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Train Departure Scheduling Using Kleene Star and Petri Net Queue Model (Cicalengka–Padalarang)

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Abstract: The train is one of the public transport used by people to move between regions. PT Kereta Api Indonesia (PT KAI) is an Indonesian State-Owned Enterprise (BUMN) that provides rail transportation services. One of the railway lines belonging to KAI is a line which connects the towns of a province and is called the local railway. An example of a local train in the town is the one which connects Padalarang and Cicalengka subdistricts with the Bandung Raya Ekonomi Railway. The use of this local train is of high enough interest in the community because the price of train tickets is relatively cheap and it is relatively easy for the public to purchase train tickets. Train tickets can be purchased via the KAI Access application online or directly at the counter available at the local station. The entrance to the station is also very practical for the public in general, but because there are two methods of buying train tickets, it sometimes results in many queues in the ticket scanning process. In this paper, we discussed the ticket flow model for entering the Padalarang station in order to avoid any queue during the process of scanning the ticket. In addition, this paper discussed the effective time of train departure so that the time used for train operation is the optimal time. The method applied in this research is a Petri net for station entry flow modeling and the Kleene star algorithm for the effectiveness of train scheduling. The results of this study are useful for reducing queuing time before train departure and for effective train operating time. This research can be a reference for other researchers to develop a Petri net model and a reference for the government to optimize train departure time.

Keywords: *Petri net; queuing system; train; Kleene star algorithm; mathematical model.*

Mathematics Subject Classification (2020): 70K20, 93C85, 93C65, 90B35, 15A80, 68M20.

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1 Introduction

Carl Adam Petri introduced Petri nets in 1962 as a graphical modeling tool similar to flowcharts, capable of representing activities occurring simultaneously [1]. Petri nets employ modeling techniques and network analysis both qualitatively and quantitatively [2], and can be used to model algorithms for specific processes such as rail transportation from the station entry to boarding and traveling to the destination. They are powerful tools for modeling and analyzing discrete event systems and are widely applied in scheduling, planning, queuing systems, deadlock control, stochastic processes, and performance evaluation in various resource allocation contexts [3, 4].

Max-plus algebra is a powerful framework for representing and analyzing discrete event systems, particularly those involving synchronization without concurrency [5, 6]. Built upon the max-plus semiring $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with operations $\oplus$ (maximum) and $\otimes$ (addition), it extends naturally to vectors and matrices [7]. Linear time-invariant systems can also be modeled within this algebraic structure. Eigenvalue and eigenvector problems in max-plus algebra are defined similarly to classical linear algebra but utilize max-plus operations, where $A \otimes v = \lambda \otimes v$ [8]. Combined with Petri nets, max-plus algebra effectively addresses discrete event system challenges. This study applies both approaches to real-world problems, specifically to determine train departure scheduling and rail transport procedures.

Research on max-plus algebra and Petri nets has been widely conducted to address real-world problems related to scheduling and queuing systems. Its applications include improving railway maintenance planning in the UK, analyzing student enrollment patterns, managing spare parts procurement for marine engines, and optimizing queuing systems in banks and healthcare facilities [9–12]. Other studies utilize Petri nets for cyber-physical systems, quantum annealing models, flexible manufacturing systems, railway delay simulations, business process error detection, and human-agent interaction planning [13–15]. These works demonstrate the effectiveness of combining Petri nets and max-plus algebra in improving efficiency, concurrency, and optimization across diverse domains.

Previous studies have explored the use of Petri nets in general contexts, but none have addressed entrance queues in train stations under current conditions involving both online and offline ticketing systems, nor have they focused on optimizing local train departure schedules in West Java, particularly for the Bandung Raya trains. This paper investigates public rail transportation scheduling using Petri net modeling and max-plus algebra. The Petri net models the complete passenger flow—from entering the station and purchasing tickets to boarding the train and departing. Max-plus algebra, using the Kleene star algorithm, is applied to calculate effective train departure time, with a case study on the Bandung Raya Ekonomi Train from Padalarang to Rancaekek. The study results include a Petri net model that accommodates both ticketing systems and an optimized train schedule that reduces passenger waiting time. This research aims to support the government and PT KAI in developing more efficient train schedules, ultimately enhancing the appeal and effectiveness of rail-based public transportation.

This study addresses a key problem in discrete event dynamic systems, which are a fundamental subset of nonlinear dynamical systems due to their event-driven, non-continuous behavior. The queuing and scheduling processes in railway systems inherently involve nonlinear interactions, particularly when modeling passenger flow and train departure synchronization. By employing Petri nets and max-plus algebra—both beeing widely recognized tools in the analysis of nonlinear discrete systems, we provide a structured approach to model, simulate, and optimize the complex dynamics of train operations. The use of the Kleene star algorithm for computing train departure time further emphasizes the nonlinear temporal dependencies present in such systems. Thus, this research contributes to the nonlinear dynamics field by offering a mathematical and systems theoretical framework to address transport scheduling challenges, aligning with the focus on nonlinear systems and their applications.

2 Methodology

In this paper, train scheduling data from the Padalarang-Cicalengka route was retrieved using the KAI Access application, complemented by on-site observations at the nearest station. The research followed a structured flow: first, a Petri net was drawn to represent the ticket purchase and train entry process; then, this Petri net was tested and evaluated using the online tool at http://petri.hp102.ru/pnet.html. Based on the evaluation, a new Petri net was developed, which was then transformed into backward, forward, and incidence matrices. Using the incidence matrix, calculations were performed by analyzing enabled transitions. When these calculations returned to their initial form, an effective path through the Petri net was identified.

The eigenvalue and eigenvector calculation flow for a max-plus matrix involves several steps. First, compute the eigenvalues λ using the formula $\lambda = \bigotimes_{k=1}^{n} \left(\frac{1}{k}\,\mathrm{trace}\big(A^{\otimes k}\big)\right)$. Then, define $B = -\lambda \otimes A$. Next, calculate the star and plus closures of B, namely $B^* = E \oplus B \oplus \cdots \oplus B^{\otimes(n-1)}$ and $B^+ = B \otimes B^*$, where E is the max-plus identity matrix. The critical points of the associated graph $G(A)$ are identified by checking when the diagonal elements of B^+, i.e., $[B^+]_{vv}$, are equal to zero; such v indicates a critical node. The eigenvectors corresponding to the eigenvalue λ are then given by the v-th row of B^*. Finally, these eigenvalues and eigenvectors satisfy the max-plus eigen-equation $A \otimes v = \lambda \otimes v$, where A is the max-plus matrix, v is an eigenvector, and λ is the corresponding eigenvalue.

The train scheduling flow begins by calculating the eigenvalues and eigenvectors of the matrix that models the travel time between stations. These eigenvalues and eigenvectors are then used to determine a new train departure schedule, selecting a non-negative eigenvector since it corresponds to feasible scheduling. The departure time from each station is calculated using Eq. (1).

$$d(k) = d(0) \otimes \lambda^{\otimes k} = \begin{bmatrix} v_0 + k \cdot \lambda \\ v_1 + k \cdot \lambda \\ \vdots \\ v_{n-1} + k \cdot \lambda \end{bmatrix}, \tag{1}$$

where λ is an eigenvalue of the max-plus matrix A, and $v_0, v_1, \cdots, v_{n-1}$ are components of the corresponding eigenvector. The departure schedule is determined by substituting $k = 0$ for the first station, $k = 1$ for the second, and so on, up to $k = n - 1$ for the last station. This approach yields a new, optimized train schedule, incorporating assumptions about waiting time at each station along the route.

3 Results

3.1 Petri net from the flow of buying tickets

In both offline and online ticketing processes, passengers ultimately follow a similar journey to board the train, with the main difference being how the ticket is obtained. Offline ticketing requires passengers to visit the counter, present their KTP, and purchase a ticket directly from the counter keeper before proceeding to the station. In contrast, online ticketing allows passengers to buy tickets in advance via the KAI application. Upon arriving at the station in either case, a train scan officer verifies the ticket, after which passengers enter the waiting area and board the train when it arrives. Based on Figure 1, the flow of buying tickets until the passengers manage to get on the train is as follows.

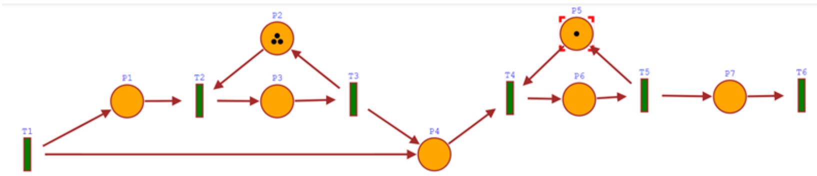

Figure 1: The Petri net from the flow of buying tickets to entering the train station.

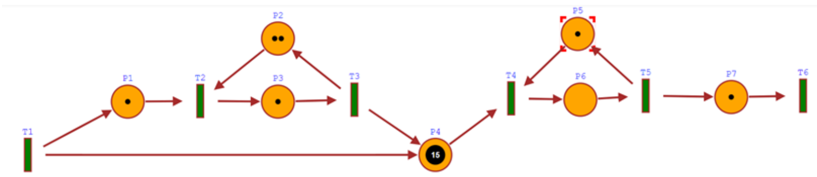

Figure 2: Evaluation of the Petri net.

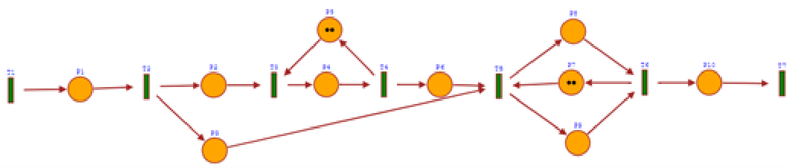

Figure 3: The new Petri net.

In Figure 1, the process begins at T1, when passengers arrive at the entrance counter (P1). At T2, they interact with the counterkeeper (P2) at the ticket sales transaction counter (P3) to purchase a ticket. After completing the transaction, T3 marks the point where passengers leave the counter and head toward the station entrance (P4). At T4, passengers move to the ticket scanning area (P6), where the scan officer (P5) verifies their tickets. Following this, at T5, passengers who have completed the scanning process proceed to the train departure station (P7). Finally, at T6, they board the train and depart from the station.

During the simulation, congestion occurred at the station entrance near the scanning area due to the formation of two separate queues—one for passengers with online tickets and another for those with offline tickets — as illustrated in Figure 2. Based on this evaluation, it is recommended to add separate scanning points for online and offline tickets to reduce congestion, as shown in Figure 3.

The process begins at T1, when passengers arrive in the station area (P1).For offline ticket buyers, T2 marks their entry through the offline ticket purchase entrance (P2), followed by T3, where they interact with the counter officer (P5) at the ticket sales transaction counter (P4) to purchase a ticket. At T4, these passengers leave the counter and head toward the ticket scanning station (P6). For online ticket buyers, the entry is through the online ticket purchase entrance (P3), and both groups proceed to T5, the ticket scanning station (P6), where scanning is conducted by the ticket scanning officer (P7). Offline and online tickets are scanned separately at P8 and P9, respectively. After successful scanning at T6, passengers move to the train departure area, and at T7, they board the train and depart the station.

The new form of Petri net assumes that passengers using the online and offline ticket purchase methods must pass through the counter staff first. So, from the shape of the Petri net, the backward, forward, and incidence matrices will be obtained as follows:

$$\tilde{A}_f = \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tilde{A}_b = \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{A} = \tilde{A}_f - \tilde{A}_b = \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

The initial state of the Petri net is $x_0 = \begin{bmatrix}0 & 2 & 0 & 0 & 0 & 2 & 0 & 0\end{bmatrix}^T$. The possible firing sequences to return to the initial state are as follows:

1. $$x_2 = \begin{bmatrix}0\\1\\1\\0\\2\\0\\2\\0\\0\\0\end{bmatrix} \xrightarrow{t_3} \begin{bmatrix}0\\0\\1\\1\\1\\0\\2\\0\\0\\0\end{bmatrix} \xrightarrow{t_4} \begin{bmatrix}0\\0\\1\\0\\2\\1\\2\\0\\0\\0\end{bmatrix} \xrightarrow{t_5} \begin{bmatrix}0\\0\\0\\0\\2\\0\\1\\1\\1\\0\end{bmatrix} \xrightarrow{t_6} \begin{bmatrix}0\\0\\0\\0\\2\\0\\2\\0\\0\\1\end{bmatrix} \xrightarrow{t_7} \begin{bmatrix}0\\0\\0\\0\\2\\0\\2\\0\\0\\0\end{bmatrix} = x_1,$$

(passengers who buy tickets offline enter the scan area to enter the station)

2. $$x_2 = \begin{bmatrix}0\\1\\1\\0\\2\\0\\2\\0\\0\\0\end{bmatrix} \xrightarrow{t_5} \begin{bmatrix}0\\0\\0\\0\\2\\0\\1\\1\\1\\0\end{bmatrix} \xrightarrow{t_6} \begin{bmatrix}0\\0\\0\\0\\2\\0\\2\\0\\0\\1\end{bmatrix} \xrightarrow{t_7} \begin{bmatrix}0\\0\\0\\0\\2\\0\\2\\0\\0\\0\end{bmatrix} = x_1$$

(passengers who buy tickets online enter the scan area to enter the station).

Based on the calculation above, there are two critical paths in the Petri net, namely $t_1 \to t_2 \to t_3 \to t_4 \to t_5 \to t_6 \to t_7$, where the path is for passengers with offline ticket purchases, and $t_1 \to t_2 \to t_5 \to t_6 \to t_7 \to t_8$, where the path is for passengers with online ticket purchases. With the simulation conducted, it is found that the Petri net that has been made is an effective Petri net because there is no long queue when the officer scans the ticket.

3.2 Train departure scheduling using the Kleene star algorithm

In Figure 4, the stations included in the route are as follows: PDL refers to Padalarang Station, GK to Gadobangkong Station, CMI to Cimahi Station, CMD to Cimindi Station, CIR to Ciroyom Station, BD to Bandung Station, CTH to Cikudapateuh Station, KAC to Kiaracondong Station, CMK to Cimekar Station, RCK to Rancaekek Station, HRP to Haurpugur Station, and CCL to Cicalengka Station.

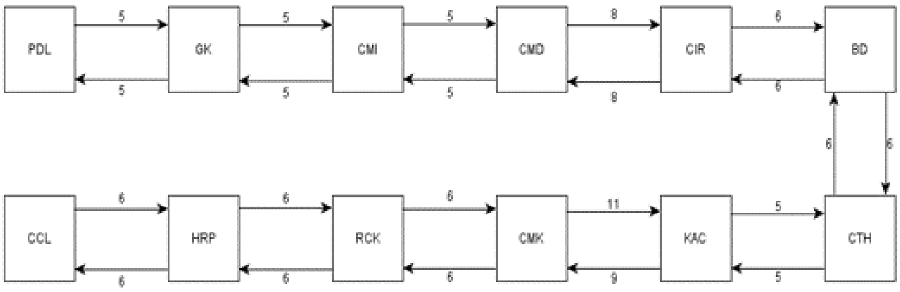

Figure 4: Graph of the Cicalengka Train Station - Padalarang Station route.

The model is based on several assumptions: there are two trains operating on the route; the travel time for each train between stations is fixed; arrival, passenger drop-off, boarding, and departure time are assumed to be zero; and each train's departure must wait for the arrival of the preceding train from the previous departure.

For example:
$x(k)$: The $k-th$ departure time of all trains
$d(k)$: Schedule of the $k-th$ departure of all trains
$x_1(k)$: The $k-th$ departure time from PDL station to GK station
$x_2(k)$: The $k-th$ departure time from GK station to CMI station
$x_3(k)$: The $k-th$ departure time from CMI station to CMD station
$\vdots$
$x_{22}(k)$: The $k-th$ departure time from GK station to PDL station.

So, from Figure 4, it can be modeled as follows :

$$\begin{aligned} x_1(k+1) &= (5 \otimes x_{22}(k)) \oplus d_1(k+1), \\ x_2(k+1) &= (5 \otimes x_1(k)) \oplus d_2(k+1), \\ x_3(k+1) &= (5 \otimes x_2(k)) \oplus d_3(k+1), \\ &\vdots \\ x_{22}(k+1) &= (5 \otimes x_{21}(k)) \oplus d_{22}(k+1). \end{aligned}$$

So, the matrix can be formed as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{22}(k+1) \end{bmatrix} = \tilde{A} \otimes \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{22}(k) \end{bmatrix} \oplus \vec{d}(k+1), \tag{2}$$

where

$$\tilde{A} = \begin{bmatrix}
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 5 \\
5 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & 5 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & 5 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & 8 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
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\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 9 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
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\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 11 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 5 & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 6 & \lambda & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 8 & \lambda & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 5 & \lambda & \lambda \\
\lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & \lambda & 5 & \lambda
\end{bmatrix}.$$

After doing calculations using Python, the eigenvalue is $\frac{68}{11} = 6.1818181818$, and the positive eigenvector is $\begin{bmatrix} 3.545454 \\ 2.363636 \\ 1.181818 \\ 0 \\ 1.818181 \\ 1.636363 \\ 1.454545 \\ 2.272727 \\ 3.090909 \\ 2.909090 \\ 2.727272 \\ 2.545454 \\ 2.363636 \\ 2.181818 \\ 2 \\ 6.818181 \\ 5.636363 \\ 5.454545 \\ 5.272727 \\ 7.090909 \\ 5.909090 \\ 4.727272 \end{bmatrix} = \begin{bmatrix} \frac{39}{11} \\ \frac{26}{11} \\ \frac{13}{11} \\ 0 \\ \frac{20}{11} \\ \frac{18}{11} \\ \frac{16}{11} \\ \frac{3}{11} \\ \frac{34}{11} \\ \frac{32}{11} \\ \frac{30}{11} \\ \frac{28}{11} \\ \frac{26}{11} \\ \frac{24}{11} \\ 2 \\ \frac{75}{11} \\ \frac{62}{11} \\ \frac{60}{11} \\ \frac{58}{11} \\ \frac{78}{11} \\ \frac{65}{11} \\ \frac{52}{11} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 0 \\ 2 \\ 2 \\ 2 \\ 1 \\ 4 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 2 \\ 7 \\ 6 \\ 6 \\ 6 \\ 8 \\ 6 \\ 5 \end{bmatrix}.$

After we got the eigenvalue and eigenvector, then the calculated train departure schedules from each station can be

$$d(k) = d(0) \otimes 6^{\otimes k} = \begin{bmatrix} v_0 + (6 \cdot \lambda) \\ v_1 + (6 \cdot \lambda) \\ \vdots \\ v_{22} + (6 \cdot \lambda) \end{bmatrix}. \tag{3}$$

Table 1 and Table 2 present the optimal departure schedules for Bandung Raya local trains at Padalarang Station and Cicalengka Station, respectively. These schedules were obtained through a mathematical optimization process aimed at minimizing delays and maximizing efficiency within the current railway operation framework. The schedules generated from this optimization model take into account various operational constraints and objectives to ensure that train departures are well-coordinated.

Railway Station	Railway Schedule							
PDL	00.04	01.34	03.04	04.34	06.04	07.34	09.04	10.34
GD	00.10	01.40	03.10	04.40	06.10	07.40	09.10	10.40
CMI	00.16	01.46	03.16	04.46	06.16	07.46	09.16	10.46
CMD	00.21	01.51	03.21	04.51	06.21	07.51	09.21	10.51
CRM	00.30	02.00	03.30	05.00	06.30	08.00	09.30	11.00
BD	00.37	02.07	03.37	05.07	06.37	08.07	09.37	11.07
CKD	00.44	02.14	03.44	05.14	06.44	08.14	09.44	11.14
KAC	00.50	02.20	03.50	05.20	06.50	08.20	09.50	11.20
CMR	01.00	02.30	04.00	05.30	07.00	08.30	10.00	11.30
RCK	01.06	02.36	04.06	05.36	07.06	08.36	10.06	11.36
HPG	01.13	02.43	04.13	05.43	07.13	08.43	10.13	11.43
CCK	01.19	02.49	04.19	05.49	07.19	08.49	10.19	11.49
PDL	12.04	13.34	15.04	16.34	18.04	19.34	21.04	22.34
GD	12.10	13.40	15.10	16.40	18.10	19.40	21.10	22.40
CMI	12.16	13.46	15.16	16.46	18.16	19.46	21.16	22.46
CMD	12.21	13.51	15.21	16.51	18.21	19.51	21.21	22.51
CRM	12.30	14.00	15.30	17.00	18.30	20.00	21.30	23.00
BD	12.37	14.07	15.37	17.07	18.37	20.07	21.37	23.07
CKD	12.44	14.14	15.44	17.14	18.44	20.14	21.44	23.14
KAC	12.50	14.20	15.50	17.20	18.50	20.20	21.50	23.20
CMR	13.00	14.30	15.00	17.30	19.00	20.30	22.00	23.30
RCK	13.06	14.36	15.06	17.36	19.06	20.36	22.06	23.36
HPG	13.13	14.43	15.13	17.43	19.13	20.43	22.13	23.43
CCK	13.19	14.49	15.19	17.49	19.19	20.49	22.19	23.49

Table 1: The new train schedule from Padalarang to Cicalengka Station.

When compared with the existing schedules available through the KAI Access platform, the optimized results show only minor differences. This close alignment between the calculated and actual schedules indicates that the current operational departure times used by the railway operator are already functioning near an optimal level.Thus, the results validate the effectiveness of the existing train scheduling practices at both Padalarang and Cicalengka Stations.

Railway Station	Railway Schedule							
CCK	00.04	01.34	03.04	04.34	06.04	07.34	09.04	10.34
HPG	00.11	01.41	03.11	04.41	06.11	07.41	09.11	10.41
RCK	00.18	01.48	03.18	04.48	06.18	07.48	09.18	10.48
CMR	00.24	01.54	03.24	04.54	06.24	07.54	09.24	10.54
KAC	00.36	02.06	03.36	05.06	06.36	08.06	09.36	11.06
CKD	00.42	02.12	03.42	05.12	06.42	08.12	09.42	11.12
BD	00.49	02.19	03.49	05.19	06.49	08.19	09.49	11.19
CRM	00.56	02.26	03.56	05.26	06.56	08.26	09.56	11.26
CMD	01.05	02.35	04.05	05.35	07.05	08.35	10.05	11.35
CMI	01.10	02.40	04.10	05.40	07.10	08.40	10.10	11.40
GD	01.16	02.46	04.16	05.46	07.16	08.46	10.16	11.46
PDL	01.21	02.51	04.21	05.51	07.21	08.51	10.21	11.51
CCK	12.04	13.34	15.04	16.34	18.04	19.34	21.04	22.34
HPG	12.11	13.41	15.11	16.41	18.11	19.41	21.11	22.41
RCK	12.18	13.48	15.18	16.48	18.18	19.48	21.18	22.48
CMR	12.24	13.54	15.24	16.54	18.24	19.54	21.24	22.54
KAC	12.36	14.06	15.36	17.06	18.36	20.06	21.36	23.06
CKD	12.42	14.12	15.42	17.12	18.42	20.12	21.42	23.12
BD	12.49	14.19	15.49	17.19	18.49	20.19	21.49	23.19
CRM	12.56	14.26	15.56	17.26	18.56	20.26	21.56	23.26
CMD	13.05	14.35	15.05	17.35	19.05	20.35	22.05	23.35
CMI	13.10	14.40	15.10	17.40	19.10	20.40	22.10	23.40
GD	13.16	14.46	15.16	17.46	19.16	20.46	22.16	23.46
PDL	13.21	14.51	15.21	17.51	19.21	20.51	22.21	23.51

Table 2: The new train schedule from Cicalengka to Padalarang Station.

4 Discussion

The application of Petri nets in modeling the passenger entry flow at train stations provides an effective framework for visualizing and analyzing discrete event systems. In this study, the passenger process from ticket purchase to boarding was modeled using a Petri net that distinguishes between online and offline ticket buyers. Simulation results revealed congestion at the ticket scanning area, particularly due to the merging of online and offline passenger flows. By refining the Petri net to include separate scanning points, the revised model demonstrated reduced queuing, thus suggesting a more efficient station layout and process. These findings highlight the utility of Petri nets in optimizing station entry systems and guiding infrastructure improvements.

The use of the Kleene Star algorithm and Max-Plus Algebra provided a mathematical basis for calculating an optimal train departure schedule. By modeling travel time between stations using a max-plus matrix, the eigenvalues and eigenvectors were computed to determine departure intervals that minimize delays and maximize efficiency. The new schedule, compared against real data from KAI Access, showed a high level of alignment with current operational times, reinforcing the accuracy of the model. This not only validates the Kleene Star approach but also confirms its practical potential for supporting real-world railway scheduling systems.

Furthermore, the integration of Petri nets and max-plus algebra reveals a promising interdisciplinary methodology for transport systems analysis. The Petri net effectively captures system behavior and interaction among components, while the max-plus algebra offers a structured approach for computing optimal timing. This combination enables the modeling of complex queuing behavior and time synchronization, which are critical in public transport environments. As demonstrated in the case of the Bandung Raya local train line, this approach can be tailored to other routes and regions with similar scheduling constraints and commuter behaviors.

The proposed models also offer policy implications for PT KAI and local government transportation agencies. By identifying critical points of congestion and providing data-driven departure scheduling, the methods in this study can be used to enhance commuter satisfaction and system efficiency. In particular, separating the service flow between online and offline ticket users may be a low-cost intervention with significant impact. Future work may include implementing real-time data and machine learning integration to further refine the models and adapt them dynamically based on daily traffic fluctuations or unexpected delays.

5 Conclusion

In this paper, we have used a combined modeling framework that integrates Petri nets for station entry flow and the Kleene star algorithm within max-plus algebra for train departure scheduling—an approach that, to our knowledge, has not been applied in previous studies of railway systems. The novelty also lies in the modeling of parallel online and offline ticketing flows as distinct Petri net structures, which reflect operational conditions and directly inform infrastructure improvements. Furthermore, the derived train schedule aligns with existing KAI departure data, indicating that the model is both theoretically rigorous and practically applicable. These contributions offer a new methodology for optimizing discrete-event transportation systems within the field of nonlinear dynamics.

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References

[1] Y. Lei and H. Mu. Analysis and optimization of a stochastic Petri net for air–rail intermodal transportation. *PLOS One* **19** (7) (2024) e0307647.

[2] I. Kabashkin and Z. Sansyzbayeva. Methodological framework for sustainable transport corridor modeling using Petri nets. *Sustainability* **16** (2) (2024) 489.

[3] A. Boukoftane, M. A. Kalwar, N. Oukid, K. Zennir, M. S. Memon and M. A. Khan. Application of discrete event simulation and system dynamics modeling in optimizing the perfor-

mance of outpatient department. *Nonlinear Dynamics and Systems Theory* **24** (4) (2024) 354–365.

[4] A. Chahad, M. Bassoudi and Z. C. Elmezouar. Spectral density estimation in time series analysis for dynamical systems. *Nonlinear Dynamics and Systems Theory* **25** (2) (2025) 128–143.

[5] A. Sambas, X. Zhang, I. A. Moghrabi, S. Vaidyanathan, K. Benkouider, M. Alçın, M. Tuna and M. D. Johansyah. ANN-based chaotic PRNG in the novel jerk chaotic system and its application for image encryption via 2-D Hilbert curve. *Scientific Reports* **14** (1) (2024) 29602.

[6] A. Sambas, M. Miroslav, S. Vaidyanathan, B. Ovilla-Martínez, E. Tlelo-Cuautle, A. A. Abd El-Latif and T. Bonny. A new hyperjerk system with a half line equilibrium: multistability, period-doubling reversals, antimonotonicity, electronic circuit, FPGA design, and an application to image encryption. *IEEE Access* **12** (2024) 9177–9194.

[7] S. Gaubert and S. Sergeev. The level set method for the two-sided max-plus eigenproblem. *Discrete Event Dynamic Systems* **23** (2) (2013) 105–134.

[8] M. Umer, U. Hayat, F. Abbas, A. Agarwal and P. Kitanov. An efficient algorithm for eigenvalue problem of Latin squares in a bipartite min-max-plus system. *Symmetry* **12** (2) (2020) 311.

[9] N. Gkonou, E. Nisyrios and K. Gkiotsalitis. Combined optimization of maintenance works and crews in railway networks. *Applied Sciences* **13** (18) (2023) 10503.

[10] M. Raji, J. Duggan, B. DeCotes, J. Huang and B. Vander Zanden. Modeling and visualizing student flow. *IEEE Transactions on Big Data* **7** (3) (2018) 510–523.

[11] F. Azizah and S. Subiono. Max-plus algebra and Petri net application on scheduling of ship engine component's spare part ordering. *International Journal of Computing Science and Applied Mathematics* **4** (1) (2018) 1–9.

[12] T. Gonsalves and K. Itoh. Service optimization with patient satisfaction in healthcare systems. *Journal of Simulation* **3** (3) (2009) 150–162.

[13] H. Orojloo and M. A. Azgomi. Modelling and evaluation of the security of cyber-physical systems using stochastic Petri nets. *IET Cyber-Physical Systems: Theory & Applications* **4** (1) (2019) 50–57.

[14] M. Nakamura, K. Kaneshima and T. Yoshida. Petri net modeling for Ising model formulation in quantum annealing. *Applied Sciences* **11** (16) (2021) 7574.

[15] Y. Yang, J. Yang, N. Liang and C. Zhong. Control law for two-process flexible manufacturing systems modeled using Petri nets. *Mathematics* **13** (4) (2025) 611.