Nonlinear Dynamics and Systems Theory, **26** (1) (2026) 41–55

Fractional Boole Type Inequalities for Differentiable s-Convex Functions

W. Kaidouchi [1*], M. Benssaad [2] and B. Meftah [2]

[1] *Higher Normal School of Technological Education-Skikda, Department of Mathematics and Informatics. Azzaba, Algeria.*
[2] *Department of Mathematics, Faculty of Mathematics, Computer Science and Material Sciences, University 8 Mai 1945, Guelma, Algeria.*

Received: March 6, 2025; Revised: February 10, 2026

Abstract: We introduce a new identity involving five-point Newton-Cotes inequalities called Boole's inequalities. By employing this identity, we establish some Boole-type inequalities for functions whose first derivatives are s-convex via Riemann-Liouville fractional integral operators. The results provide a generalization of classical inequalities by using tools from fractional calculus. Additionally, we present applications that highlight the utility and relevance of the obtained inequalities in various mathematical contexts.

Keywords: *Boole's inequality, s-convex functions, Hölder inequality, Riemann-Liouville fractional integrals.*

Mathematics Subject Classification (2020): 26D10, 26D15, 26A51, 70K45, 70K75.

1 Introduction

Let $I \subset \mathbb{R}$ be an interval. A function $\eth : I \to \mathbb{R}$ is said to be convex if

$$\eth(\varrho x + (1-\varrho)y) \le \varrho \eth(x) + (1-\varrho)\eth(y)$$

holds for all $x, y \in I$ and $\varrho \in [0,1]$.

The first finding between convex functions and integrals was the Hermite–Hadamard inequality, which can be stated as follows: for every convex function $\eth$ on the interval $[\xi_1, \xi_2]$ with $\xi_1 < \xi_2$, we have

$$\eth\left(\frac{\xi_1+\xi_2}{2}\right) \le \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} \eth(x)\,dx \le \frac{\eth(\xi_1)+\eth(\xi_2)}{2}. \tag{1}$$

* Corresponding author: mailto:kaidouchi.wahida@gmail.com

The above inequality has many advantages and is broadly utilized, especially in numerical integration, operator theory, approximation theory, and engineering problems. Because of its numerous applications, mathematicians have extensively worked on it and obtained numerous results related to (1), see [3, 5, 6, 10, 15, 24, 26].

The idea of convexity has been generalized in different ways. One of these generalizations is presented by Breckner, called the s-convexity and declared as follows.

A nonnegative function $\eth : I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$ if

$$\eth(\varrho x + (1-\varrho)y) \leq \varrho^s \eth(x) + (1-\varrho)^s \eth(y)$$

holds for all $x, y \in I$ and $\varrho \in [0,1]$.

Dragomir and Fitzpatrick [8] proved the following variant of inequality (1) which holds for s-convex functions in the second sense. For every positive and s-convex function in the second sense $\eth$ on the interval $[\xi_1, \xi_2]$ with $\xi_1 < \xi_2$, we have

$$2^{s-1}\eth\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2} \eth(x)\,dx \leq \frac{\eth(\xi_1)+\eth(\xi_2)}{s+1}. \tag{2}$$

Fractional calculus has become an important topic in mathematical analysis nowdays, and has attracted the attention of many researchers due to its wide applications in pure and applied mathematics as well as other sciences. Indeed, it provides an excellent tool for the description of memory and heritable properties of various materials and processes. In the literature, several fractional operators have been introduced [1, 7, 16, 17, 25], however the most used is that attributed to Liouville.

The Riemann-Liouville operator is defined as follows.

Definition 1.1 Let $\eth \in L^1[\xi_1, \xi_2]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{\xi_1^+}\eth$ and $I^{\alpha}_{\xi_2^-}\eth$ of order $\alpha > 0$ with $\xi_1 \geq 0$ are defined by

$$\begin{aligned} I^{\alpha}_{\xi_1^+}\eth(x) &= \frac{1}{\Gamma(\alpha)}\int_{\xi_1}^{x} (x-\varrho)^{\alpha-1}\eth(\varrho)d\varrho, \quad x > \xi_1, \\ I^{\alpha}_{\xi_2^-}\eth(x) &= \frac{1}{\Gamma(\alpha)}\int_{x}^{\xi_2} (\varrho-x)^{\alpha-1}\eth(\varrho)d\varrho, \quad \xi_2 > x, \end{aligned}$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-\varrho}\varrho^{\alpha-1}d\varrho$ is the Gamma function and $I^0_{\xi_1^+}\eth(x) = I^0_{\xi_2^-}\eth(x) = \eth(x)$.

For papers dealing with the applications and other characteristics of convex functions and their variant forms via various types of fractional integral operators, we refer the readers to [2, 4, 5, 9, 12, 13, 14, 18-23, 27] and the references therein.

In this paper, we propose to study one of the five-point Newton-Cotes inequalities given as follows:

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth(u)\,du\right| \leq \frac{4(\xi_2-\xi_1)^6}{945}\left\|\eth^{(6)}\right\|,$$

where $\Game$ is a 6-times differentiable mapping and $\left\|\Game^{(6)}\right\| = \sup\limits_{x\in[\xi_1,\xi_2]} \left|\Game^{(6)}(x)\right| < +\infty$ (see [11]). For this, we first prove a new identity involving Riemann-Liouville fractional integrals. By using this identity, we establish some new fractional Boole-type integral inequalities for functions whose first derivatives are s-convex in the second sense. We end this work by some applications to quadrature formulas and inequalities for special means.

2 Main Results

Before giving our main results, let us recall some special functions.

Definition 2.1 For any complex number and non-positive integers x, y such that $Re(x) > 0$ and $Re(y) > 0$, the Beta function is defined by

$$B(x,y) = \int_0^1 \varrho^{x-1}(1-\varrho)^{y-1}\, d\varrho.$$

The definition of the incomplete Beta function is given as follows.

Definition 2.2 For any complex number and non-positive integers x, y such that $Re(x) > 0$ and $Re(y) > 0$, we have

$$B_{\xi_1}(x,y) = \int_0^{\xi_1} \varrho^{x-1}(1-\varrho)^{y-1}\, d\varrho, \quad \xi_1 < 1.$$

Definition 2.3 The hypergeometric function is defined for $Re(c) > Re(\xi_2) > 0$ and $|z| < 1$, as follows:

$${}_2F_1(\xi_1,\xi_2,c,z) = \frac{1}{B(\xi_2, c-\xi_2)} \int_0^1 \varrho^{\xi_2-1}(1-\varrho)^{c-\xi_2-1}(1-z\varrho)^{-\xi_1} d\varrho,$$

where $c > \xi_2 > 0, |z| < 1$ and $B(.,.)$ is the Beta function.

Lemma 2.1 *Let $\Game : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\xi_1, \xi_2 \in I^\circ$ with $\xi_1 < \xi_2$, and $\Game' \in L^1[\xi_1,\xi_2]$, then the following equality holds:*

$$\begin{aligned} &\frac{7\Game(\xi_1)+32\Game\left(\frac{\xi_1+3\xi_2}{4}\right)+12\Game\left(\frac{\xi_1+\xi_2}{2}\right)+32\Game\left(\frac{3\xi_1+\xi_2}{4}\right)+7\Game(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\mathcal{B}\left(\xi_1,\xi_2,I^\alpha\Game\right) \\ = &\frac{\xi_2-\xi_1}{16}\left(\int_0^1 \left(\varrho^\alpha - \frac{14}{45}\right)\Game'\left((1-\varrho)\xi_1 + \varrho\frac{3\xi_1+\xi_2}{4}\right) d\varrho\right.\end{aligned}$$

$$-\int_0^1 \left((1-\varrho)^\alpha - \tfrac{4}{15}\right) \eth'\left((1-\varrho)\,\tfrac{3\xi_1+\xi_2}{4} + \varrho\tfrac{\xi_1+\xi_2}{2}\right) d\varrho$$
$$+\int_0^1 \left(\varrho^\alpha - \tfrac{4}{15}\right) \eth'\left((1-\varrho)\,\tfrac{\xi_1+\xi_2}{2} + \varrho\tfrac{\xi_1+3\xi_2}{4}\right) d\varrho$$
$$\left.-\int_0^1 \left((1-\varrho)^\alpha - \tfrac{14}{45}\right) \eth'\left((1-\varrho)\,\tfrac{\xi_1+3\xi_2}{4} + \varrho\xi_2\right) d\varrho\right),$$

where

$$\mathcal{B}\left(\xi_1,\xi_2,I^\alpha\eth\right) =$$
$$= I^\alpha_{\left(\frac{3\xi_1+\xi_2}{4}\right)^-}\eth(\xi_1) + I^\alpha_{\left(\frac{3\xi_1+\xi_2}{4}\right)^+}\eth(\tfrac{\xi_1+\xi_2}{2}) + I^\alpha_{\left(\frac{\xi_1+3\xi_2}{4}\right)^-}\eth(\tfrac{\xi_1+\xi_2}{2}) + I^\alpha_{\left(\frac{\xi_1+3\xi_2}{4}\right)^+}\eth(\xi_2). \quad (3)$$

Proof. Let

$$I = I_1 - I_2 + I_3 - I_4, \quad (4)$$

where

$$I_1 = \int_0^1 \left(\varrho^\alpha - \tfrac{14}{45}\right) \eth'\left((1-\varrho)\,\xi_1 + \varrho\tfrac{3\xi_1+\xi_2}{4}\right) d\varrho,$$
$$I_2 = \int_0^1 \left((1-\varrho)^\alpha - \tfrac{4}{15}\right) \eth'\left((1-\varrho)\,\tfrac{3\xi_1+\xi_2}{4} + \varrho\tfrac{\xi_1+\xi_2}{2}\right) d\varrho,$$
$$I_3 = \int_0^1 \left(\varrho^\alpha - \tfrac{4}{15}\right) \eth'\left((1-\varrho)\,\tfrac{\xi_1+\xi_2}{2} + \varrho\tfrac{\xi_1+3\xi_2}{4}\right) d\varrho$$

and

$$I_4 = \int_0^1 \left((1-\varrho)^\alpha - \tfrac{14}{45}\right) \eth'\left((1-\varrho)\,\tfrac{\xi_1+3\xi_2}{4} + \varrho\xi_2\right) d\varrho.$$

Integrating by parts I_1, we get

$$\begin{aligned} I_1 &= \tfrac{4}{\xi_2-\xi_1}\left(\varrho^\alpha - \tfrac{14}{45}\right)\eth\left((1-\varrho)\,\xi_1 + \varrho\tfrac{3\xi_1+\xi_2}{4}\right)\Big|_{\varrho=0}^{\varrho=1} - \tfrac{4\alpha}{\xi_2-\xi_1}\int_0^1 \varrho^{\alpha-1}\eth\left((1-\varrho)\,\xi_1 + \varrho\tfrac{3\xi_1+\xi_2}{4}\right) d\varrho \\ &= \tfrac{124}{45(\xi_2-\xi_1)}\eth\left(\tfrac{3\xi_1+\xi_2}{4}\right) + \tfrac{56}{45(\xi_2-\xi_1)}\eth\left(\xi_1\right) - \tfrac{4\alpha}{\xi_2-\xi_1}\int_0^1 \varrho^{\alpha-1}\eth\left((1-\varrho)\,\xi_1 + \varrho\tfrac{3\xi_1+\xi_2}{4}\right) d\varrho \\ &= \tfrac{124}{45(\xi_2-\xi_1)}\eth\left(\tfrac{3\xi_1+\xi_2}{4}\right) + \tfrac{56}{45(\xi_2-\xi_1)}\eth\left(\xi_1\right) - \tfrac{4^{\alpha+1}\alpha}{(\xi_2-\xi_1)^{\alpha+1}}\int_{\xi_1}^{\frac{3\xi_1+\xi_2}{4}} \left(u-\xi_1\right)^{\alpha-1}\eth\left(u\right) du \\ &= \tfrac{124}{45(\xi_2-\xi_1)}\eth\left(\tfrac{3\xi_1+\xi_2}{4}\right) + \tfrac{56}{45(\xi_2-\xi_1)}\eth\left(\xi_1\right) - \tfrac{4^{\alpha+1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha+1}} I^\alpha_{\left(\frac{3\xi_1+\xi_2}{4}\right)^-}\eth(\xi_1). \end{aligned} \quad (5)$$

Similarly, I_2, I_3 and I_4 can be found as

$$I_2 = -\frac{16}{15(\xi_2-\xi_1)}\eth\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{44}{15(\xi_2-\xi_1)}\eth\left(\frac{3\xi_1+\xi_2}{4}\right) + \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha+1}} I^{\alpha}_{\left(\frac{3\xi_1+\xi_2}{4}\right)^+}\eth\left(\frac{\xi_1+\xi_2}{2}\right), \tag{6}$$

$$I_3 = \frac{44}{15(\xi_2-\xi_1)}\eth\left(\frac{\xi_1+3\xi_2}{4}\right) + \frac{16}{15(\xi_2-\xi_1)}\eth\left(\frac{\xi_1+\xi_2}{2}\right) - \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha+1}} I^{\alpha}_{\left(\frac{\xi_1+3\xi_2}{4}\right)^-}\eth\left(\frac{\xi_1+\xi_2}{2}\right), \tag{7}$$

$$I_4 = -\frac{56}{45(\xi_2-\xi_1)}\eth(\xi_2) - \frac{124}{45(\xi_2-\xi_1)}\eth\left(\frac{\xi_1+3\xi_2}{4}\right) + \frac{4^{\alpha+1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha+1}} I^{\alpha}_{\left(\frac{\xi_1+3\xi_2}{4}\right)^+}\eth(\xi_2). \tag{8}$$

Substituting (5)-(8) into (4), and then multiplying the resulting equality by $\frac{\xi_2-\xi_1}{16}$, we get the desired result.

Theorem 2.1 *Let $\eth : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\eth' \in L^1[\xi_1, \xi_2]$ with $0 \le \xi_1 < \xi_2$. If $|\eth'|$ is s-convex in the second sense for some fixed $s \in (0, 1]$, then we have*

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}(\xi_1,\xi_2,I^{\alpha}\eth)\right| \\
\le\ & \frac{\xi_2-\xi_1}{16}\left(\left(\frac{14}{45(s+1)}\left(1-2\left(1-\left(\frac{14}{45}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right) - \chi\left(\alpha, s, \frac{14}{45}\right)\right)\left(|\eth'(\xi_1)| + |\eth'(\xi_2)|\right)\right. \\
&+ \frac{2}{(s+1)(\alpha+s+1)}\left(\frac{32(s+1)-13\alpha}{45} + \frac{\alpha(14)^{1+\frac{s+1}{\alpha}}+\alpha(12)^{1+\frac{s+1}{\alpha}}}{(45)^{1+\frac{s+1}{\alpha}}}\right) \times \\
&\times \left(\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right| + \left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|\right) \\
&\left.+ 2\left(\frac{4}{15(s+1)}\left(1-2\left(1-\left(\frac{4}{15}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right) - \chi\left(\alpha, s, \frac{4}{15}\right)\right)\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right),
\end{aligned}$$

where $\alpha > 0, \mathcal{B}(\xi_1, \xi_2, I^{\alpha}\eth)$ is defined by (3) and

$$\chi(\alpha, s, x) = B_{x^{\frac{1}{\alpha}}}(\alpha+1, s+1) - B_{1-x^{\frac{1}{\alpha}}}(s+1, \alpha+1). \tag{9}$$

Proof. From Lemma 2.1, the modulus and s-convexity of $|\eth'|$, we have

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}(\xi_1,\xi_2,I^{\alpha}\eth)\right| \\
\le\ & \frac{\xi_2-\xi_1}{16}\left(\int_0^1 \left|\varrho^{\alpha} - \frac{14}{45}\right| \left|\eth'\left((1-\varrho)\xi_1 + \varrho\frac{3\xi_1+\xi_2}{4}\right)\right| d\varrho\right. \\
&+ \int_0^1 \left|(1-\varrho)^{\alpha} - \frac{4}{15}\right| \left|\eth'\left((1-\varrho)\frac{3\xi_1+\xi_2}{4} + \varrho\frac{\xi_1+\xi_2}{2}\right)\right| d\varrho
\end{aligned}$$

$$+\int_0^1 \left|\varrho^{\alpha}-\tfrac{4}{15}\right| \left|\eth'\left((1-\varrho)\,\tfrac{\xi_1+\xi_2}{2}+\varrho\tfrac{\xi_1+3\xi_2}{4}\right)\right| d\varrho$$

$$\left.+\int_0^1 \left|(1-\varrho)^{\alpha}-\tfrac{14}{45}\right| \left|\eth'\left((1-\varrho)\,\tfrac{\xi_1+3\xi_2}{4}+\varrho\xi_2\right)\right| d\varrho\right)$$

$$\leq \tfrac{\xi_2-\xi_1}{16}\left(\left(\tfrac{14}{45(s+1)}\left(1-2\left(1-\left(\tfrac{14}{45}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-\chi\left(\alpha,s,\tfrac{14}{45}\right)\right)\left(\left|\eth'(\xi_1)\right|+\left|\eth'(\xi_2)\right|\right)\right.$$

$$+\tfrac{2}{(s+1)(\alpha+s+1)}\left(\tfrac{32(s+1)-13\alpha}{45}+\tfrac{\alpha(14)^{1+\frac{s+1}{\alpha}}+\alpha(12)^{1+\frac{s+1}{\alpha}}}{(45)^{1+\frac{s+1}{\alpha}}}\right)\times$$

$$\times\left(\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|+\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|\right)$$

$$\left.+\,2\left(\tfrac{4}{15(s+1)}\left(1-2\left(1-\left(\tfrac{4}{15}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-\chi\left(\alpha,s,\tfrac{4}{15}\right)\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|\right),$$

where we have used

$$\int_0^1 \left|\varrho^{\alpha}-\tfrac{14}{45}\right|(1-\varrho)^s\,d\varrho = \tfrac{14}{45(s+1)}\left(1-2\left(1-\left(\tfrac{14}{45}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-\chi\left(\alpha,s,\tfrac{14}{45}\right), \quad (10)$$

$$\int_0^1 \left|\varrho^{\alpha}-\tfrac{14}{45}\right|\varrho^s d\varrho = \tfrac{31(s+1)-14\alpha}{45(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{14}{45}\right)^{1+\frac{s+1}{\alpha}}, \quad (11)$$

$$\int_0^1 \left|(1-\varrho)^{\alpha}-\tfrac{4}{15}\right|(1-\varrho)^s\,d\varrho = \tfrac{11(s+1)-4\alpha}{15(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{4}{15}\right)^{1+\frac{s+1}{\alpha}}, \quad (12)$$

$$\int_0^1 \left|(1-\varrho)^{\alpha}-\tfrac{4}{15}\right|\varrho^s d\varrho = \tfrac{4}{15(s+1)}\left(1-2\left(1-\left(\tfrac{4}{15}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-\chi\left(\alpha,s,\tfrac{4}{15}\right). \quad (13)$$

The proof is finished.

Remark 2.1 By simple calculation, we easily obtain

$$\chi(1,s,x)=\tfrac{1}{(s+1)(s+2)}+\tfrac{2}{s+2}(1-x)^{s+2}-\tfrac{2}{s+1}(1-x)^{s+1},$$

$$\chi(\alpha,1,x)=\tfrac{2}{\alpha+1}x^{1+\frac{1}{\alpha}}-\tfrac{2}{\alpha+2}x^{1+\frac{2}{\alpha}}-\tfrac{1}{(\alpha+1)(\alpha+2)},\quad \chi(1,1,x)=\tfrac{6x^2-4x^3-1}{6}.$$

Corollary 2.1 *In Theorem 2.1, if we take $\alpha=1$, then we obtain*

$$\left|\tfrac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\tfrac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth(w)\,dw\right|$$

$$\leq \tfrac{\xi_2-\xi_1}{16}\left(\left(\tfrac{14s-17}{45(s+1)(s+2)}+\tfrac{2}{(s+1)(s+2)}\left(\tfrac{31}{45}\right)^{s+2}\right)\left(\left|\eth'(\xi_1)\right|+\left|\eth'(\xi_2)\right|\right)\right.$$

$$+\tfrac{2}{(s+1)(s+2)}\left(\tfrac{32s+19}{45}+\tfrac{(14)^{s+2}+(12)^{s+2}}{(45)^{s+2}}\right)\left(\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|+\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|\right)$$

$$\left.+\,2\left(\tfrac{4s-7}{15(s+1)(s+2)}+\tfrac{2}{(s+1)(s+2)}\left(\tfrac{11}{15}\right)^{s+2}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|\right).$$

Corollary 2.2 *In Theorem 2.1, if we take $s = 1$, then we obtain*

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right| \\
\leq\ & \frac{\xi_2-\xi_1}{16}\left(\left(\frac{45-7(\alpha+1)(\alpha+2)}{45(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1}\left(\frac{14}{45}\right)^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\left(\frac{14}{45}\right)^{1+\frac{2}{\alpha}}\right)\left(\left|\eth'\left(\xi_1\right)\right| + \left|\eth'\left(\xi_2\right)\right|\right)\right. \\
&+\frac{\alpha}{45(\alpha+2)}\left(\frac{64-13\alpha}{\alpha} + \frac{(14)^{1+\frac{2}{\alpha}}+(12)^{1+\frac{2}{\alpha}}}{(45)^{\frac{2}{\alpha}}}\right)\left(\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right| + \left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|\right) \\
&\left.+ 2\left(\frac{15-2(\alpha+1)(\alpha+2)}{15(\alpha+1)(\alpha+2)} + \frac{2\alpha}{\alpha+1}\left(\frac{4}{15}\right)^{1+\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\left(\frac{4}{15}\right)^{1+\frac{2}{\alpha}}\right)\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right).
\end{aligned}$$

Corollary 2.3 *In Theorem 2.1, if we take $\alpha = s = 1$, then we obtain*

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth\left(w\right)dw\right| \\
\leq\ & \frac{239(\xi_2-\xi_1)}{3240}\times \\
&\left(\frac{53507\left|\eth'(\xi_1)\right|+215494\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|+107298\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|+215494\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|+53507\left|\eth'(\xi_2)\right|}{645300}\right).
\end{aligned}$$

Theorem 2.2 *Let $\eth : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\eth' \in L^1[\xi_1, \xi_2]$ with $0 \leq \xi_1 < \xi_2$. If $|\eth'|^q$ is s-convex in the second sense for some fixed $s \in (0, 1]$ and $q > 1$ with $\frac{1}{q} + \frac{1}{p} = 1$, then we have*

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right| \\
\leq\ & \frac{\xi_2-\xi_1}{16}\left(\left(\left(\frac{14}{45}\right)^{p+\frac{1}{\alpha}}\frac{1}{\alpha}B\left(\frac{1}{\alpha},p+1\right) + \left(\frac{31}{45}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{31}{45}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right. \\
&\times\left(\left(\frac{\left|\eth'(\xi_1)\right|^q+\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+\left|\eth'(\xi_2)\right|^q}{s+1}\right)^{\frac{1}{q}}\right) \\
&+\left(\left(\frac{4}{15}\right)^{p+\frac{1}{\alpha}}\frac{1}{\alpha}B\left(\frac{1}{\alpha},p+1\right) + \left(\frac{11}{15}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{11}{15}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}} \\
&\left.\times\left(\left(\frac{\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}$$

where $\mathcal{B}(\xi_1, \xi_2, I^{\alpha}\eth)$ is defined by (3) and B and ${}_2F_1$ are the Beta and hypergeometric functions, respectively.

Proof. From Lemma 2.1, the modulus, Hölder's inequality and s-convexity of

$|\eth'|^q$, we have

$$\begin{aligned}
&\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\mathcal{B}\left(\xi_1,\xi_2,I^\alpha\eth\right)\right|\\
\leq\ & \frac{\xi_2-\xi_1}{16}\left(\left(\int_0^1\left|\varrho^\alpha-\tfrac{14}{45}\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\eth'\left((1-\varrho)\xi_1+\varrho\tfrac{3\xi_1+\xi_2}{4}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|(1-\varrho)^\alpha-\tfrac{4}{15}\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\eth'\left((1-\varrho)\tfrac{3\xi_1+\xi_2}{4}+\varrho\tfrac{\xi_1+\xi_2}{2}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\varrho^\alpha-\tfrac{4}{15}\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\eth'\left((1-\varrho)\tfrac{\xi_1+\xi_2}{2}+\varrho\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\left|(1-\varrho)^\alpha-\tfrac{14}{45}\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\eth'\left((1-\varrho)\tfrac{\xi_1+3\xi_2}{4}+\varrho\xi_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right)\\
=\ & \frac{\xi_2-\xi_1}{16}\left(\left(\left(\tfrac{14}{45}\right)^{p+\frac{1}{\alpha}}\tfrac{1}{\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\left(\tfrac{31}{45}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{31}{45}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right.\\
&\times\left(\left(\frac{|\eth'(\xi_1)|^q+\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+|\eth'(\xi_2)|^q}{s+1}\right)^{\frac{1}{q}}\right)\\
&+\left(\left(\tfrac{4}{15}\right)^{p+\frac{1}{\alpha}}\tfrac{1}{\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\left(\tfrac{11}{15}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{11}{15}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\\
&\left.\times\left(\left(\frac{\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
\int_0^1\left|\varrho^\alpha-\tfrac{14}{45}\right|^p d\varrho &= \int_0^{\left(\frac{14}{45}\right)^{\frac{1}{\alpha}}}\left(\tfrac{14}{45}-\varrho^\alpha\right)^p d\varrho+\int_{\left(\frac{14}{45}\right)^{\frac{1}{\alpha}}}^1\left(\varrho^\alpha-\tfrac{14}{45}\right)^p d\varrho\\
&= \left(\tfrac{14}{45}\right)^{p+\frac{1}{\alpha}}\tfrac{1}{\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\left(\tfrac{31}{45}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{31}{45}\right)}{\alpha(p+1)}
\end{aligned}$$

and

$$\int_0^1\left|\varrho^\alpha-\tfrac{4}{15}\right|^p d\varrho=\left(\tfrac{4}{15}\right)^{p+\frac{1}{\alpha}}\tfrac{1}{\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\left(\tfrac{11}{15}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{11}{15}\right)}{\alpha(p+1)}.$$

The proof is finished.

Corollary 2.4 *In Theorem 2.2, if we take $\alpha = 1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth(w)\,dw\right|$$
$$\leq \frac{\xi_2-\xi_1}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{14^{p+1}+31^{p+1}}{45^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{|\eth'(\xi_1)|^q+\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+|\eth'(\xi_2)|^q}{s+1}\right)^{\frac{1}{q}}\right)+\left(\frac{4^{p+1}+11^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left(\frac{\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.5 *In Theorem 2.2, if we take $s = 1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right|$$
$$\leq \frac{\xi_2-\xi_1}{16}\left(\left(\left(\tfrac{14}{45}\right)^{p+\frac{1}{\alpha}}\frac{1}{\alpha}B\left(\frac{1}{\alpha},p+1\right)+\left(\tfrac{31}{45}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{31}{45}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\right.$$
$$\times\left(\left(\frac{|\eth'(\xi_1)|^q+\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+|\eth'(\xi_2)|^q}{2}\right)^{\frac{1}{q}}\right)$$
$$+\left(\left(\tfrac{4}{15}\right)^{p+\frac{1}{\alpha}}\frac{1}{\alpha}B\left(\frac{1}{\alpha},p+1\right)+\left(\tfrac{11}{15}\right)^{p+\frac{1}{\alpha}}\frac{{}_2F_1\left(1-\frac{1}{\alpha},1,p+2;\frac{11}{15}\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left(\frac{\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.6 *In Theorem 2.2, if we take $\alpha = s = 1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90} - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth(w)\,dw\right|$$
$$\leq \frac{\xi_2-\xi_1}{16}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{14^{p+1}+31^{p+1}}{45^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{|\eth'(\xi_1)|^q+\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+|\eth'(\xi_2)|^q}{2}\right)^{\frac{1}{q}}\right)+\left(\frac{4^{p+1}+11^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left(\frac{\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Theorem 2.3 *Let $\eth : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) such that $\eth' \in L^1[\xi_1, \xi_2]$ with $0 \leq \xi_1 < \xi_2$. If $|\eth'|^q$ is s-convex in the second sense for some fixed*

$s\in(0,1]$ and $q\geq 1$, then we have

$$\begin{aligned}
&\left|\tfrac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right|\\
\leq\ &\tfrac{\xi_2-\xi_1}{16}\left(\left(\mathcal{V}\left(\alpha,0,\tfrac{14}{45}\right)\right)^{1-\frac{1}{q}}\left(\left(\mathcal{K}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\xi_1\right)\right|^{q}+\mathcal{V}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\mathcal{K}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^{q}+\mathcal{V}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\\
&+\left(\mathcal{V}\left(\alpha,0,\tfrac{4}{15}\right)\right)^{1-\frac{1}{q}}\left(\left(\mathcal{V}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^{q}+\mathcal{K}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\mathcal{V}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^{q}+\mathcal{K}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\xi_2\right)\right|^{q}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}$$

where $\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)$ is defined by (3),

$$\mathcal{K}\left(\alpha,s,x\right)=\tfrac{x}{s+1}\left(1-2\left(1-x^{\frac{1}{\alpha}}\right)^{s+1}\right)-\chi\left(\alpha,s,x\right)\tag{14}$$

and

$$\mathcal{V}\left(\alpha,s,x\right)=\tfrac{(1-x)(s+1)-x\alpha}{(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}x^{1+\frac{s+1}{\alpha}}.\tag{15}$$

Proof. From Lemma 2.1, the modulus, power mean inequality and s-convexity of $|\eth'|$, we have

$$\begin{aligned}
&\left|\tfrac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\tfrac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right|\\
\leq\ &\tfrac{\xi_2-\xi_1}{16}\left(\left(\left(\int_0^1\left|\varrho^{\alpha}-\tfrac{14}{45}\right|d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\varrho^{\alpha}-\tfrac{14}{45}\right|\left|\eth'\left(\left(1-\varrho\right)\xi_1+\varrho\tfrac{3\xi_1+\xi_2}{4}\right)\right|^{q}d\varrho\right)^{\frac{1}{q}}\right.\right.\\
&+\left(\int_0^1\left|(1-\varrho)^{\alpha}-\tfrac{4}{15}\right|d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|(1-\varrho)^{\alpha}-\tfrac{4}{15}\right|\left|\eth'\left(\left(1-\varrho\right)\tfrac{3\xi_1+\xi_2}{4}+\varrho\tfrac{\xi_1+\xi_2}{2}\right)\right|^{q}d\varrho\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 \left|\varrho^\alpha-\tfrac{4}{15}\right| d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\varrho^\alpha-\tfrac{4}{15}\right| \left|\eth'\left((1-\varrho)\tfrac{\xi_1+\xi_2}{2}+\varrho\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q d\varrho\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|(1-\varrho)^\alpha-\tfrac{14}{45}\right| d\varrho\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1 \left|(1-\varrho)^\alpha-\tfrac{14}{45}\right| \left|\eth'\left((1-\varrho)\tfrac{\xi_1+3\xi_2}{4}+\varrho\xi_2\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right)$$

$$= \frac{\xi_2-\xi_1}{16}\left(\left(\mathcal{V}\left(\alpha,0,\tfrac{14}{45}\right)\right)^{1-\frac{1}{q}}\left(\left(\mathcal{K}\left(\alpha,s,\tfrac{14}{45}\right)|\eth'(\xi_1)|^q+\mathcal{V}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\mathcal{K}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^q+\mathcal{V}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\left(\mathcal{V}\left(\alpha,0,\tfrac{4}{15}\right)\right)^{1-\frac{1}{q}}\left(\left(\mathcal{V}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^q+\mathcal{K}\left(\alpha,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mathcal{V}\left(\alpha,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q+\mathcal{K}\left(\alpha,s,\tfrac{14}{45}\right)|\eth'(\xi_2)|^q\right)^{\frac{1}{q}}\right),$$

where $\mathcal{K}$ and $\mathcal{V}$ are defined by (14) and (15), respectively. The proof is completed.

Corollary 2.7 *In Theorem 2.3, if we take $\alpha=1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth(w)\,dw\right|$$

$$\leq \frac{\xi_2-\xi_1}{16}\left(\left(\tfrac{1157}{4050}\right)^{1-\frac{1}{q}}\left(\left(\mathcal{K}\left(1,s,\tfrac{14}{45}\right)|\eth'(\xi_1)|^q+\mathcal{V}\left(1,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\mathcal{K}\left(1,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^q+\mathcal{V}\left(1,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right)$$

$$+\left(\tfrac{137}{450}\right)^{1-\frac{1}{q}}\left(\left(\mathcal{V}\left(1,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{3\xi_1+\xi_2}{4}\right)\right|^q+\mathcal{K}\left(1,s,\tfrac{4}{15}\right)\left|\eth'\left(\tfrac{\xi_1+\xi_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\mathcal{V}\left(1,s,\tfrac{14}{45}\right)\left|\eth'\left(\tfrac{\xi_1+3\xi_2}{4}\right)\right|^q+\mathcal{K}\left(1,s,\tfrac{14}{45}\right)|\eth'(\xi_2)|^q\right)^{\frac{1}{q}}\right),$$

where

$$\mathcal{K}\left(1,s,\tfrac{14}{45}\right)=\frac{14s-17}{45(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\tfrac{31}{45}\right)^{s+2},$$

$$\mathcal{V}\left(1,s,\tfrac{14}{45}\right)=\frac{31s+17}{45(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\tfrac{14}{45}\right)^{s+2},$$

$$\mathcal{K}\left(1,s,\tfrac{4}{15}\right)=\frac{4s-7}{15(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\tfrac{11}{15}\right)^{s+2},$$

$$\mathcal{V}\left(1,s,\tfrac{4}{15}\right)=\frac{11s+7}{15(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\tfrac{4}{15}\right)^{s+2}.$$

Corollary 2.8 *In Theorem 2.3, if we take $s=1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\xi_2-\xi_1)^{\alpha}}\mathcal{B}\left(\xi_1,\xi_2,I^{\alpha}\eth\right)\right|$$

$$\leq \frac{\xi_2-\xi_1}{16}\left(\left(\frac{31-14\alpha}{45(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{14}{45}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\left(\mathcal{K}\left(\alpha,1,\tfrac{14}{45}\right)\left|\eth'\left(\xi_1\right)\right|^q+\mathcal{V}\left(\alpha,1,\tfrac{14}{45}\right)\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\mathcal{K}\left(\alpha,1,\tfrac{4}{15}\right)\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\mathcal{V}\left(\alpha,1,\tfrac{4}{15}\right)\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right)$$
$$+\left(\frac{11-4\alpha}{15(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{4}{15}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\mathcal{V}\left(\alpha,1,\tfrac{4}{15}\right)\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\mathcal{K}\left(\alpha,1,\tfrac{4}{15}\right)\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\mathcal{V}\left(\alpha,1,\tfrac{14}{45}\right)\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+\mathcal{K}\left(\alpha,1,\tfrac{14}{45}\right)\left|\eth'\left(\xi_2\right)\right|^q\right)^{\frac{1}{q}}\right)\right),$$

where

$$\mathcal{K}\left(\alpha,1,\tfrac{14}{45}\right)=\frac{45-7(\alpha+1)(\alpha+2)}{45(\alpha+1)(\alpha+2)}+\frac{28\alpha}{45(\alpha+1)}\left(\frac{14}{45}\right)^{\frac{1}{\alpha}}-\frac{14\alpha}{45(\alpha+2)}\left(\frac{14}{45}\right)^{\frac{2}{\alpha}},$$

$$\mathcal{V}\left(\alpha,1,\tfrac{14}{45}\right)=\frac{31-7\alpha}{45(\alpha+2)}+\frac{14\alpha}{54(\alpha+2)}\left(\frac{14}{45}\right)^{\frac{2}{\alpha}},$$

$$\mathcal{K}\left(\alpha,1,\tfrac{4}{15}\right)=\frac{15-2(\alpha+1)(\alpha+2)}{15(\alpha+1)(\alpha+2)}+\frac{8\alpha}{15(\alpha+1)}\left(\frac{4}{15}\right)^{\frac{1}{\alpha}}-\frac{4\alpha}{15(\alpha+2)}\left(\frac{4}{15}\right)^{\frac{2}{\alpha}},$$

$$\mathcal{V}\left(\alpha,1,\tfrac{4}{15}\right)=\frac{11-2\alpha}{15(\alpha+2)}+\frac{4\alpha}{15(\alpha+2)}\left(\frac{4}{15}\right)^{\frac{2}{\alpha}}.$$

Corollary 2.9 *In Theorem 2.3, if we take $\alpha=s=1$, we obtain*

$$\left|\frac{7\eth(\xi_1)+32\eth\left(\frac{\xi_1+3\xi_2}{4}\right)+12\eth\left(\frac{\xi_1+\xi_2}{2}\right)+32\eth\left(\frac{3\xi_1+\xi_2}{4}\right)+7\eth(\xi_2)}{90}-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\eth\left(w\right)dw\right|$$

$$\leq \frac{\xi_2-\xi_1}{16}\left(\left(\frac{1157}{4050}\right)^{1-\frac{1}{q}}\left(\left(\frac{53507}{546750}\left|\eth'\left(\xi_1\right)\right|^q+\frac{102688}{546750}\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{1987}{20250}\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q+\frac{4178}{20250}\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q\right)^{\frac{1}{q}}\right)$$
$$+\left(\frac{137}{450}\right)^{1-\frac{1}{q}}\left(\left(\frac{4178}{20250}\left|\eth'\left(\frac{3\xi_1+\xi_2}{4}\right)\right|^q+\frac{1987}{20250}\left|\eth'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\frac{102688}{546750}\left|\eth'\left(\frac{\xi_1+3\xi_2}{4}\right)\right|^q+\frac{53507}{546750}\left|\eth'\left(\xi_2\right)\right|^q\right)^{\frac{1}{q}}\right)\right),$$

3 Applications

Let Υ be the partition of the points $\xi_1 = x_0 < x_1 < ... < x_n = \xi_2$ of the interval $[\xi_1, \xi_2]$, and consider the quadrature formula

$$\int_{\xi_1}^{\xi_2} \Game(u)\, du = \lambda(\Game, \Upsilon) + R(\Game, \Upsilon),$$

where

$$\begin{aligned}\lambda(\Game, \Upsilon) &= \sum_{i=0}^{n-1} \tfrac{x_{i+1}-x_i}{90} \Big(7\Game(x_i) + 32\Game\left(\tfrac{3x_i+x_{i+1}}{4}\right) + 12\Game\left(\tfrac{x_i+x_{i+1}}{2}\right) \\ &\quad + 32\Game\left(\tfrac{x_i+3x_{i+1}}{4}\right) + 7\Game(x_{i+1})\Big)\end{aligned}$$

and $R(\Game, \Upsilon)$ denotes the associated approximation error.

Proposition 3.1 *Let $n \in \mathbb{N}$ and $\Game : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable function on (ξ_1, ξ_2) with $0 \le \xi_1 < \xi_2$ and $\Game' \in L^1[\xi_1, \xi_2]$. If $|\Game'|$ is an s-convex function in the second sense for some fixed $s \in (0, 1]$, we have*

$$\begin{aligned}|R(\Game, \Upsilon)| &\le \sum_{i=0}^{n-1} \tfrac{(x_{i+1}-x_i)^2}{16} \Big(\Big(\tfrac{14s-17}{45(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{31}{45}\right)^{s+2}\Big)\left(|\Game'(x_i)| + |\Game'(x_{i+1})|\right) \\ &\quad + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{32s+19}{45} + \tfrac{(14)^{s+2}+(12)^{s+2}}{(45)^{s+2}}\right)\left(\left|\Game'\left(\tfrac{3x_i+x_{i+1}}{4}\right)\right| + \left|\Game'\left(\tfrac{x_i+3x_{i+1}}{4}\right)\right|\right) \\ &\quad + 2\left(\tfrac{4s-7}{15(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{11}{15}\right)^{s+2}\right)\left|\Game'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right|\Big).\end{aligned}$$

Proof. Applying Corollary 2.1, on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, ..., n-1)$ of the partition Υ, we get

$$\begin{aligned}&\left|\frac{7\Game(x_i)+32\Game\left(\frac{x_i+3x_{i+1}}{4}\right)+12\Game\left(\frac{x_i+x_{i+1}}{2}\right)+32\Game\left(\frac{3x_i+x_{i+1}}{4}\right)+7\Game(x_{i+1})}{90} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} \Game(w)\, dw\right| \\ &\le \tfrac{x_{i+1}-x_i}{16} \Big(\Big(\tfrac{14s-17}{45(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{31}{45}\right)^{s+2}\Big)\left(|\Game'(x_i)| + |\Game'(x_{i+1})|\right) \\ &\quad + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{32s+19}{45} + \tfrac{(14)^{s+2}+(12)^{s+2}}{(45)^{s+2}}\right)\left(\left|\Game'\left(\tfrac{3x_i+x_{i+1}}{4}\right)\right| + \left|\Game'\left(\tfrac{x_i+3x_{i+1}}{4}\right)\right|\right) \\ &\quad + 2\left(\tfrac{4s-7}{15(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)}\left(\tfrac{11}{15}\right)^{s+2}\right)\left|\Game'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right|\Big). \qquad (16)\end{aligned}$$

Multiplying both sides of (16) by $(x_{i+1} - x_i)$, and then summing the obtained inequalities for all $i = 0, 1, ..., n-1$, and using the triangular inequality, we get the desired result.

Application to special means.
For arbitrary real numbers ξ_1, ξ_2, we have:
The Arithmetic mean: $A(\xi_1, \xi_2) = \frac{\xi_1+\xi_2}{2}$.
The Geometric mean: $G(\xi_1, \xi_2) = \sqrt{\xi_1\xi_2}$, $\xi_1, \xi_2 > 0$.

The p-Logarithmic mean: $L_p(\xi_1, \xi_2) = \left(\frac{\xi_2^{p+1}-\xi_1^{p+1}}{(p+1)(\xi_2-\xi_1)}\right)^{\frac{1}{p}}$, $\xi_1, \xi_2 > 0, \xi_1 \ne \xi_2$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 3.2 *Let $\xi_1, \xi_2 \in \mathbb{R}$ with $0 < \xi_1 < \xi_2$, then we have*

$$\left|7A\left(\xi_1^{\frac{1}{2}}, \xi_2^{\frac{1}{2}}\right) + 32A^2\left(\left(\frac{\xi_1+3\xi_2}{4}\right)^{\frac{1}{2}}, \left(\frac{3\xi_1+\xi_2}{4}\right)^{\frac{1}{2}}\right) + 6A^{\frac{1}{2}}(\xi_1, \xi_2) - 45L_2^2(\xi_1, \xi_2)\right|$$
$$\leq \frac{3\sqrt{62}(\xi_2-\xi_1)}{16}\left(\left(\frac{241}{279}\right)^{\frac{1}{2}}\left(\left(\xi_1^2 + \left(\frac{3\xi_1+\xi_2}{4}\right)^2\right)^{\frac{1}{2}} + \left(\left(\frac{\xi_1+3\xi_2}{4}\right)^2 + \xi_2^2\right)^{\frac{1}{2}}\right)\right.$$
$$\left. + \left(\left(\left(\frac{3\xi_1+\xi_2}{4}\right)^2 + \left(\frac{\xi_1+\xi_2}{2}\right)^2\right)^{\frac{1}{2}} + \left(\left(\frac{\xi_1+\xi_2}{2}\right)^2 + \left(\frac{\xi_1+3\xi_2}{4}\right)^2\right)^{\frac{1}{2}}\right)\right).$$

Proof. The assertion follows from Theorem 2.3 applied to the function $\partial(x) = x^2$ with $p = q = 2$.

4 Conclusion

This study is part of the approach of applying analysis tools to differential and integral equations as well as other branches. So, we first established a new identity involving five points, based on this identity and playing on the s-convexity of the first derivative, we obtained new inequalities of Boole type. Finally, we gave some applications to quadrature formula and inequalities for special means.

References

[1] T. Abdeljawad. On conformable fractional calculus. *J. Comput. Appl. Math.* **279** (2015) 57–66.

[2] P. Agarwal, M. Jleli and M. Tomar. Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. *J. Inequal. Appl.* **55** (2017) 10 pp.

[3] M. B. Almatrafi, W. Saleh, A. Lakhdari, F. Jarad and B. Meftah. On the multiparameterized fractional multiplicative integral inequalities. *J. Inequal. Appl.* **(52)** (2024) 27 pp.

[4] M. U. Awan, S. Talib, Y.-M. Chu, M. A. Noor and K. I. Noor. Some new refinements of Hermite-Hadamard-type inequalities involving ψ_k-Riemann-Liouville fractional integrals and applications. *Math. Probl. Eng. Art.* (2020) 10 pp.

[5] M. Benssaad, B. Meftah, S. Ghomrani, W. Kaidouchi. Fractional Hermite-Hadamard type inequalities for functions whose mixed derivatives are co-ordinated $(log, (s, m))$-convex. *Int. J. Nonlinear Anal. Appl.* **13** (2) (2022) 159–171.

[6] B. Bin-Mohsin, A. Lakhdari, N. E. I. Karabadji, M. U. Awan, A. B. Makhlouf, B., Meftah and S. S. Dragomir. An Extension of Left Radau Type Inequalities to Fractal Spaces and Applications. *Axioms.* **13** (9) (2024) 653.

[7] S. Dhar and J. T. Neugebauer. Lyapunov-Type Inequalities for a Fractional Boundary Value Problem with a Fractional Boundary Condition. *Nonlinear Dynamics and Systems Theory.* **22** (2) (2022) 133–143.

[8] S. S. Dragomir and S. Fitzpatrick. The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (4) (1999) 687–696.

[9] T. Du, C. Luo and Z. Cao. On the Bullen-type inequalities via generalized fractional integrals and their applications. *Fractals* **29** (07) (2021) 2150188.

[10] T. Du and X. Yuan. On the parameterized fractal integral inequalities and related applications. *Chaos Solitons Fractals* **170** (2023) Paper No. 113375 22 pp.

[11] I. Franjić, J. Pečarić and I. Perić. General Euler-Boole's and dual Euler-Boole's formulae. *Math. Inequal. Appl.* **8** (2) (2005) 287–303.

[12] S. Ghomrani, B. Meftah, W. Kaidouchi, M. Benssaad. Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated $(\log, (\alpha, m))$-preinvex. *Afr. Mat.* **32** (5-6) (2021) 925–940.

[13] S. Hamida and B. Meftah. Fractional Bullen type inequalities for differentiable preinvex functions. *ROMAI J.* **16** (2) (2020) 63–74.

[14] W. Kaidouchi, B.Meftah, M. Benssaad and S. Ghomrani. Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated extended (s_1, m_1)-(s_2, m_2)-preinvex. *Real Anal. Exchange.* **44** (2) (2019) 305-332.

[15] A. Kashuri, B. Meftah and P.O. Mohammed. Some weighted Simpson type inequalities for differentiable s-convex functions and their applications. *J. Frac. Calc. and Nonlinear Sys.* **1** (1) (2020) 75-94.

[16] U. N. Katugampola. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **6** (4) (2014) 1–15.

[17] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh. A new definition of fractional derivative. *J. Comput. Appl. Math.* **264** (2014) 65–70.

[18] A. Lakhdari, B. Bin-Mohsin, F. Jarad, H. Xu and B. Meftah. A parametrized approach to generalized fractional integral inequalities: Hermite-Hadamard and Maclaurin variants. *Journal of King Saud University-Science* **(36)** (11) (2024) 103523.

[19] A. Lakhdari, H. Budak, M. U. Awan and B. Meftah. Extension of Milne-type inequalities to Katugampola fractional integrals. *Bound. Value Probl.* **100** (2024) 16 pp.

[20] Q. R. Liu, M. U. Awan, B. Bin-Mohsin, M. Z. Javed, L. Ciurdariu and B. Meftah. Bridging Pre-Invex Mappings and Fractional Integrals: A Pathway to Iterative Schemes via Error Boundaries of Maclaurin's Rule. *Fractal and Fractional* **8** (12) (2024) 734.

[21] B. Meftah, M. Benssaad,W. Kaidouchi and S. Ghomrani. Conformable fractional Hermite-Hadamard type inequalities for product of two harmonic s-convex functions. *Proc. Amer. Math. Soc.* **149** (4) (2021) 1495–1506.

[22] J. E. V. Nápoles and B. Bayraktar. On the generalized inequalities of the Hermite-Hadamard type. *Filomat* **35** (14) (2021) 4917–4924.

[23] J. E. V. Nápoles and F. Rabossi. A note on some fractional integral inequalities via generalized fractional integral. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **68** (1) (2022) 79–89.

[24] Soufiane Benyoussef, Oumkeltoum Benhamouda, Mohamed Dalah and Khaled Zennir. A New Numerical Scheme for Solving Time-Fractional Variable-Order Partial Differential Equations. *Nonlinear Dynamics and Systems Theory* **24** (4) (2024) 340–353.

[25] H. Xu, A. Lakhdari, F. Jarad, T. Abdeljawad and B. Meftah. On multiparametrized integral inequalities via generalized α-convexity on fractal set. *Math. Methods Appl. Sci.* **48** (1) (2025) 980–1002.

[26] W. S. Zhu, B. Meftah, H. Xu, F. Jarad and A. Lakhdari. On parameterized inequalities for fractional multiplicative integrals. *Demonstr. Math.* **57** (1) (2024) Paper No. 20230155, 17 pp.