Mykulyak and I. Skurativska

Subbotin Institute of Geophysics of NAS of Ukraine, Palladin av. 32, Kyiv, 03142, Ukraine

Received: November 20, 2024; Revised: October 27, 2025

Abstract: This paper deals with a mathematical model for oil filtration in a porous medium and its self-similar and traveling wave regimes. The model consists of the equation of mass conservation and dependencies of porosity, permeability, and oil density on pressure. The oil viscosity is considered to be the experimentally determined parabolic relationship with respect to pressure. To close the model, two types of the Darcy law are used: the classic one and the dynamic one describing the relaxation processes during filtration. In the former case, self-similar solutions are studied, while in the latter case, traveling wave solutions are the focus. Using the invariant solutions, the initial model is reduced to the nonlinear ordinary differential equations possessing the trajectories vanishing at infinity and representing the moving liquid fronts in porous media. To approximate these solutions, we elaborate the semi-analytic procedure based on modified Padé approximants. In fact, we calculate sequentially Padé approximants up to the 3-rd order for a two-point boundary value problem on the semi-infinite domain. A good agreement of evaluated Padé approximants and numerical solutions is observed. The approach provides relatively simple quasi-rational expressions of solutions and can be easily adapted for other types of model's nonlinearity.

Keywords: *nonlinear filtration; self-similar solution; relaxation; traveling wave; Padé approximant.*

Mathematics Subject Classification (2020): 35K55, 35L75, 35C06, 35C07, 41A21, 70K99, 93-10.

* Corresponding author: mailto:skurserg@gmail.com

1 Introduction

The filtration processes are studied in many branches of science, including geophysics, biology, ecology, medicine, etc. Control of the filtration processes is at the heart of technologies applied for enhancing oil and gas recovery [2, 5], cleaning polluted gas-liquid substances, and providing high-quality drugs in the biopharmaceutical industry. Due to the significance and prevalence of these processes in nature and technological developments, theoretical studies of filtration in porous media are relevant, especially regarding the deviation of filtration flow dynamics from linear patterns.

The complete formulation of filtration problems, incorporating process nonlinearity, high intensity, and multiphasicity of liquid flows, interacting effects, and complex initial and boundary conditions, presents significant challenges. This necessitates the development of new or improved tools for study.

In this research, we consider the oil filtration in a porous medium within the framework of continuous mechanics [2], taking into account several nonlinear effects. The filtration model consists of the equation of motion representing the conservation of mass, the equation of state for oil, the dependencies of porosity and permeability on pressure, and finally, the Darcy law, which is considered in its classical or generalized form. Model nonlinearity originates from the nonlinear dependence of oil viscosity on pressure which is discovered experimentally. Note that the viscosity of reservoir oil is an important characteristic that affects the proper functioning of producing wells [4,10]. Other fluid dynamics problems in porous media also seek to determine the influence of variable viscosity and permeability of filtrating liquids on flow behavior [15].

In the case of the classic Darcy law, the model in the one-dimensional case is reduced to the nonlinear filtration equation, which can be regarded as a weakly nonlinear diffusion equation [11,19,20] $p_t = (k(p)p_x)_x$, where the function $k(p)$ is the diffusion coefficient or hydraulic conductivity. A vast number of studies concern the boundary value problem (BVP) on a semi-infinite domain when the model admits self-similar regimes.

Another interesting class of filtration models known as relaxation models or models with memory [8, 14] has been formed when considering the filtration processes with a relatively rapid change in parameters, the flows of non-Newtonian liquids (heavy oil, solutions of polymers, mixtures, emulsions, multiphase liquids with mass exchange between phases), and filtration in layers with a particularly complex structure (crack-porous media) [2]. In such conditions, a delay is observed in the response of the filtration flow; in other words, there is a local nonequilibrity of the filtration process accompanied by the relaxation of pressure and velocity. To incorporate process nonequilibrity, the classical Darcy law is generalized by adding the terms with the first temporal derivatives [5,12,18] describing the approach of pressure and velocity to their equilibrium values. As a rule, the nonequilibrium (or relaxation) filtration models do not admit the same self-similar solutions as the classical Darcy-type models. Instead, the relaxation models possess the traveling wave solutions, the structure of which is richer.

Despite an immense number of research on the nonlinear diffusion-type equations, they rarely succeed in obtaining a general exact solution of equations, especially their hyperbolic generalizations. Therefore, there is still a need to improve existing and develop more general methods of derivation of solutions, including the development of the asymptotic approach [1] in combination with the extensive involvement of numerical methods [13].

Thus, we aim to develop a semi-analytic approach based on the Padé approximants, which were proven to be effective in many applications [1,3,17], and use it for calculating the invariant solutions describing the filtration of oil with variable viscosity.

2 The Model of Oil Filtration and Its Reduction to a Single Equation

The mathematical model for the elastic mode of filtration reads as follows:

$$\begin{aligned} (m\rho)_t + (\rho v)_x = 0, \qquad & \rho = \rho_0 \left(1 + C_f \left(p - p_0\right)\right), \\ m = m_0 \left(1 + C_m \left(p - p_0\right)\right), \quad & k = k_0 \left(1 + C_k \left(p - p_0\right)\right). \end{aligned} \tag{1}$$

Here, the system (1) consists of the continuity equation expressing the mass conservation law, equation of state for a fluid, and dependencies of porosity and permeability on pressure. The traditional designations used are: ρ is the fluid density, p is the pressure, v is the filtration velocity, C_f, C_m, and C_k are the compression coefficients of the fluid, porosity, and permeability.

To close the system, we used the generalized Darcy law containing the description of the nonequilibrium (or relaxation) filtration process [5, 12, 18]

$$\tau \left(v + K_\infty p_x\right)_t + v + K_0 p_x = 0, \tag{2}$$

where the hydraulic conductivity functions $K_0 = \frac{k}{\mu}$ and $K_\infty = \theta K_0$ are related to the equilibrium and frozen diffusion coefficients, τ and θ are constants. In this research, we pay more attention to the oil viscosity μ, assuming that it varies significantly with pressure, which prompts the consideration of nonlinear pressure dependencies for the function $\mu(p)$.

It is obvious that by dropping the relaxing terms in (2), we arrive at the classical Darcy law

$$v = -K_0 p_x. \tag{3}$$

To simplify the problem, we reduce the model of filtration to a single equation with respect to p.

Let us start with considering the model using the classic Darcy law while justifying the quadratic pressure dependence of oil viscosity μ.

To specify the function μ, we consider the process of oil filtration in a reservoir in the range of pressures when the oil is close to the phase transition zone which can form during depletion away from a wellbore ([4], p.42). We are interested in the vicinity of the phase transition point, where a single phase of oil transforms into a gas-liquid mixture. Assume that in this zone, the amount of gas phase is not enough to influence the filtration dynamics, but the oil viscosity undergoes significant changes, which are taken into account in the model.

We consider the experimental data concerning the measurements of the viscosities of oils under reservoir conditions [10]. Several experimental points from the paper [10] (see Fig.3, curve 4) are depicted in Fig.1. These data confidently show the convex character of the graph of oil viscosity at pressure variations.

In this study, we pay attention to the vicinity of the point of minimum known as a bubble point and describe the oil viscosity μ as a quadratic function of pressure,

$$\mu = \mu_0 \left(1 + a \left(p - p_0\right)^2\right), \tag{4}$$

where μ_0 is the viscosity at $p = p_0$, a is a positive constant.

To specify the function μ, we approximate the experimental data in Fig.1 by the parabola (4) whose vertex coincides with the minimum value of the data (Fig.1). The coordinates of the parabola vertex $(p_0; \mu_0)$ are evaluated from the experimental data quite accurately providing that $\mu_0 = 0.005$ Pa·s and $p_0 = 41.6855$ bar (or 4.169 MPa). The evaluation of the parameter a leads to the following value $a = 1.507 \cdot 10^{-14}$ Pa^{-2}.

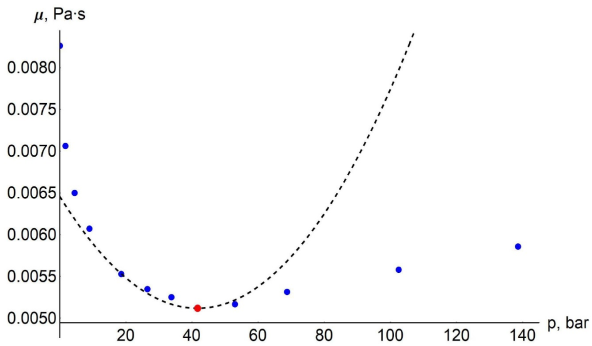


Figure 1: The approximation of viscosity by the parabola $\mu = \mu_0 \left(1 + a\,(p - p_0)^2\right)$ with the vertex $(p_0; \mu_0) = (4.16855 \cdot 10^6, 0.005)$ and $a = 1.507 \cdot 10^{-14}$ Pa^{-2}. The experiment of Hocott et al. [10] is marked with filled circles, while their parabolic approximation is drawn with the dashed line.

Applying the auxiliary constraints $C_f C_m \ll 1$, the filtration model (1) closed by the dynamic Darcy law (2) is reduced to the single partial differential equation

$$\tau p_{tt} - \tau\theta[D(p)p_x]_{xt} + p_t - [D(p)p_x]_x = 0, \tag{5}$$

where

$$D(p) = \kappa \frac{1 + C_k\,(p - p_0)}{1 + a\,(p - p_0)^2} \text{ and } \kappa = \frac{k_0}{\mu_0 m_0 (C_f + C_m)}.$$

When $\tau = 0$, that is, the classic Darcy law (3) is used, it follows from (5) that

$$p_t = (D(p)p_x)_x\,. \tag{6}$$

Next, we consider the invariant solutions of equations (5) and (6) and develop the semi-analytical procedure for their approximation using Padé approximations. Let us start from the more straightforward equation (6) and solve a BVP possessing the self-similar invariant solutions.

3 BVP for the Filtration Model with the Classical Darcy Law and Its Self-Similar Solutions

When equation (6) is subject to the following initial and boundary conditions:

$$p(x, t = 0) = p_1, \quad p(x = 0, t) = p_2, \qquad p_{1,2} = \text{const}, \tag{7}$$

we arrive at the classical BVP [2, 7, 16] that admits self-similar solutions. To continue the theoretical studies, let us perform the substitution

$$p(x,t) = \Omega(P + y_1), \tag{8}$$

where $\Omega = 1/\sqrt{a}$ and $p_{0,1,2} = \Omega y_{0,1,2}$.

Then equation (6) and conditions (7) can be written in the form of the dimensionless BVP

$$\begin{gathered} P_t = \left(D(P)P_x\right)_x, \\ P(x,t=0) = 0, \qquad P(x=0,t) = y_2 - y_1, \end{gathered} \tag{9}$$

where

$$D(P) = D(0)G(P), \qquad G(P) = \frac{1+\beta_1 P}{1+2\beta_3 P + \beta_2 P^2}, \qquad D(0) = \kappa \frac{1 + C_k\Omega\,(y_1 - y_0)}{1 + (y_1 - y_0)^2},$$

$$\beta_1 = \frac{C_k\Omega}{1 + C_k\Omega\,(y_1 - y_0)}, \qquad \beta_2 = \frac{1}{1 + (y_1 - y_0)^2}, \qquad \beta_3 = \frac{(y_1 - y_0)}{1 + (y_1 - y_0)^2}.$$

Further studies do not require the value of $D(0)$ due to the special selection of the solution form, while β_j affects the solution characteristics.

Since $a \sim 10^{-14}$ Pa^{-2}, one has $\Omega \sim 10^7 = 10$ MPa. The values of C_f and C_k are of order $10^{-10} - 10^{-8}$ Pa^{-1} [2], therefore, β_1 may not be small, especially if we take into account the possibility of a negative value of $y_1 - y_0$.

The remarkable feature of the model (9) is that this problem possesses the well-known self similar solution

$$P(x,t) = P(\xi), \quad \xi = \frac{x}{2\sqrt{D(0)t}}, \tag{10}$$

reducing (9) to the ordinary differential equation

$$\frac{d}{d\xi}\left(\frac{1+\beta_1 P}{1+2\beta_3 P+\beta_2 P^2}\frac{dP}{d\xi}\right) = -2\xi\frac{dP}{d\xi} \tag{11}$$

subjected to the conditions

$$P(\xi = 0) = y_2 - y_1, \qquad P(\xi = \infty) = 0. \tag{12}$$

Equation (11) has a long history that can be traced through the works [7, 16]. Here, we briefly remark that the construction of the solution of (11) depends on the form of hydraulic conductivity function $D(P)$. It is known [7, 16] that the analytical representation of the solution can be obtained for the cases $\beta_1 = \beta_2 = 0$, $\beta_3 = -q, \beta_2 = q^2$, and $\beta_1 = 0$. In other cases, equation (11) is analyzed by alternative methods.

In what follows, we develop the semi-analytic procedure for the evaluation of solutions to BVP (11) – (12) utilizing the Padé approximants.

4 The Padé Approximant Construction for the BVP Self-Similar Solutions

To do this, we need the integral relations representing a certain type of conservation laws for equation (11). In particular, integrating (11) over the interval $(0; \infty)$, we obtain

$$\frac{dP}{d\xi}(0) = -2\frac{1 + 2\beta_3 P(0) + \beta_2 P(0)^2}{1+\beta_1 P(0)}\int_0^\infty P d\xi. \tag{13}$$

In essence, the procedure is the adaptation of the approaches developed in [1,17]. The direct application of the procedures outlined in the above-mentioned papers encounters massive symbolic calculations that do not allow to obtain desired results or significantly exploit the peculiarities of the model. In this research, we use the specific conservation laws and quasi-fractional Padé approximants [1].

Thus, we are looking for the solution of the problem in the form of the Taylor series

$$P = \sum_{i=0}^{N} r_i \xi^i, \tag{14}$$

where $P(0) = r_0$ is evaluated from the initial condition at $\xi = 0$.

Inserting it into equation (11), we derive the coefficients r_i, $i \geqslant 2$, as the functions of $r_1 = P'(0)$ only. As ξ increases, series (14) diverges and does not describe the solution properly. Therefore, we approximate it by a Padé approximant, i.e., the rational approximation for a series. To specify the form of the Padé approximant, we use the additional information on the solution's behavior at infinity.

Assume that $P(\xi)$ is vanishing as ξ tends to infinity. Then, from equation (11), it follows that asymptotics is defined by the equation $\frac{d^2P}{d\xi^2} = -2\xi\frac{dP}{d\xi}$, whose vanishing solution is $P(\xi) = const \cdot \operatorname{erfc}(\xi) \equiv const \cdot \int_{\xi}^{\infty} \exp(-z^2)dz$. In turn, the asymptotics of the function $\operatorname{erfc}(\xi) \sim Q(\xi)\exp(-\xi^2)$ is also valid.

Thus, to construct the solution valid for all ξ, we approximate the Taylor series (14) by the quasi-fractional Padé approximant [1] combining the rational Padé approximant and the asymptotics $\sim \exp(-\xi^2)$

$$PA_{[M/M]} = \frac{\sum_{i=0}^{M} A_i \xi^i}{\sum_{j=0}^{M} B_j \xi^j} e^{-\xi^2}, \tag{15}$$

where M is the order of the Padé approximant; A_i and B_j are constants.

Let us recall that A_i and B_j depend on r_1 only. To evaluate M coefficients of A_i and B_j, we need to derive $N = 2M$ coefficients of the Taylor series (14). Relation (15) is inserted in integral relations (13) which are solved with respect to r_1. Note that the form of $PA_{[M/M]}$ can be modified further by incorporating the polynomial $\sum_k^L C_k \xi^k$ into the exponent of the exponential function. The constants C_k can be calculated by the auxiliary integral equations deduced from the starting equation by multiplying by ξ^n and integration over $(0;\infty)$ [1].

4.1 Application of the procedure of BVP solving

To apply the procedure developed above, we consider BVP (11) – (12) at a and p_0 evaluated for the parabola of Fig. 1. We choose the value $p_1 = 2$MPa, which lies to the left of the point $p = p_0$ in Fig. 1. Then we obtain $\Omega = 1/\sqrt{a} = 0.81 \cdot 10^7$; $(y_1 - y_0) = (20 \cdot 10^5 - p_0)/\Omega = -0.2677$. To evaluate β_1, we fix the product $C_k\Omega$ that varies in a wide range due to the significant variations of C_k as mentioned above. Then, for instance, when $C_k\Omega = 0.001$, then $\beta_1 = 0.001$, while at $C_k\Omega = 0.4$, we have $\beta_1 = 0.447973$. Therefore, let us consider two cases. The former when β_1 is small enough to be neglected and the latter when β_1 is not small. For the sake of simplicity, the initial condition $y_2 - y_1$ is assumed to be 1. Thus, $P(0) = r_0 = 1$ in (14) for all further studies.

Thus, the coefficients r_i of series (14) are as follows:

$$r_2 = r_1^2 \frac{\beta_1(\beta_2 - 1) + 2(\beta_2 + \beta_3)}{2(1+\beta_1)(1+\beta_2+2\beta_3}, \qquad r_3 = -r_1 \frac{1+\beta_2+2\beta_3}{3(1+\beta_1)} +$$
$$r_1^3 \frac{\beta_1^2(3+\beta_2^2+4\beta_3) + 2(\beta_2+3\beta_2^2+6\beta_2\beta_3+2\beta_3^2) + 4\beta_1(\beta_2^2-\beta_2-2\beta_3(1+\beta_3))}{6(1+\beta_1)^2(1+\beta_2+2\beta_3)^2}, \dots$$

The coefficients r_i, $i \geqslant 2$, depend only on r_1. However, they quickly become cumbersome when the number i increases.

Next, by relation (15), we construct the expression $PA_{[M/M]}$ for the Taylor series (14) in a conventional way taking into account in addition the expansion $\exp(-\xi^2) = \sum_{n=0}^{\infty}(-1)^n \xi^{2n}/n!$. Let us start from the simplest case when $M = 1$ and $PA_{[1/1]} = (1 + A_1\xi)\exp(-\xi^2)/(1 + B_1\xi)$. Then the relation for specifying A_1 and B_1 reads as follows:

$$(1 + r_1\xi + r_2\xi^2 + \dots)(1 + B_1\xi) - (1 - \xi^2 + \xi^4/2 + \dots)(1 + A_1\xi) = 0.$$

Nullifying the coefficients at ξ and ξ^2, we obtain a pair of equations whose roots are as follows:

$$A_1 = -\frac{1 + r_2 - r_1^2}{r_1}, \qquad B_1 = -\frac{1 + r_2}{r_1}. \tag{16}$$

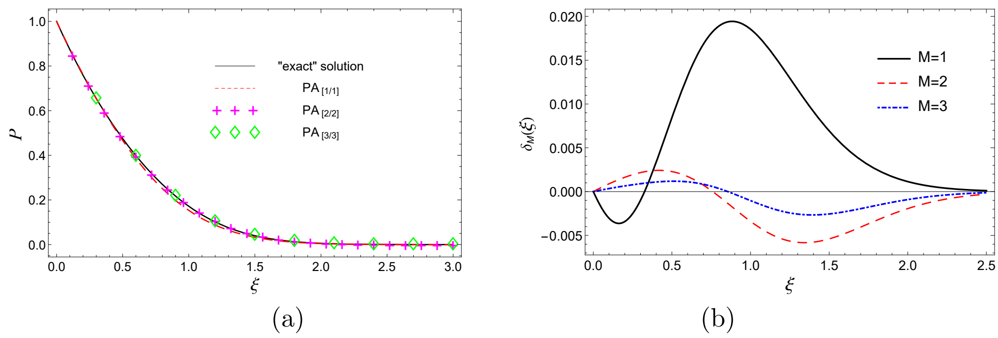


Figure 2: a: The $P(\xi)$ profiles for the solutions of BVPs (11) – (12) evaluated numerically (regarded as the "exact" solution and marked by the solid curve) and corresponding $PA_{[M/M]}$. b: The differences $\delta_M(\xi) = P - PA_{[M/M]}$ vs. ξ for the profiles from the left panel.

Let the parameter values be as follows:

$$\beta_1 = 0.447973; \qquad \beta_2 = 0.933119; \qquad 2\beta_3 = -0.249816. \tag{17}$$

Using the *Mathematica* commands NDSolve[Equation (11), P[0]==1, P[5]==0}, P, ξ, Method $\to$ {"Shooting", "StartingInitialConditions" $\to$ {P[0] == 1, P'[0] == −2}, we calculate the trajectory $P(\xi)$ (Fig.2a, solid curve) and its derivative at zero $(r_1)_{ex} = -1.32175$, which is regarded as exact. The coefficients r_i of the series (14) are as follows:

$$r_0 = 1, \qquad r_1 - \text{unknown}, \qquad r_2 = 0.3254 r_1^2, \qquad r_3 = -0.3875 r_1 + 0.1883 r_1^3,$$
$$r_4 = -0.3783 r_1^2 + 0.0608 r_1^4, \dots \tag{18}$$

M	1	2	3
$(r_1)_M$ by (13)	-1.28324	-1.33149	-1.32556
$\eta = \left\| \frac{(r_1)_M-(r_1)_{ex}}{(r_1)_{ex}} \right\|$	0.0291	0.0073	0.0028

Table 1: Values of $(r_1)_M$ and their relative errors depending on M ($(r_1)_{ex} = -1.32175$).

We construct $PA_{[M/M]}$ using the coefficients (18) and insert the resulting Padé approximants into the integral law (13). Solving the resulting equations with respect to r_1, we obtain the successive approximations for r_1 shown in Table 1, the second row.

Then, using the evaluated values of r_1 and inserting them into (18), we write the corresponding Pade approximants (it is especially easy to obtain $PA_{[1/1]}$ by evaluating (16))

$$PA_{[1/1]} = \frac{1-0.0863411x}{1+1.1969x}e^{-x^2}, \qquad PA_{[2/2]} = \frac{1+0.141081x+0.414061x^2}{1+1.47257x+0.797799x^2}e^{-x^2},$$
$$PA_{[3/3]} = \frac{1+0.0632479x+0.632533x^2-0.0548591x^3}{1+1.38881x+0.901649x^2+0.207737x^3}e^{-x^2}.$$

To find out the quality of the approximation, we compare the numerically derived profile $P(\xi)$ and the profiles $PA_{[M/M]}$ (Fig. 2a). For convenience, we depict the differences $\delta_M(\xi) = P(\xi) - PA_{[M/M]}$ in Fig.2b. Obviously, the deviations of the profiles from zero decrease when the order M grows. In particular, at $M = 3$, the difference $\delta_3(\xi)$ varies in the interval $[-0.0026; 0.0012]$, i.e., $P(\xi)$ and $PA_{[M/M]}$ are almost indistinguishable.

The convergence of the iteration procedure of the Padé approximant evaluation is monitored by calculating the relative error $\eta = |\{(r_1)_M - (r_1)_{ex}\}/(r_1)_{ex}|$, where $(r_1)_M = dPA_{[M/M]}/d\xi$ is the derivative of the Padé approximant at $\xi = 0$ and $(r_1)_{ex} = -1.32175$. The results of the calculations are presented in Table 1, the third row. Analyzing the behavior of relative errors, we see that η decreases when M grows. This allows one to conclude that the iteration process converges to the value $(r_1)_{ex}$.

5 Traveling Wave Solutions of the Filtration Model with the Relaxation Darcy Law and Their Padé Approximants

Now, consider equation (5), assuming that the pressure approaches p_1 as $x \to \infty$, and transform it using the substitution $p = \Omega P + p_1 \equiv \Omega(P + y_1)$ similar to (8). The resulting equation is as follows:

$$\tau P_{tt} - \tau\theta[D(P)P_x]_{xt} + P_t - [D(P)P_x]_x = 0, \tag{19}$$

where $D(P) = D(0)G(P)$ is defined in (9). For our studies, we can further put $D(0) = 1$ without loss of generality.

Equation (19) does not admit the self-similar regimes (10), instead, among invariant solutions, there are traveling wave regimes. Therefore, in what follows, we consider the traveling wave solution

$$p = P(\xi), \qquad \xi = x - ct, \tag{20}$$

where c is the phase velocity.

Inserting (20) into (19), we get the ordinary differential equation of the third order. After integration under the condition $P \to 0$ when ξ tends to infinity, we get the second order differential equation

$$\tau c^2 P' + c\tau\theta[G(P)P']' - cP - G(P)P' = 0. \tag{21}$$

Thus, the problem is to approximate the forward semi-trajectory starting from $P(0)$ and approaching zero at infinity by the Padé approximant. Note that such solutions can be helpful for the description of moving fronts in models for heat and mass transfer [6].

To develop the Padé approximations for this solution, the conservation law is required. To derive it, we integrate equation (21) from 0 to infinity and arrive at the resulting relation

$$\int_0^\infty P d\xi = \left(-\tau c^2 - c\theta G(1)P'(0) + \int_0^1 G(x)dx\right)/c \equiv \Delta. \tag{22}$$

Using the approach described in Section 4, in the vicinity of the point $\xi = 0$, we look for the Taylor series expansion $P = 1 + \sum_{j=1}^N r_j \xi^j$ inserting it into relation (21). All coefficients r_j, $j \geq 2$, are the functions of $r_1 = P'(0)$ only. The Padé approximant now is written in the following form:

$$PA_{[M/M]} = \frac{\sum_{i=0}^M A_i \xi^i}{\sum_{j=0}^M B_j \xi^j} e^{H\xi},$$

where the multiplier $e^{H\xi}$ describes the asymptotic solution's behavior as $\xi \to \infty$. To evaluate H, we linearize equation (21) arriving to

$$\tau\theta P'' + (\tau c^2 - 1)P' - cP = 0,$$

and then the simplest solution vanishing at infinity is $e^{H\xi}$, where $H = (1 - \tau c^2 - \sqrt{(\tau c^2 - 1)^2 + 4\tau\theta c})/(2\tau\theta) < 0$.

The coefficients of the $PA_{[M/M]}$ nullify the relation for all ξ,

$$\sum_{j=0}^{2M} \frac{(H\xi)^j}{j!} \left(1 + \sum_{j=1}^M A_j \xi^j\right) - \left(1 + \sum_{j=1}^{2M} r_j \xi^j\right)\left(1 + \sum_{j=1}^M B_j \xi^j\right) = 0.$$

From this relation, the system of equations with respect to $2M$ variables A_j and B_j can be extracted. For instance, when $M = 1$, we obtain

$$A_1 - B_1 = r_1 - H, \qquad HA_1 - r_1 B_1 = r_2 - H^2/2, \tag{23}$$

and at $M = 2$,

$$\begin{aligned} &A_1 - B_1 = r_1 - H, \qquad A_2 - r_1 B_1 - B_2 + A_1 H = r_2 + H^2/2, \\ &- r_2 B_1 - r_1 B_2 + A_2 H + A_1 H^2/2 = r_3 + H^3/6, \\ &- r_3 B_1 - r_2 B_2 + A_2 H^2/2 + A_1 H^3/6 = r_4 + H^4/24. \end{aligned} \tag{24}$$

Systems (23) and (24) are linear and consistent. Thus, they possess unique solutions.

After identifying Padé approximant (15), we insert it into the conservation law (22), where $P'(0) = r_1$, $G(1) = \frac{1+\beta_1}{1+2\beta_3+\beta_2}$ = const, and $\int_0^1 G(x)dx$ = const. The resulting integral equation serves for the evaluation of the unknown quantity r_1. It is hard to solve such an equation even numerically.

To overcome this, we use the analytical representations for the integral term. Specifically, for $M = 1$ or $M = 2$, the expression $\int_0^\infty P dx = \int_0^\infty PA_{[M/M]} dx$ can be derived analytically. Using the *Mathematica* command, we obtain

$$\int_0^\infty PA_{[1,1]} dx = \int_0^\infty e^{Hx} \frac{1 + A_1 x}{1 + B_1 x} dx = -\frac{A_1}{B_1 H} + \frac{A_1 - B_1}{B_1^2} \operatorname{Ei}\left(\frac{H}{B_1}\right) e^{-H/B_1}, \quad (25)$$

where Ei(·) is the exponential integral function. A similar, but a bit cumbersome expression can also be computed for $M = 2$. In this case, it is hard to derive the improper integral. Instead, the definite integral on the interval $[0, L]$ (L is large enough, $L = 4$ is used in this case) fits well.

5.1 Padé approximant construction

Now, consider the application of the approach we developed at the fixed parameters $\tau = 1$, $\theta = 1.5$, $c = 2.7$, and the initial condition $P(0) = 1$. The parameters $\beta_{1,2,3}$ for the function $G(p)$ coincide with (17).

To justify the existence of a trajectory vanishing at infinity, let us integrate equation (21) under the second initial condition for the derivative $P'(0)$, which varies in the range $[-2.218, -2.215]$ with the step 0.0005. The resulting bundle of solutions, depicted in the inset of Fig.3a, contains the trajectories unbounded from above and others – from below. Then we can conclude that a unique trajectory exists, vanishing at infinity at a certain $P'(0)$. To control the conservation law implementation, we also attach the equation $dY/d\xi = P(\xi)$ with the initial condition $Y(0) = 0$ to equation (21) and calculate the trajectory $Y(\xi)$ which approaches to Δ as $\xi \to \infty$, as shown in Fig. 3b.

The numerically evaluated trajectory is regarded as an "exact" solution with which we will compare its Padé approximant. Using the *Mathematica* command NDSolve[·, Method → {"Shooting", "StartingInitialConditions" → {P[0] == 1, P'[0] == -1.7}, the trajectory we are looking for is estimated with good accuracy (Fig.3a) providing $P'(0) = -2.21658 \equiv (r_1)_{ex}$.

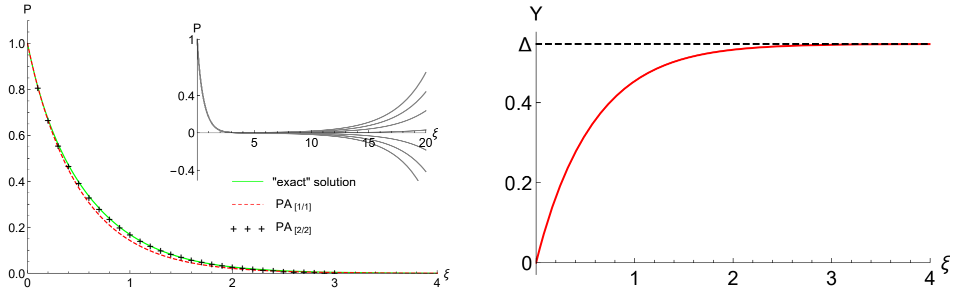
Finally, when the coefficients of the Taylor series $r_{1,2,3,4}$ and Padé approximant A_1, B_1, and relation (25) are inserted into the conservation law (22), we arrive at the equation with respect to r_1 possessing the root $r_1 = -2.18753$. The corresponding Padé approximant is as follows:

$$PA_{[1/1]} = \frac{0.17637 + 0.94988\xi}{0.17637 + \xi} e^{H\xi}, \qquad H = -1.90335. \quad (26)$$

Proceeding in the same manner, we calculate the next $r_1 = -2.22365$ and the corresponding Padé approximant

$$PA_{[2/2]} = \frac{1 + 2.00241\xi + 1.16097\xi^2}{1 + 2.32541\xi + 0.28312\xi^2} e^{H\xi}. \quad (27)$$

Figure 3a exhibits the comparison of $PA_{[1/1]}$ (dashed line), $PA_{[2/2]}$ (crosses), and "exact" solution (solid curve). It is obvious that $PA_{[2/2]}$ is indistinguishable from the "exact" solution.



(a) (b)

Figure 3: (a) The profiles of the solution of equation (21) and Padé approximants $PA_{[1/1]}$ and $PA_{[2/2]}$, defined by (26) and (27), respectively. The inset shows the bundle of trajectories starting from the initial conditions $P(0) = 1$ and $P'(0)$ from the range $[-2.218, -2.215]$. (b) The profile of $Y(\xi)$ describing the approach of a conservative quantity to its limit value Δ.

6 Conclusion

Thus, this research considered the nonlinear model describing the filtration of oil with variable viscosity in the semi-infinite domain. Model's nonlinearity was mostly determined by the quadratic pressure dependence of oil viscosity, the parameters of which were estimated from experimental data. The model incorporating the classical Darcy law possesses self-similar invariant solutions, which allow one to reduce the initial BVP to the nonlinear BVP for an ODE on a semi-infinite domain. The traveling wave solutions are considered when the filtration model is closed by the relaxation Darcy law. The research focused on the solutions vanishing at infinity. We developed the semi-analytical procedure for approximating the self-similar and traveling wave solutions utilizing the modified Padé approximant approach. The results of the procedure's application were compared with numerical solutions. It was shown that excellent results of approximation can be achieved even when using the low-order Padé approximations (up to the 3-rd order) in contrast to the use of conventional rational Padé approximations. Note also that the proposed procedure can be applied to the filtration equation with another form of hydraulic conductivity $D(p)$. Note also that the low order modified Padé approximants represent relatively simple and useful expressions for the model's solutions, which are preferable to use even when the exact but cumbersome solution exists. Moreover, quasi-rational approximations are indispensable when further manipulations on solutions are performed. Specifically, this is important for modeling well operation and liquid front propagation in porous media. Similar models and their solutions are also encountered in heat transfer theory [7, 16], demonstrating the multidisciplinary nature of the research.

Acknowledgment

The research is partly supported by the NAS of Ukraine, Project 0123U100183.

References

[1] I. Andrianov and A. Shatrov. Padé approximation to solve the problems of aerodynamics and heat transfer in the boundary layer. In: *Mathematical Theorems - Boundary Value*

Problems and Approximations (Ed.: L. Alexeyeva), Ch. 2. IntechOpen, 2020, 1–21.

[2] G. I. Barenblatt, V. M. Entov and V. M. Ryzhik. *The motion of fluids and gases in natural strata.* Moscow, Nedra Publishing House, 1984. [Russian]

[3] J. P. Boyd. Padé approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain. *Computers in Physics* **11** (3) (1997) 299 –303.

[4] A. Danesh. PVT tests and correlations. In: *PVT and Phase Behaviour of Petroleum Reservoir Fluids* (Ed.: A. Danesh), **47** Developments in Petroleum Science. Elsevier, 1998, 33–104.

[5] I. I. Denysiuk, I. A. Skurativska and I. M. Hubar. Attenuation of velocity fields during non-equilibrium filtration in a half-space medium for harmonic action on it. *J. Phys. Stud.* **27** (3) (2023) 3801.

[6] N. T. Fadai and M. J. Simpson. New travelling wave solutions of the porous–fisher model with a moving boundary. *J. Phys. A-Math.* **53** (2020) 095601.

[7] H. Fujita. The exact pattern of a concentration-dependent diffusion in a semi-infinite medium. Part III. *Textile Research Journal* **24** (3) (1954) 234–240.

[8] C. Guenoune, A. Bachmar and S. Boutechebak. A dynamic problem with wear involving thermoviscoelastic materials with a long memory. *Nonlinear Dyn. Syst. Theory* **24** (5) (2024) 473–484.

[9] D. F. Hays and H. N. Curd. Concentration-dependent diffusion in a semi-infinite medium. *J. Frank. Inst.* **283** (4) (1967) 300–307.

[10] C. R. Hocott and S. E. Buckley. Measurements of the viscosities of oils under reservoir conditions. *Transactions of the AIME* **142** (01) (1941) 131–136.

[11] K. Huber, J. Nowack and D. Göritz. Pervaporation of alcohols in PBT near the glass temperature: An analysis of the concentration-dependent diffusion. *J. Appl. Polym. Sci.* **68** (9) (1998) 1503–1515.

[12] J. Jäckle and H. L. Frisch. Properties of a generalized diffusion equation with a memory. *J. Chem. Phys.* **85** (1986) 1621–1627.

[13] A. M. Jasim, A. A. Ali, I. H. Halil, F. Z. Ahmed and M. A. Shallal. Numerical solution for Benjamin-Bona-Mahony-Burgers equation using septic b-spline Galerkin method *Nonlinear Dyn. Syst. Theory* **24** (3) (2024) 267–274.

[14] B. Khuzhayorov, J. Auriault and P. Royer. Derivation of macroscopic filtration law for transient linear viscoelastic fluid flow in porous media. *Int. J. Engineering Science* **38** (5) (2000) 487–504.

[15] N. A. Kudryashov, M. B. Kochanov and V. L. Savatorova. Approximate self-similar solutions for the boundary value problem arising in modeling of gas flow in porous media with pressure dependent permeability. *Appl. Math. Model.* **37** (7) (2013) 5563–5573.

[16] A. V. Lykov. *Theory of Heat Conduction.* Moscow, Vysshaya Shkola, 1967. [Russian]

[17] Y. V. Mikhlin and G. V. Manucharyan. Construction of homoclinic and heteroclinic trajectories in mechanical systems with several equilibrium positions. *Chaos, Solitons and Fractals* **16** (2) (2003) 299–309.

[18] S. Skurativskyi and I. Skurativska. Solutions of the model of liquid and gas filtration in the elastic mode with dynamic filtration law. *Ukr. J. Phys.* **64** (1) (2019) 19–26.

[19] J. Philip. General method of exact solution of the concentration-dependent diffusion equation. *Aust. J. Phys.* **13** (1960) 1–12.

[20] L. F. Shampine. Exact solutions for concentration dependent diffusion and the inverse complementary error function. *J. Frank. Inst.* **295** (3) (1973) 239–247.