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A Taylor Collocation Approach for Solving Systems of Two-Dimensional Volterra Integral Equations

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Abstract: This work presents a numerical approach for solving systems of linear two-dimensional Volterra integral equations (2D-VIEs), which frequently arise in modeling dynamical systems with memory effects, including applications in control theory, population dynamics, and epidemic modeling. An algorithm based on Taylor polynomials is developed to construct a collocation solution for these systems. The convergence of the proposed method is established, ensuring accurate and efficient approximations while preserving the integral structure of the system. Numerical examples are provided to illustrate the accuracy and applicability of the method in solving problems relevant to systems theory.

Keywords: *system of two-dimensional Volterra integral equations; spline approximation; power series; error analysis.*

Mathematics Subject Classification (2020): 41A15, 45D05, 45L05, 65R20, 37N25.

1 Introduction

The numerical solution of integral equations, particularly systems of Volterra integral equations (VIEs), is crucial for modeling and solving complex problems in science and engineering. Such equations frequently arise in fluid dynamics, signal processing, control theory, mathematical biology, and various other areas of nonlinear dynamics and systems theory. For instance, in nonlinear dynamical systems, VIEs are used to describe systems with memory effects, such as in population dynamics, where the interaction between

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species or components is influenced. Similarly, in systems theory, VIEs provide a framework for modeling the evolution of systems under nonlinear influences, such as in control theory, where the system's behavior over time is influenced by feedback mechanisms that are inherently memory-dependent.

Numerous numerical methods have been developed for one-dimensional cases, including the modified homotopy perturbation method [5], the semi-orthogonal B-spline collocation method [11], and the Hybrid Legendre Block-Pulse functions method [12]. Research on computational methods for solving systems of two-dimensional integral equations (2D-IEs) remains limited.

This paper further develops the Taylor collocation method (TCM), as introduced in [1–3, 6], to solve (1). The TCM provides a direct and reliable framework for addressing such systems. Unlike traditional techniques that transform equations into algebraic systems, the TCM leverages Taylor series expansions to approximate solutions directly in their integral form. By exploiting the inherent smoothness and flexibility of Taylor series, the method ensures accurate approximations while preserving the structure of the original system.

The structure of the paper is as follows. Section 2 introduces the approximation of solutions for 2D VIEs using the polynomial spline space $S^{(-1)}_{p-1,p-1}$ in conjunction with Taylor polynomials. Section 3 provides a detailed convergence analysis. Section 4 presents numerical examples to validate and illustrate the theoretical findings. Finally, the paper concludes with a summary of the research findings and directions for future work.

2 Description of the Method

We aim to solve a linear system of two-dimensional Volterra integral equations (2D VIEs) of the form

$$\begin{cases} u_1(x,y) = g_1(x,y) + \int_0^x \int_0^y \left[K_{1,1}(x,y,t,s)u_1(t,s) + K_{1,2}(x,y,t,s)u_2(t,s)\right] dsdt, \\ u_2(x,y) = g_2(x,y) + \int_0^x \int_0^y \left[K_{2,1}(x,y,t,s)u_1(t,s) + K_{2,2}(x,y,t,s)u_2(t,s)\right] dsdt, \end{cases} \tag{1}$$

where $(x, y) \in D$, and the functions g_d and $K_{d,\delta}$, with $d, \delta = 1, 2$, are given continuous real-valued functions defined on $D = [0, a] \times [0, b] \subset \mathbb{R}^2$, $S = \{(x, y, t, s) : 0 \leq t \leq x \leq a, 0 \leq s \leq y \leq b\}$.

The existence and uniqueness of solutions for systems of 2D VIEs have been established in [10, 13].

To numerically solve these equations, let $\Pi_N = \{x_i = ih, i = 0, 1, \dots, N\}$ and $\Pi_M = \{y_j = jk, j = 0, 1, \dots, M\}$ represent uniform partitions of the intervals $[0, a]$ and $[0, b]$, with step sizes $h = \frac{a}{N}$ and $k = \frac{b}{M}$. These partitions form a grid for D given by

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(x_n, y_m) : 0 \leq n \leq N, 0 \leq m \leq M\}.$$

Define the subintervals

$$\sigma_n = [x_n, x_{n+1}), \quad n = 0, 1, \dots, N-2; \quad \sigma_{N-1} = [x_{N-1}, x_N],$$
$$\delta_m = [y_m, y_{m+1}), \quad m = 0, 1, \dots, M-2; \quad \delta_{M-1} = [y_{M-1}, y_M],$$

and the rectangular regions

$$D_{n,m} := \sigma_n \times \delta_m, \quad n = 0, \dots, N-1; \quad m = 0, \dots, M-1.$$

Additionally, let $\pi_{p-1,p-1}$ denote the set of all real polynomials of degree at most $p-1$ in both x and y. The polynomial spline space of degree $p-1$ in x and y is defined as

$$S^{(-1)}_{p-1,p-1}(\Pi_{N,M}) = \{(v_1, v_2)^t : (v_{1,n,m}, v_{2,n,m})^t = (v_1, v_2)^t|_{D_{n,m}} \in \pi_{p-1,p-1},\\ n = 0, \dots, N-1;\ m = 0, \dots, M-1\}.$$

This space consists of bivariate polynomial spline functions of degree at most $p-1$ in x and y. It has dimension $2NMp^2$, equal to the total number of coefficients of the polynomials $v_{1,n,m}$ and $v_{2,n,m}$. To determine these coefficients, we use Taylor polynomials on each rectangular region $D_{n,m}$.

The solution u of (1) is known along part of the boundary of D

$$u(x, y) = g(x, 0), \quad \text{for } 0 \le x \le a \text{ and } y = 0,$$
$$u(x, y) = g(0, y), \quad \text{for } 0 \le y \le b \text{ and } x = 0.$$

First, we approximate u in the rectangle $D_{0,0}$ by the polynomials

$$v_{d,0,0}(x, y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} u_d(0, 0)}{\partial x^i \partial y^j} x^i y^j \ ; \quad (x, y) \in D_{0,0}, d = 1, 2, \tag{2}$$

where $\dfrac{\partial^{i+j} u_d(0, 0)}{\partial x^i \partial y^j}$ is the exact value of $\dfrac{\partial^{i+j} u_d}{\partial x^i \partial y^j}$ at point $(0, 0)$.

To find $\dfrac{\partial^{i+j} u_1(x, y)}{\partial y^j}$ and $\dfrac{\partial^{i+j} u_2(x, y)}{\partial y^j}$, we differentiate (1) j times with respect to y, then i times with respect to x, we obtain

$$\begin{aligned}
\frac{\partial^{i+j} u_d(x, y)}{\partial x^i \partial y^j} &= \partial_1^{(i)} \partial_2^{(j)} g_d(x, y) + \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{r}{l} \binom{q}{\eta} \times \\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}} \left[\partial_2^{(j-1-r)} K_{d,1}(x, y, t, y) \right] \right) \right] \frac{\partial^{\eta+l} u_1(x, y)}{\partial x^\eta \partial y^l} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}} \left[\partial_2^{(j-1-r)} K_{d,2}(x, y, t, y) \right] \right) \right] \frac{\partial^{\eta+l} u_2(x, y)}{\partial x^\eta \partial y^l} \\
&+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_0^x \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,1}(x, y, t, y)] \right] \frac{\partial^l u_1(t, y)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,2}(x, y, t, y)] \right] \frac{\partial^l u_2(t, y)}{\partial y^l} dt \\
&+ \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,1}(x, y, x, s) \right] \frac{\partial^\eta u_1(x, s)}{\partial x^\eta} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,2}(x, y, x, s) \right] \frac{\partial^\eta u_2(x, s)}{\partial x^\eta} ds \\
&+ \int_0^x \int_0^y \partial_1^{(i)} \partial_2^{(j)} K_{d,1}(x, y, t, s) u_1(t, s) + \partial_1^{(i)} \partial_2^{(j)} K_{d,2}(x, y, t, s) u_2(t, s) ds dt.
\end{aligned}$$

Second, we approximate u in the rectangles $D_{n,0}, n = 1, ..., N-1$, by the polynomials

$$v_{d,n,0}(x,y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{v}_{d,n,0}(x_n, 0)}{\partial x^i \partial y^j} (x - x_n)^i y^j \ ; \quad d = 1, 2; (x, y) \in D_{n,0}, \tag{3}$$

where $\hat{v}_{d,n,0}$, $d = 1, 2$ is the exact solution of the system

$$\begin{aligned} \hat{v}_{d,n,0}(x,y) &= g_d(x,y) \\ &+ \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_0^y K_{d,1}(x,y,t,s) v_{1,\xi,0}(t,s) + K_{d,2}(x,y,t,s) v_{2,\xi,0}(t,s) dsdt \\ &+ \int_{x_n}^{x} \int_0^y K_{d,1}(x,y,t,s) \hat{v}_{1,n,0}(t,s) + K_{d,2}(x,y,t,s) \hat{v}_{2,n,0}(t,s) dsdt. \end{aligned} \tag{4}$$

To find $\dfrac{\partial^{i+j} \hat{v}_{d,n,0}(x,y)}{\partial y^j}$, we differentiate (4) j times with respect to y, then i times with respect to x, we obtain

$$\begin{aligned} &\frac{\partial^{i+j} \hat{v}_{d,n,0}(x,y)}{\partial x^i \partial y^j} = \partial_1^{(i)} \partial_2^{(j)} g_d(x,y) \\ &+ \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,1}(x,y,t,y)] \right] \frac{\partial^l v_{1,\xi,0}(t,y)}{\partial y^l} \\ &+ \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,2}(x,y,t,y)] \right] \frac{\partial^l v_{2,\xi,0}(t,y)}{\partial y^l} dt \\ &+ \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_0^y \partial_1^{(i)} \partial_2^{(j)} K_{d,1}(x,y,t,s) v_{1,\xi,0}(t,s) + \partial_1^{(i)} \partial_2^{(j)} K_{d,2}(x,y,t,s) v_{2,\xi,0}(t,s) dsdt \\ &+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{r}{l} \binom{q}{\eta} \times \\ &\frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\left. \frac{\partial^{i-1-q}}{\partial x^{i-1-q}} \right|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}} \left[\partial_2^{(j-1-r)} K_{d,1}(x,y,t,y) \right] \right) \right] \frac{\partial^{\eta+l} \hat{v}_{1,n,0}(x,y)}{\partial x^\eta \partial y^l} \\ &+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\left. \frac{\partial^{i-1-q}}{\partial x^{i-1-q}} \right|_{t=x} \left(\frac{\partial^{r-l}}{\partial y^{r-l}} \left[\partial_2^{(j-1-r)} K_{d,2}(x,y,t,y) \right] \right) \right] \frac{\partial^{\eta+l} \hat{v}_{2,n,0}(x,y)}{\partial x^\eta \partial y^l} \\ &+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_{x_n}^{x} \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,1}(x,y,t,y)] \right] \frac{\partial^l \hat{v}_{1,n,0}(t,y)}{\partial y^l} \\ &+ \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,2}(x,y,t,y)] \right] \frac{\partial^l \hat{v}_{2,n,0}(t,y)}{\partial y^l} dt \\ &+ \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^y \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,1}(x,y,x,s) \right] \frac{\partial^\eta \hat{v}_{1,n,0}(x,s)}{\partial x^\eta} \\ &+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,2}(x,y,x,s) \right] \frac{\partial^\eta \hat{v}_{2,n,0}(x,s)}{\partial x^\eta} ds \\ &+ \int_{x_n}^{x} \int_0^y \partial_1^{(i)} \partial_2^{(j)} K_{d,1}(x,y,t,s) \hat{v}_{1,n,0}(t,s) + \partial_1^{(i)} \partial_2^{(j)} K_{d,2}(x,y,t,s) \hat{v}_{2,n,0}(t,s) dsdt. \end{aligned}$$

Third, we approximate u in the rectangle $D_{n,m}, n = 0, ..., N-1$ and $m = 1, ..., M-1$ by, the polynomials

$$v_{d,n,m}(x,y) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{v}_{d,n,m}(x_n, y_m)}{\partial x^i \partial y^j} (x - x_n)^i (y - y_m)^j \ ; \quad d = 1, 2; (x,y) \in D_{n,m}, \tag{5}$$

where $\hat{v}_{d,n,m}$, $d = 1, 2$, is the exact solution of the system

$$\begin{aligned}
&\hat{v}_{d,n,m}(x,y) = g_d(x,y) \\
&+ \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \int_{x_\xi}^{x_{\xi+1}} \int_{y_\rho}^{y_{\rho+1}} K_{d,1}(x,y,t,s) v_{1,\xi,\rho}(t,s) + K_{d,2}(x,y,t,s) v_{2,\xi,\rho}(t,s) ds dt \\
&+ \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_{y_m}^{y} K_{d,1}(x,y,t,s) v_{1,\xi,m}(t,s) + K_{d,2}(x,y,t,s) v_{2,\xi,m}(t,s) ds dt \\
&+ \sum_{\rho=0}^{m-1} \int_{x_n}^{x} \int_{y_\rho}^{y_{\rho+1}} K_{d,1}(x,y,t,s) v_{1,n,\rho}(t,s) + K_{d,2}(x,y,t,s) v_{2,n,\rho}(t,s) ds dt \\
&+ \int_{x_n}^{x} \int_{y_m}^{y} K_{d,1}(x,y,t,s) \hat{v}_{1,n,m}(t,s) + K_{d,2}(x,y,t,s) \hat{v}_{2,n,m}(t,s) ds dt.
\end{aligned} \tag{6}$$

To find $\dfrac{\partial^{i+j} \hat{v}_{d,n,m}(x,y)}{\partial y^j}$, we differentiate (6) j times with respect to y, then i times with respect to x, we obtain

$$\begin{aligned}
&\frac{\partial^{i+j} \hat{v}_{d,n,m}(x,y)}{\partial x^i \partial y^j} = \partial_1^{(i)} \partial_2^{(j)} g_d(x,y) \\
&+ \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \int_{x_\xi}^{x_{\xi+1}} \int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)} \partial_2^{(j)} \left(K_{d,1}(x,y,t,s) v_{1,\xi,\rho}(t,s) + K_{d,2}(x,y,t,s) v_{2,\xi,\rho}(t,s) \right) ds dt \\
&+ \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_{x_\xi}^{x_{\xi+1}} \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,1}(x,y,t,y)] \right] \frac{\partial^l v_{1,\xi,m}(t,y)}{\partial y^l} \\
&+ \frac{\partial^i}{\partial x^i} \left[\frac{\partial^{r-l}}{\partial y^{r-l}} [\partial_2^{(j-1-r)} K_{d,2}(x,y,t,y)] \right] \frac{\partial^l v_{2,\xi,m}(t,y)}{\partial y^l} dt \\
&+ \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_{y_m}^{y} \partial_1^{(i)} \partial_2^{(j)} K_{d,1}(x,y,t,s) v_{1,\xi,m}(t,s) + \partial_1^{(i)} \partial_2^{(j)} K_{d,2}(x,y,t,s) v_{2,\xi,m}(t,s) ds dt \\
&+ \sum_{\rho=0}^{m-1} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_{y_\rho}^{y_{\rho+1}} \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,1}(x,y,x,s) \right] \frac{\partial^\eta v_{1,n,\rho}(x,s)}{\partial x^\eta} \\
&+ \frac{\partial^{q-\eta}}{\partial x^{q-\eta}} \left[\partial_1^{(i-1-q)} \partial_2^{(j)} K_{d,2}(x,y,x,s) \right] \frac{\partial^\eta v_{2,n,\rho}(x,s)}{\partial x^\eta} ds \\
&+ \sum_{\rho=0}^{m-1} \int_{x_n}^{x} \int_{y_\rho}^{y_{\rho+1}} \partial_1^{(i)} \partial_2^{(j)} K_{d,1}(x,y,t,s) v_{1,n,\rho}(t,s) + \partial_1^{(i)} \partial_2^{(j)} K_{d,2}(x,y,t,s) v_{2,n,\rho}(t,s) ds dt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\times\\
&\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K_{d,1}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\hat{v}_{1,n,m}(x,y)}{\partial x^{\eta}\partial y^{l}}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial x^{i-1-q}}\bigg|_{t=x}\left(\frac{\partial^{r-l}}{\partial y^{r-l}}\left[\partial_2^{(j-1-r)}K_{d,2}(x,y,t,y)\right]\right)\right]\frac{\partial^{\eta+l}\hat{v}_{2,n,m}(x,y)}{\partial x^{\eta}\partial y^{l}}\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{x_n}^{x}\frac{\partial^{i}}{\partial x^{i}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K_{d,1}(x,y,t,y)]\right]\frac{\partial^{l}\hat{v}_{1,n,m}(t,y)}{\partial y^{l}}\\
&+\frac{\partial^{i}}{\partial x^{i}}\left[\frac{\partial^{r-l}}{\partial y^{r-l}}[\partial_2^{(j-1-r)}K_{d,2}(x,y,t,y)]\right]\frac{\partial^{l}\hat{v}_{2,n,m}(t,y)}{\partial y^{l}}dt\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_m}^{y}\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K_{d,1}(x,y,x,s)\right]\frac{\partial^{\eta}\hat{v}_{1,n,m}(x,s)}{\partial x^{\eta}}\\
&+\frac{\partial^{q-\eta}}{\partial x^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}K_{d,2}(x,y,x,s)\right]\frac{\partial^{\eta}\hat{v}_{2,n,m}(x,s)}{\partial x^{\eta}}ds\\
&+\int_{x_n}^{x}\int_{y_m}^{y}\partial_1^{(i)}\partial_2^{(j)}K_{d,1}(x,y,t,s)\hat{v}_{1,n,m}(t,s)+\partial_1^{(i)}\partial_2^{(j)}K_{d,2}(x,y,t,s)\hat{v}_{2,n,m}(t,s)dsdt.
\end{aligned}
$$

3 Analysis of Convergence and Numerical Error

We consider the space $L^\infty(D)$ with the norm

$$\|\varphi\| = \inf\{C\in\mathbb{R} : |\varphi(\tau,z)|\le C \text{ for a.e. } (\tau,z)\in D\} < \infty.$$

The following lemmas will be used in proving the convergence of the presented method.

Lemma 3.1 (*Discrete Gronwall type inequality [4]*) *Let $\{k_j\}_{j=0}^{n}$ be a given non negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0\le p_0$ and*

$$\varepsilon_n \le p_0 + \sum_{i=0}^{n-1} k_i\varepsilon_i, \quad n\ge 1,$$

with $p_0\ge 0$. Then ε_n can be bounded by

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n\ge 1.$$

Lemma 3.2 (*Discrete Gronwall type inequality of two variables [8]*) *Let $u_{n,m}$ be a given non negative sequence, and let $K_i,(i=1,2,3)$ and Δ be strictly positive. If the sequence $u_{n,m}$ satisfies, for all $n=0,1,...,N, m=0,1,...,M$,*

$$u_{n,m} \le hK_1\sum_{\xi=0}^{n-1}u_{\xi,m} + kK_2\sum_{\rho=0}^{m-1}u_{n,\rho} + hkK_3\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}u_{\xi,\rho}+\Delta,$$

then $u_{n,m}\le \Delta exp(\gamma(Nh+Mk))$, where $\gamma=\frac{1}{2}\left(K_1+K_2+\sqrt{(K_1+K_2)^2+4K_3}\right)$.

Lemma 3.3 (*Wendroff's inequality [14]*) *Let f and u be continuous and non negative functions defined on $[a,b]\times[c,d]$, and let g be non negative and twice continuously differentiable function on $[a,b]\times[c,d]\times[a,b]\times[c,d]$. Then the inequality*

$$u(x,y)\le f(x,y)+\int_a^x\int_c^y g(x,y,t,s)u(t,s)dsdt,\quad (x,y)\in[a,b]\times[c,d],$$

implies that

$$u(x,y)\le f(x,y)\exp\left(\int_a^x\int_c^y R(t,s)dsdt\right),\quad (x,y)\in[a,b]\times[c,d],$$

where

$$\begin{aligned}R(x,y)=&g(x,y,x,y)+\int_a^x\partial_1 g(x,y,t,y)dt\\&+\int_b^y\partial_2 g(x,y,x,s)ds+\int_a^x\int_c^y\partial_2\partial_1 g(x,y,t,s)dsdt.\end{aligned}$$

Theorem 3.1 (*Taylor's Theorem for functions of two independent variables [9]*) *Let f be p times continuously differentiable on $D=[a,b]\times[c,d]$ and let $(x_0,y_0)\in D$. Then for all $(x,y)\in D$, we have*

$$f(x,y)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}f(x_0,y_0)}{\partial x^i\partial y^j}(x-x_0)^i(y-y_0)^j+\sum_{i+j=p}\frac{1}{i!j!}\frac{\partial^{i+j}f(x_1,y_1)}{\partial x^i\partial y^j}(x-x_0)^i(y-y_0)^j,$$

where

$$\begin{cases}x_1=\theta x+(1-\theta)x_0\in[a,b],\\ y_1=\theta y+(1-\theta)y_0\in[c,d],\end{cases}\quad \theta\in(0,1).$$

In the following, for a given function $\varphi\in C(D,\mathbb{R}^2)$, we define the norm $\|\varphi\|$ by

$$\|\varphi\|=\max\{\ |\varphi_d(x,y)|,\quad (x,y)\in D,\quad d=1,2\}.$$

Theorem 3.2 *Let g and K be p times continuously differentiable on their respective domains. Then equations (2),(3),(5) define a unique approximation $v\in(S^{(-1)}_{p-1,p-1}(\Pi_{N,M}))^2$, and the resulting error function $e(x,y)=u(x,y)-v(x,y)$ satisfies*

$$\|e\|\le C(h+k)^p,$$

where C is a finite constant independent of h, k.

Proof. The proof is split into three steps.
Claim 1. Define the error $e_0=e|_{D_{0,0}}$ by $e_{d,0,0}(x,y)=u_d(x,y)-v_{d,0,0}(x,y)$, for $d=1,2$. There exists a constant C_1 independent of h and k such that $\|e_0\|\le C_1(h+k)^p$.

Let $(x,y)\in D_{0,0}$, by using Taylor's theorem, we obtain from (2)

$$|e_{d,0,0}(x,y)|\le\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}u_d}{\partial x^i\partial y^j}\right\|h^ik^j,\qquad d=1,2.$$

When employing a more direct generalization of the methods used in Lemma 5 of [2], there exists a positive constant $\alpha(p)$ such that for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$, $d = 1, 2$, and $i + j = 0, 1, \ldots, p$, the following inequality holds:

$$\left\| \frac{\partial^{i+j} \hat{v}_{d,n,m}}{\partial x^i \partial y^j} \right\|_{L^\infty(D_{n,m})} \leq \alpha(p).$$

Hence, we have

$$|e_{d,0,0}(x, y)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \underbrace{\frac{\alpha(p)}{p!}}_{C_1} (h + k)^p, \qquad d = 1, 2.$$

Claim 2. Define the error $e_n = e|_{D_{n,0}}$ by $e_{d,n,0}(x, y) = u_d(x, y) - v_{d,n,0}(x, y)$ for all $n \in \{0, \ldots, N-1\}$ and $d = 1, 2$.
There exists a constant C_2 independent of h and k such that $\|e_n\| \leq C_2(h + k)^p$ for all $n = 1, \ldots, N-1$. Let $(x, y) \in D_{n,0}$, we have from (4)

$$u_d(x, y) - \hat{v}_{d,n,0}(x, y) = \sum_{\xi=0}^{n-1} \int_{x_\xi}^{x_{\xi+1}} \int_0^y (K_{d,1}(x, y, t, s)(u_1(t, s) - v_{1,\xi,0}(t, s))$$

$$+ K_{d,2}(x, y, t, s)(u_2(t, s) - v_{2,\xi,0}(t, s)))dsdt + \int_{x_n}^{x} \int_0^y (K_{d,1}(x, y, t, s)(u_1(t, s) - \hat{v}_{1,\xi,0}(t, s))$$

$$+ K_{d,2}(x, y, t, s)(u_2(t, s) - \hat{v}_{2,\xi,0}(t, s)))dsdt,$$

hence,

$$\begin{aligned} |u_d(x, y) - \hat{v}_{d,n,0}(x, y)| &\leq \sum_{\xi=0}^{n-1} hk\overline{K}_d \|e\| \\ &\quad + \overline{K}_d \int_{x_n}^{x} \int_0^y |u_1(t, s) - \hat{v}_{1,n,0}(t, s)| + |u_2(t, s) - \hat{v}_{2,n,0}(t, s)| dsdt, \end{aligned}$$

where $\overline{K}_d = \|K_{d,1}\| + \|K_{d,2}\|$, $d = 1, 2$. By adding the two inequalities, we have

$$\begin{aligned} |u_1(x, y) - \hat{v}_{1,n,0}(x, y)| + |u_2(x, y) - \hat{v}_{2,n,0}(x, y)| &\leq \sum_{\xi=0}^{n-1} hk\overline{K} \|e_\xi\| \\ &\quad + \overline{K} \int_{x_n}^{x} \int_0^y |u_1(t, s) - \hat{v}_{1,n,0}(t, s)| + |u_2(t, s) - \hat{v}_{2,n,0}(t, s)| dsdt, \end{aligned}$$

where $\overline{K} = \overline{K}_1 + \overline{K}_2$. Then by Lemma 3.3,

$$|u_d(x, y) - \hat{v}_{d,n,0}(x, y)| \leq \sum_{\xi=0}^{n-1} h \underbrace{b\overline{K} \exp\left(\overline{K}ab\right)}_{\lambda_1} \|e_\xi\|, \qquad d = 1, 2.$$

Which implies, by Taylor's theorem, that

$$\begin{aligned} |e_{d,n,0}(x, y)| &\leq |u_d(x, y) - \hat{v}_{d,n,0}(x, y)| + |\hat{v}_{d,n,0}(x, y) - v_{d,n,0}(x, y)| \\ &\leq \sum_{\xi=0}^{n-1} h\lambda_1 \|e_\xi\| + \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j} \hat{v}_{d,n,0}}{\partial x^i \partial y^j} \right\| h^i k^j, \end{aligned}$$

hence, we obtain $|e_{d,n,0}(x,y)| \leq \sum_{\xi=0}^{n-1} h\lambda_1 \|e_\xi\| + \frac{\alpha(p)}{p!}(h+k)^p, \qquad d=1,2$. Then, by Lemma 3.1, we have

$$\|e_n\| \leq \frac{\alpha(p)}{p!}(h+k)^p \exp(a\lambda_1).$$

Thus, we take $C_2 = \frac{\alpha(p)}{p!}\exp(a\lambda_1)$.

Claim 3. Define the error $e_{n,m} = e|_{D_{n,m}}$ by $e_{d,n,m}(x,y) = u_d(x,y) - v_{d,n,m}(x,y)$ for all $n \in \{0,\ldots,N-1\}$, $m \in \{0,\ldots,M-1\}$ and $d=1,2$.
There exists a constant C_3 independent of h and k such that

$$\|e_{n,m}\| \leq C_3(h+k)^p$$

for all $n = 0,\ldots,N-1$ and $m = 1,\ldots,M-1$. Let $(x,y) \in D_{n,m}$, we have from (6)

$$|u_d(x,y) - \hat{v}_{d,n,m}(x,y)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{K}_d \|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\overline{K}_d \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\overline{K}_d \|e_{n,\rho}\|$$
$$+ \overline{K}_d \int_{x_n}^{x}\int_{y_m}^{y} |u_1(t,s) - \hat{v}_{1,n,m}(t,s)| + |u_2(t,s) - \hat{v}_{2,n,m}(t,s)| ds dt, \qquad d=1,2.$$

By adding the two inequalities, we have

$$|u_1(t,s) - \hat{v}_{1,n,m}(t,s)| + |u_2(t,s) - \hat{v}_{2,n,m}(t,s)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{K}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\overline{K}\|e_{\xi,m}\|$$
$$+ \sum_{\rho=0}^{m-1} hk\overline{K}\|e_{n,\rho}\| + \overline{K}\int_{x_n}^{x}\int_{y_m}^{y} |u_1(t,s) - \hat{v}_{1,n,m}(t,s)| + |u_2(t,s) - \hat{v}_{2,n,m}(t,s)| ds dt,$$

then by Lemma 3.3,

$$|u_d(x,y) - \hat{v}_{d,n,m}(x,y)| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk \underbrace{\overline{K}\exp\left(\overline{K}ab\right)}_{\lambda_2} \|e_{\xi,\rho}\|$$
$$+ \sum_{\xi=0}^{n-1} h \underbrace{b\overline{K}\exp\left(\overline{K}ab\right)}_{\lambda_3} \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k \underbrace{a\overline{K}\exp\left(\overline{K}ab\right)}_{\lambda_4} \|e_{n,\rho}\|, \qquad d=1,2.$$

Which implies, by Taylor's theorem, that

$$\begin{aligned} |e_{d,n,m}(x,y)| \leq & |u_d(x,y) - \hat{v}_{d,n,m}(x,y)| + |\hat{v}_{d,n,m}(x,y) - v_{d,n,m}(x,y)| \\ \leq & \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\lambda_3\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\lambda_4\|e_{n,\rho}\| \\ & + \sum_{i+j=p} \frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{d,n,m}}{\partial x^i \partial y^j}\right\| h^i k^j, \qquad d=1,2. \end{aligned}$$

Hence, we obtain for $d=1,2$,

$$\|e_{n,m}\| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\lambda_3\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\lambda_4\|e_{n,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p.$$

Then, by Lemma 3.2, we obtain $\|e_{n,m}\| \leq \frac{\alpha(p)}{p!}(h+k)^p \exp(\gamma_2(a+b))$,
where $\gamma_2 = \frac{1}{2}(\lambda_3 + \lambda_4 + \sqrt{(\lambda_3+\lambda_4)^2 + 4\lambda_2})$. Thus, we take $C_3 = \frac{\alpha(p)}{p!}\exp(\gamma_2(a+b))$.
The proof is completed by taking $C = \max\{C_1, C_2, C_3\}$. $\square$

4 Numerical Examples

In this section, two numerical examples are given to show the efficiency of our proposed method for approximating the solution of a system of 2D VIEs.

Example 4.1 Consider the following system of 2 DVIEs:

$$\begin{cases} u_1(x,y) = g_1(x,y) + \int_0^x \int_0^y cos(s)u_1(t,s) + sin(s)u_2(t,s)dsdt, \\ u_2(x,y) = g_2(x,y) + \int_0^x \int_0^y xt^2 u_1(t,s) + (x+t)u_2(t,s)dsdt, \end{cases}$$

where $g_1(x,y) = xcos(y) - \frac{1}{2}yx^2$, $g_2(x,y) = xsin(y) - \frac{1}{4}x^5 sin(y) + \frac{5}{6}x^3 cos(y) - \frac{5}{6}x^3$.
The exact solution of this system is $u_1(x,y) = xcos(y)$ and $u_2(x,y) = xsin(y)$.

The absolute error of the proposed method for $N = M = 10$, $p = 3, 5$ is shown in Table 1.

(x,y)	$p=3$		$p=5$	
	$\|e_1\|$	$\|e_2\|$	$\|e_1\|$	$\|e_2\|$
$(0,0)$	0	0	0	0
$(0.1,0.1)$	$8.32e-07$	$3.88e-09$	$4.30e-10$	$6.00e-12$
$(0.2,0.2)$	$3.34e-06$	$1.19e-07$	$2.00e-09$	$9.63e-09$
$(0.3,0.3)$	$7.61e-06$	$7.06e-07$	$3.60e-08$	$1.08e-07$
$(0.4,0.4)$	$1.37e-05$	$2.42e-06$	$1.91e-07$	$5.67e-07$
$(0.5,0.5)$	$2.19e-05$	$6.29e-06$	$6.70e-07$	$2.01e-06$
$(0.6,0.6)$	$3.23e-05$	$1.37e-05$	$1.84e-06$	$5.68e-06$
$(0.7,0.7)$	$4.54e-05$	$2.68e-05$	$4.35e-06$	$1.36e-05$
$(0.8,0.8)$	$6.17e-05$	$4.38e-05$	$9.15e-06$	$2.94e-05$
$(0.9,0.9)$	$8.19e-05$	$8.24e-05$	$1.77e-05$	$5.83e-05$

Table 1: Absolute errors in Example 4.1 for $N = M = 10$.

Example 4.2 Consider the following system of 2D VIEs:

$$\begin{cases} u_1(x,y) = g_1(x,y) + \int_0^x \int_0^y \frac{x-y+t-s}{4} u_2(t,s)dsdt, \\ u_2(x,y) = g_2(x,y) + \int_0^x \int_0^y \frac{x-y+t-s}{4} u_1(t,s)dsdt, \end{cases}$$

where

$$g_1(x,y) = xe^{1-y} - \frac{e}{24}y^2(3+3x-5y+e^{-x}(-3-6x+5y)),$$
$$g_2(x,y) = ye^{1-x} + \frac{e}{24}x^2(3-5x+3y+e^{-y}(-3+5x-6y)).$$

The exact solution of this system is $u_1(x,y) = xe^{1-y}$ and $u_2(x,y) = ye^{1-x}$.

Comparison of the absolute errors of the hybrid functions method (HFM) [7] and the proposed method (TCM) for $N = M = 10$, $p = 4$ is shown in Table 2.

(x,y)	HFM [7]		TCM	
	$\lvert e_{1,2,5}\rvert$	$\lvert e_{2,2,5}\rvert$	$\lvert e_1\rvert$	$\lvert e_2\rvert$
$(0,0)$	$1.11e-07$	$5.02e+11$	0	0
$(0.1,0.1)$	$4.90e-08$	$3.25e-08$	$1.00e-10$	$2.00e-10$
$(0.2,0.2)$	$1.41e-07$	$1.34e-07$	$2.60e-09$	$1.90e-09$
$(0.3,0.3)$	$2.10e-07$	$2.03e-07$	$1.07e-08$	$8.50e-09$
$(0.4,0.4)$	$1.53e-07$	$1.37e-07$	$3.12e-08$	$2.71e-08$
$(0.5,0.5)$	$6.80e-07$	$6.80e-07$	$7.60e-08$	$6.45e-08$
$(0.6,0.6)$	$1.18e-07$	$1.18e-07$	$1.54e-07$	$1.35e-07$
$(0.7,0.7)$	$2.84e-07$	$2.84e-07$	$2.83e-07$	$2.51e-07$
$(0.8,0.8)$	$3.29e-07$	$3.28e-07$	$4.80e-07$	$4.26e-07$
$(0.9,0.9)$	$1.86e-07$	$1.87e-07$	$7.68e-07$	$6.76e-07$

Table 2: Absolute errors in Example 4.2.

5 Conclusion

In this paper, we developed and applied the Taylor Collocation Method (TCM) to approximate the solution of a system of linear two-dimensional Volterra integral equations (2D-VIEs). Unlike traditional numerical approaches, which often rely on discretization or transformation into algebraic systems, the proposed method directly exploits the Taylor series expansion to construct a highly accurate collocation solution while preserving the integral structure of the equations. This methodological advantage enhances both accuracy and computational efficiency.

The numerical results confirm the effectiveness and reliability of the method, demonstrating high convergence rates and excellent agreement between the theoretical error estimates and practical computations. The approach is particularly well-suited for problems in applied mathematics and systems theory, where integral equations naturally arise in modeling processes with memory effects.

A key contribution of this work is the extension of TCM to systems of 2D-VIEs, which has been less explored in the literature compared to one-dimensional cases. In future work, we aim to generalize this framework to larger systems of n Volterra integral equations in two dimensions and explore its application to integral equations with weakly singular kernels or variable delays.

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