Fractional-Order Shapovalov Mid-Size Firm Model: Dynamical Analysis, Chaos Control and Synchronization

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Abstract: This paper presents a novel fractional-order formulation of the Shapovalov mid-size firm model incorporating Caputo derivatives, offering a more flexible framework for capturing memory effects in economic dynamics. The fundamental dynamical properties of the system are analyzed. Stabilization of the model is achieved through the design of both linear and nonlinear feedback control strategies. In addition, the complete synchronization problem is investigated using a simple linear control law. Numerical simulations confirm the effectiveness and feasibility of the proposed methods. Beyond theoretical insights, the approach enhances forecasting reliability and enables precise tuning of the economic mechanism, thereby improving the overall efficiency of decision-making processes in complex economic systems.

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1 Introduction

Chaos theory is a fundamental branch of mathematics with diverse applications across numerous fields, including meteorology, biology, medicine, engineering, social sciences, and economics [7–9]. It has become an essential tool for studying complex systems characterized by inherent uncertainty and unpredictability.

Nonlinear dynamics investigates systems where outputs are not directly proportional to inputs, leading to phenomena such as chaos, emergence, and bifurcation. Economic models are often intricate, involving interacting economic agents whose nonlinear behaviors can result in chaotic and unstable dynamics. Consequently, economic forecasting based on these models is challenging, underscoring the critical role of chaos theory in addressing the complexities inherent in nonlinear economic systems.

Fractional-order calculus extends differentiation and integration to non-integer orders. This branch of mathematics has garnered increasing interest due to its ability to describe memory effects and hereditary properties in many real-world processes, making it a valuable tool for modeling and decision-making in physics, engineering, biology, medicine, and financial economics [10, 13, 14]. Fractional-order systems offer a more accurate representation of complex behaviors, particularly in systems exhibiting long-term dependencies.

Chaotic behavior in economic models complicates the prediction of market trends, investment risks, and policy outcomes, posing significant challenges for economists and decision-makers. Consequently, developing methods to stabilize chaotic economic systems has become a vital research focus, aiming to enhance the reliability of economic forecasts and support the design of effective control strategies [1, 3].

Numerous studies have investigated chaotic economic models, including the open economy new Keynesian model [4], the Gali and Monacelli model [5], and the Shapovalov mid-size firm model [3].

The Shapovalov model is an economic framework developed to analyze the dynamic behavior of medium-sized firms. It is formulated as a system of differential equations that describe the dynamic interactions among three key production factors: loan amount, fixed capital, and number of employees.

This model has attracted increasing academic attention, with several studies investigating its dynamic properties—such as stability, chaos, and multistability—using analytical tools including Lyapunov methods and attractor theory [2].

The Shapovalov model provides a valuable foundation for studying the nonlinear dynamics of economic systems and contributes to enhancing predictive accuracy in volatile and uncertain economic environments. Furthermore, it serves as an effective instrument for guiding strategic decision-making and informing economic policies tailored to medium-sized enterprises.

This paper aims to study the stabilization and synchronization problems of the fractional version of the Shapovalov system through the implementation of designed feedback control strategies and the application of a linear control law.

The rest of the paper is organized as follows. Section 2 provides the definition of the Caputo fractional-order derivative and theorems concerning the stability of Caputo fractional-order nonlinear dynamical systems. Section 3 presents the fractional-order Shapovalov system, a dynamical system characterized by Caputo fractional derivatives, which captures complex nonlinear behaviors. In Section 4, the stabilization problems of the proposed system are examined through the implementation of designed feedback

control strategies. Section 5 analyzes the complete synchronization of the fractional-order Shapovalov system through the application of a simple linear control law. Finally, Section 6 summarizes the findings, outlines potential directions for future research, and highlights the importance of this study.

2   Preliminaries

This section presents the definition of the Caputo fractional derivative and some stability theorems pertaining to fractional-order nonlinear dynamical systems. For more details, the reader is referred to [11].

Definition 2.1 *The α-th order Caputo fractional derivative of the function $f(t)$ with respect to t and the terminal point 0 is defined as follows:*

$$^C D_t^\alpha f = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,$$

where m is an integer such that $m-1 \leq \alpha \leq m$, and Γ is the well-known Gamma function.

We define the Caputo fractional-order nonlinear dynamical system as follows:

$$^C D_t^\alpha x_i = f_i(x_1, x_2, ..., x_n), \ \left(i = \overline{1,n}, \ 0 < \alpha < 1\right). \tag{1}$$

The equilibrium points of system (1) can be determined by solving the equations

$$f_i(x_1, x_2, ..., x_n) = 0, \ \left(i = \overline{1,n}, \ 0 < \alpha < 1\right).$$

To analyze the stability of system (1), the following theorem is essential.

Theorem 2.1 *[6] The system (1) is asymptotically stable if all the eigenvalues λ_i, $i = \overline{1,n}$, of the Jacobian matrix J evaluated at the equilibrium point of system (1) satisfy the condition*

$$\left|\arg(\lambda_i)\right| > \alpha \frac{\pi}{2}, \left(i = \overline{1,n}, 0 < \alpha < 1\right),$$

where the Jacobian matrix is defined as

$$J = \frac{\partial f}{\partial x}, \ f = (f_1, f_2, ..., f_n), \ x = (x_1, x_2, ..., x_n)^T.$$

Theorem 2.2 *[6] Let $x(t) \in \mathbb{R}^n$ be a vector of continuous and differentiable function, then the relationship*

$$\frac{1}{2} {}^C D_t^\alpha x^T(t) x(t) \leq x^T(t) {}^C D_t^\alpha x(t)$$

holds.

Theorem 2.3 *[6] The equilibrium point of system (1) is stable if for each x, $x(t)^T f(x(t)) \leq 0$, and it is asymptotically stable if $\forall x \neq 0$, $x(t)^T f(x(t)) < 0$.*

3   Fractional-Order Shapovalov Model

To facilitate a deeper understanding and enable more accurate predictions of economic management systems, Shapovalov [12] proposed a differential equation model describing the behavior of a medium-sized firm, formulated as follows:

$$\begin{cases} \dot{x} = -\sigma x + \delta y, \\ \dot{y} = \mu x + \mu y - \beta x z, \\ \dot{z} = -\gamma z + \alpha x y, \end{cases} \tag{2}$$

where $\alpha, \beta, \gamma, \delta, \mu, \sigma$ are positive parameters, and the variables x, y, z represent the growth of three key production factors: the loan amount x , the fixed capital y , and the number of employees z.

Through the coordinate transformation $(x, y, z) \rightarrow \left(\frac{\mu}{\sqrt{\alpha\beta}} x, \frac{\mu\sigma}{\delta\sqrt{\alpha\beta}} y, \frac{\mu\sigma}{\delta\beta} z \right)$ and the time rescaling $t \rightarrow \frac{t}{\mu}$, system (2) can be simplified into a Lorenz-like system

$$\begin{cases} \dot{x} = -cx + cy, \\ \dot{y} = rx + y - xz, \\ \dot{z} = -bz + xy. \end{cases} \tag{3}$$

The system described, where $c = \frac{\sigma}{\mu}, r = \frac{\delta}{\sigma}$ and $b = \frac{\gamma}{\mu}$, differs slightly from the classical Lorenz system due to the coefficient of y in the second equation. In this case, the coefficient is 1, whereas in the Lorenz system, it is -1. Despite this difference, chaos can still arise in system (3) just as it does in the Lorenz system.

Solving the following equilibrium equations:

$$\begin{cases} -cx + cy = 0 \\ rx + y - xz = 0 \\ -bz + xy = 0 \end{cases}$$

with $c = 18.3, r = 51, b = 5.7$, we can easily have three equilibrium points $E_1 = (0, 0, 0)$, $E_2 = (17.216, 17.216, 52.0)$, $E_3 = (-17.216, -17.216, 52.0)$.
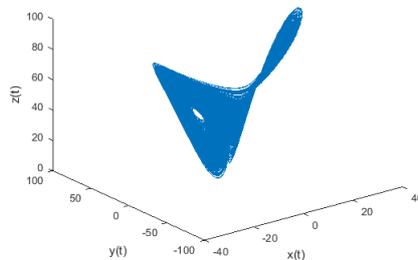The chaotic attractors are shown in Figure 1.

Figure 1: The chaotic attractor of system(3) with c=18.3,r=51,b=5.7.

This paper proposes to study the fractional system

$$
\begin{cases}
{}^C D_t^\alpha x = -cx + cy, \\
{}^C D_t^\alpha y = rx + y - xz, \\
{}^C D_t^\alpha z = -bz + xy.
\end{cases}
\tag{4}
$$

${}^C D_t^\alpha$ denotes the derivatives of order α ; $(0 < \alpha < 1;\ i = \overline{1,n})$ in the sense of Caputo, with $c = 18.3, r = 51, b = 5.7$.

4   Dynamical Analysis of Fractional-Order Shapovalov Model

Suppose system (4) has an equilibrium point $E^* = (x^*, y^*, z^*)$, then the Jacobian matrix J at the equilibrium point is

$$
J(E^*) = \begin{pmatrix} -c & c & 0 \\ r - \overset{*}{z} & 1 & -\overset{*}{x} \\ \overset{*}{y} & \overset{*}{x} & -b \end{pmatrix}.
$$

The characteristic equation $|\lambda I - J|$ at the equilibrium point E^*, for $c = 18.3, r = 51$, $b = 5.7$, is

$$
f(\lambda) = |\lambda I - J| = \begin{vmatrix} \lambda + 18.3 & -18.3 & 0 \\ -51 + z^* & \lambda - 1 & x^* \\ -y^* & -x^* & \lambda + 5.7 \end{vmatrix} = 0.
\tag{5}
$$

It is clear that the system (4) has the same equilibria

$$
E_1 = (0, 0, 0),\ \ E_2 = (17.\,216, 17.\,216, 52.0),\ \ E_3 = (-17.\,216, -17.\,216, 52.0).
$$

Lemma 4.1   *The equilibrium point $E_1 = (0, 0, 0)$ is unstable for $0 < \alpha < 1$.*

Proof. By substituting the value of E_1 into the characteristic equation (5), we obtain the following expression:

$$
f(\lambda) = \lambda^3 + 23.0\lambda^2 - 852.\,99\lambda - 5424.\,1 = 0.
$$

Thus the roots of Equation (5) are $\lambda_1 = 23.\,388$, $\lambda_2 = -5.\,7000$, $\lambda_3 = -40.\,688$. Because λ_1 is a positive real number and

$$
\min_{1 \le i \le 3} |\arg(\lambda_i)| = |\arg(\lambda_1)| = 0 < \alpha \frac{\pi}{2} \qquad (0 < \alpha < 1),
$$

by Theorem 2.1, the equilibrium point E_1 is a saddle point and it is unstable for $0 < \alpha < 1$.

Lemma 4.2 *Equilibrium points E_2, E_3 are asymptotically stable for $\alpha < 0.97$.*

Proof. By applying the same method as in the previous proof, we find

$$
|\arg(\lambda_{2,3})| = 87.65 \quad \text{for all} \quad \alpha \in (0, 1).
$$

(a) x, y, z phase plane at $\alpha = 0.965$

(b) x, y, z phase plane at $\alpha = 0.975$

(c) Time evolution of x for $\alpha = 0.965$

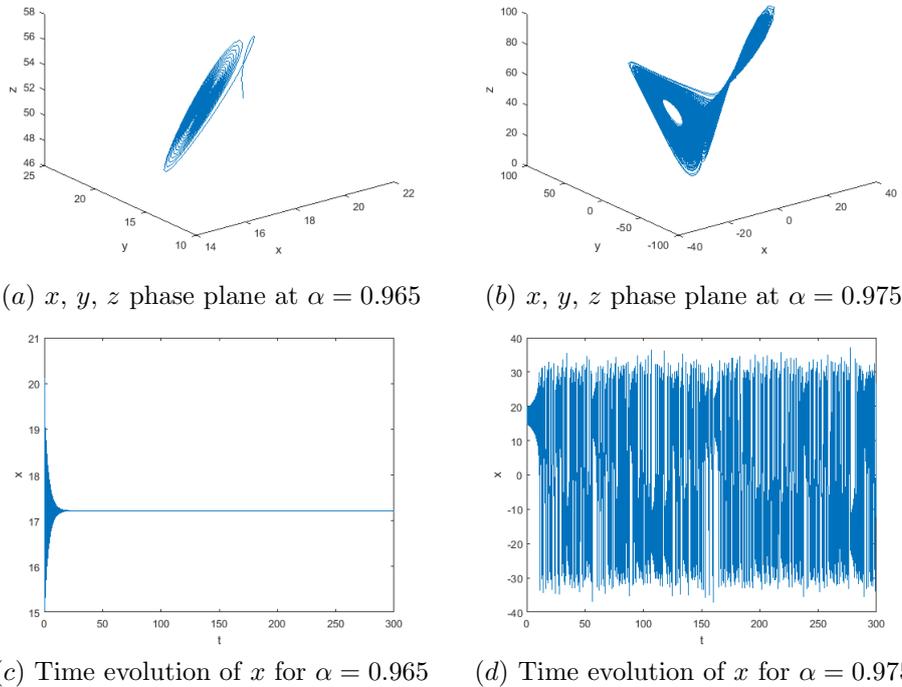(d) Time evolution of x for $\alpha = 0.975$

Figure 2: The attractors and time evolution of x of the fractional-order system for $\alpha = 0.965$ and $\alpha = 0.975$.

By Theorem 2.1, equilibrium points E_2, E_3 are asymptotically stable when

$$\alpha < \frac{2}{\pi} \left| \arg\left(\lambda_{2,3}\right) \right| \simeq 0.97.$$

For the selected parameters $c = 18.3$, $r = 51$, and $b = 5.7$, Figures 2(a)–(d) illustrate the attractors and the time evolution of x for the fractional-order system (4) at two different derivative orders, $a = 0.965$ and $a = 0.975$, with initial conditions $(x(0), y(0), z(0)) = (20, 20, 50)$. Figures 2(a) and 2(c) show that the system converges to a fixed point, whereas Figures 2(b) and 2(d) reveal the presence of chaotic attractors. These observations indicate that the fractional-order system (4) can exhibit chaotic behavior when the derivative order exceeds 0.97.

5   Feedback Control

5.1   Linear feedback control

In this section, we examine the chaotic control of system (4) through linear control techniques. This strategy facilitates the attainment of local stability at any unstable equilibrium point within system (4). The representation of the linear controlled fractional-order

system is as follows:

$$\begin{cases} {}^C D_t^\alpha x = -cx + cy - k_1 (x - x^*), \\ {}^C D_t^\alpha y = rx + y - xz - k_2 (y - y^*), \\ {}^C D_t^\alpha z = -bz + xy - k_3 (z - z^*), \end{cases} \tag{6}$$

where k_1, k_2, k_3 are three linear control parameters, the fractional order $\alpha = 0.98$, and the equilibrium point of the system is (x^*, y^*, z^*).

The Jacobian matrix of the system (6) at the equilibrium point (x^*, y^*, z^*) is given as

$$J(E^*) = \begin{pmatrix} -c - k_1 & c & 0 \\ r - z^* & 1 - k_2 & -x^* \\ y^* & x^* & -b - k_3 \end{pmatrix}.$$

The corresponding characteristic equation for $c = 18.3, r = 51, b = 5.7$, reads

$$f(\lambda) = |\lambda I - J| = \begin{vmatrix} \lambda + 18.3 + k_1 & -18.3 & 0 \\ -51 + z^* & \lambda - 1 + k_2 & x^* \\ -y^* & -x^* & \lambda + 5.7 + k_3 \end{vmatrix}. \tag{7}$$

Remark 5.1 In order to stabilize the system (4), the control parameters k_1, k_2, k_3 are chosen such that all eigenvalues λ_i of Equation (7) satisfy

$$|\arg(\lambda_i)| > \alpha \frac{\pi}{2}, \left(i = \overline{1,3}, \ 0 < \alpha < 1\right).$$

Lemma 5.1 *The system* (6) *is locally asymptotically stable at the equilibrium point* $E_1 = (0,0,0)$ *with the control parameters* $k_1 = 25, k_2 = 25, k_3 = 25$.

Proof. Substituting the given values of E_1, k_1, k_2 and k_3 into the characteristic equation (7), we obtain

$$f(\lambda) = |\lambda I - J| = \begin{vmatrix} \lambda + 18.3 + 25 & -18.3 & 0 \\ -51 & \lambda - 1 + 25 & 0 \\ 0 & 0 & \lambda + 5.7 + 25 \end{vmatrix}.$$

The characteristic roots of this equation are approximately

$$\lambda_1 = -1.6122, \lambda_2 = -30.700, \lambda_3 = -65.688.$$

By Theorem 2.1, $|\arg(\lambda_i)| = \pi > \alpha \frac{\pi}{2}, \left(i = \overline{1,3}, 0 < \alpha < 1\right)$. Hence, the controlled fractional-order system (6) is locally asymptotically stable at E_1.

Lemma 5.2 *The system* (6) *is locally asymptotically stable at the equilibrium points* $E_2 = (17.216, 17.216, 52.0)$ *and* $E_3 = (-17.216, -17.216, 52.0)$ *with the control parameters* $k_1 = 1, k_2 = 40, k_3 = 70$.

Proof. Substituting the given values of E_2, k_1, k_2 and k_3 into the characteristic equation (7), we obtain

$$f(\lambda) = |\lambda I - J| = \begin{vmatrix} \lambda + 18.3 + 1 & -18.3 & 0 \\ -51 + 52.0 & \lambda - 1 + 40 & 17.216 \\ -17.216 & -17.216 & \lambda + 5.7 + 70 \end{vmatrix}.$$

The roots of this equation are approximately

$$\lambda_1 = -26.035 \ , \lambda_2 = -38.353 \ , \ \lambda_3 = -69.612.$$

By Theorem 2.1, $|\arg(\lambda_i)| = \pi > \alpha\frac{\pi}{2}, \left(i = \overline{1,3}, \ 0 < \alpha < 1\right)$. Hence, the controlled fractional-order system (6) is locally asymptotically stable at E_2. Similarly, for $k_1 = 1, k_2 = 40, k_3 = 70$, system (6) is also locally asymptotically stable at the equilibrium point E_3.

The three linear feedback controllers are capable of effectively stabilizing the fractional-order Shapovalov system (4) at different equilibrium points.

As depicted in Figure 3 (a)-(c), the controllers are activated at $t = 20s$ with a derivative order $\alpha = 0.98$. When $t < 20s$, system (4) exhibits chaotic behavior. After the controllers are activated, the state vectors x, y, and z gradually converge to different equilibrium points, namely E_1, E_2, and E_3, respectively.

5.2   Nonlinear feedback control

The nonlinear controlled fractional-order system is defined as

$$\begin{cases} {}^{C}D_t^\alpha x = -cx + cy + u_1, \\ {}^{C}D_t^\alpha y = rx + y - xz + u_2, \\ {}^{C}D_t^\alpha z = -bz + xy + u_3, \end{cases} \tag{8}$$

where u_1, u_2 and u_3 are nonlinear controllers that will be designed later.

Lemma 5.3 *System* (4) *can achieve global asymptotic stability under the nonlinear controllers*

$$u_1 = -k_1 x - 0.1x^3, \ u_2 = -k_2\left(y+x\right) - 0.05y^3 \ and \ u_3 = -k_3 z - 0.02z^3 \tag{9}$$

if and only if the conditions $k_1 > -c, \ k_2 > c + r$ *and* $k_3 > -b$ *hold.*

Proof. Choose a positive definite Lyapunov function

$$V = \frac{1}{2}\left(x^2 + y^2 + z^2\right). \tag{10}$$

Using expressions (8) and (9), the fractional derivative of the Lyapunov function (10) is obtained by

$$^{C}D_t^\alpha V \le x \ {}^{C}D_t^\alpha x + y{}^{C}D_t^\alpha y + z \ {}^{C}D_t^\alpha z$$

$$= x\left(-cx + cy + u_1\right) + y\left(rx + y - xz + u_2\right) + z\left(-bz + xy\right)$$

$$= x\left(-cx + cy - k_1 x - 0.1x^3\right) + y\left(rx + y - xz - k_2\left(y+x\right) - 0.05y^3\right)$$

$$+z\left(-bz + xy - k_3 z - 0.02z^3\right)$$

$$= -cx^2 + cxy - k_1 x^2 - 0.1x^4 + rxy + y^2 - xyz - k_2 y^2 - k_5 xy - 0.05y^4 - bz^2 + xyz - k_3 z^2 - 0.02z^4$$

$$= -\left(c + k_1\right)x^2 - \left(k_2 - 1\right)y^2 - \left(k_3 + b\right)z^2 - \left(k_2 - c - r\right)xy - 0.1x^4 - 0.05y^4 - 0.02z^4.$$
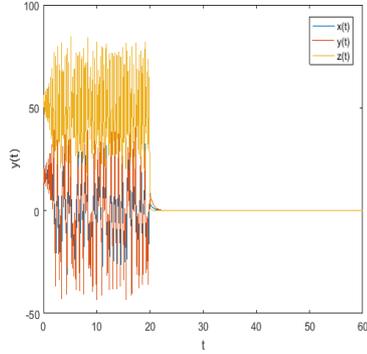
(a) Stabilization of the equilibrium point E_1
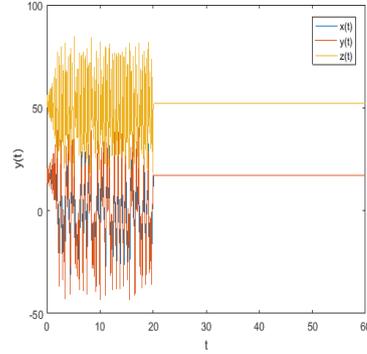
(b) Stabilization of the equilibrium point E_2
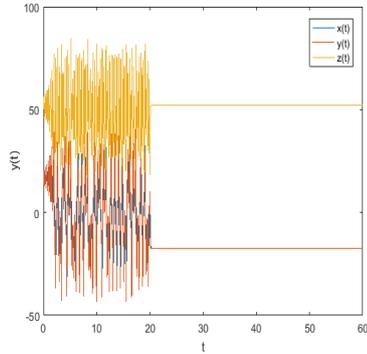
(c) Stabilization of the equilibrium point E_3
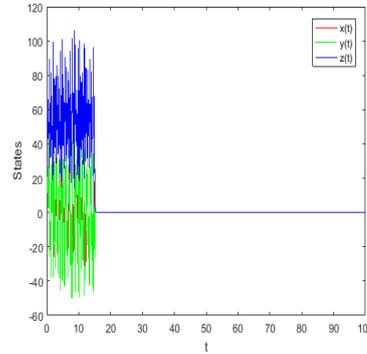
(d) Global stabilization

Figure 3: Chaotic control of the fractional-order Shapovalov system.

If $k_1 > -c$, $k_2 > c + r$, and $k_3 > -b$ hold, then the fractional derivative of the Lyapunov function is strictly negative. According to Theorem 2.3, the controlled system (8) is globally asymptotically stable.

As illustrated in Figure 3(d), with the parameters $c = 18.3$, $r = 51$, $b = 5.7$ and $\alpha = 0.98$, and when selecting $k_1 = 69.4$, $k_2 = 69.4$, and $k_3 = 2$, the system transitions from chaotic behavior to a state of global asymptotic stability when the controllers are activated at $t = 15$ s.

6   Synchronization of the Fractional-Order System

In this section, we investigate the complete synchronization of the fractional-order Shapovalov system by employing a simple linear control law. To this end, we consider the following master–slave systems:

Master system:

$$\begin{cases} {}^{C}D_t^{\alpha} x_1 = -cx_1 + cy_1, \\ {}^{C}D_t^{\alpha} y_1 = rx_1 + y_1 - x_1 z_1, \\ {}^{C}D_t^{\alpha} z_1 = -bz_1 + x_1 y_1. \end{cases} \tag{11}$$

Slave system:

$$\begin{cases} {}^{C}D_t^{\alpha}x_2 = -cx_2 + cy_2 + U_1, \\ {}^{C}D_t^{\alpha}y_2 = rx_2 + y_2 - x_2z_2 + U_2, \\ {}^{C}D_t^{\alpha}z_2 = -bz_2 + x_2y_2 + U_3, \end{cases} \tag{12}$$

where $X(t) = [x_1(t), \ y_1(t), \ z_1(t)]^T \in \mathbb{R}^3$ and $Y(t) = [x_2(t), \ y_2(t), \ z_2(t)]^T \in \mathbb{R}^3$ are the state vectors of the master and slave systems, respectively. The vector $[x_1(0), \ y_1(0), \ z_1(0)]^T$ denotes the unknown initial state of the master system, while $[x_2(0), \ y_2(0), \ z_2(0)]^T$ is the given initial state of the slave system.

To achieve complete synchronization, we design a simple control law

$$U(t) = [U_1(t), \ U_2(t), \ U_3(t)]^T \in \mathbb{R}^3,$$

which ensures that the states of the drive dynamical system (11) synchronize with those of the response dynamical system (12). Define the synchronization error vector as

$$e(t) = [e_1(t), \ e_2(t), \ e_3(t)]^T = Y(t) - X(t),$$

where $e_1(t) = x_2(t) - x_1(t)$, $e_2(t) = y_2(t) - y_1(t)$, and $e_3(t) = z_2(t) - z_1(t)$.

Definition 6.1 Systems (11) and (12) mentioned above are said to achieve complete synchronization if
$$\lim_{t \to +\infty} \|Y(t) - X(t)\| = 0.$$

Subsequently, we can deduce the following result concerning the complete synchronization between systems (11) and (12).

Theorem 6.1 *For the control law given by*

$$U_1(t) = -k_1e_1, \quad U_2(t) = -k_2e_2 + z_1e_1, \quad U_3(t) = -k_3e_3 - y_1e_1, \tag{13}$$

the master system (11) and the slave system (12) achieve complete synchronization.

Proof. From systems (11) and (12), we infer that the error system is given by

$$\begin{cases} D_t^{\alpha}e_1 = -ce_1 + ce_2 + U_1, \\ D_t^{\alpha}e_2 = re_1 + e_2 - e_1e_3 - x_1e_3 - e_1z_1 + U_2, \\ D_t^{\alpha}e_3 = -be_3 + e_1e_2 + x_1e_2 + e_1y_1 + U_3. \end{cases} \tag{14}$$

Let the Lyapunov candidate function be

$$V(t) = \frac{1}{2}\left(e_1^2(t) + e_2^2(t) + e_3^2(t)\right).$$

The Caputo fractional derivative of $V(t)$ along the trajectories of system (14) satisfies

$${}^{C}D_t^{\alpha}V(t) \le e_1 \cdot {}^{C}D_t^{\alpha}e_1 + e_2 \cdot {}^{C}D_t^{\alpha}e_2 + e_3 \cdot {}^{C}D_t^{\alpha}e_3.$$

Substitute the expressions from the error dynamics and the control law (13)

$$\begin{aligned} {}^{C}D_t^{\alpha}V(t) = {}& e_1(-ce_1 + ce_2 - k_1e_1) \\ & + e_2\left(re_1 + e_2 - e_1e_3 - x_1e_3 - e_1z_1 - k_2e_2 + z_1e_1\right) \\ & + e_3\left(-be_3 + e_1e_2 + x_1e_2 + e_1y_1 - k_3e_3 - y_1e_1\right). \end{aligned}$$

Simplifying, we obtain

$$^{C}D_t^\alpha V(t) = -(k_1 + c)e_1^2 - (k_2 - 1)e_2^2 - (k_3 + b)e_3^2$$
$$+ (c + r)e_1 e_2 - z_1 e_1 e_2 + z_1 e_1 e_2 + y_1 e_1 e_3 - y_1 e_1 e_3.$$

Hence, cross terms cancel out, and we are left with

$$^{C}D_t^\alpha V(t) \leq -(k_1 + c)e_1^2 - (k_2 - 1)e_2^2 - (k_3 + b)e_3^2 + \frac{1}{2}(c + r)(e_1^2 + e_2^2).$$

Rewriting this

$$^{C}D_t^\alpha V(t) = -\left(k_1 + c - \frac{1}{2}(c + r)\right)e_1^2 - \left(k_2 - 1 - \frac{1}{2}(c + r)\right)e_2^2 - (k_3 + b)e_3^2$$
$$= -\left(k_1 + \frac{1}{2}c - \frac{1}{2}r\right)e_1^2 - \left(k_2 - \frac{1}{2}(c + r) - 1\right)e_2^2 - (k_3 + b)e_3^2.$$

The derivative $^{C}D_t^\alpha V(t) < 0$ holds if the following conditions are satisfied:

$$k_1 > -\frac{1}{2}c + \frac{1}{2}r, \quad k_2 > \frac{1}{2}(c + r) + 1, \quad k_3 > -b.$$

For example, when the parameters are given as $c = 18.3$, $r = 51$, $b = 5.7$, and when selecting $k_1 = 17$, $k_2 = 36$, $k_3 = 10$, the derivative of the Lyapunov function becomes negative definite.

Therefore, according to Theorem 2.3, the origin of the error system (14) is asymptotically stable, and the synchronization errors $e_1(t)$, $e_2(t)$, and $e_3(t)$ converge to zero. This confirms that the master and slave systems achieve complete synchronization.

To assess the effectiveness of complete synchronization through numerical simulation, we consider the control law given in equation (13), with control gains selected as $k_1 = 17$, $k_2 = 36$, and $k_3 = 10$. The fractional derivative order is set to $\alpha = 0.99$.

The initial state vector of the master system (11) is chosen as $[x_1(0),\ y_1(0),\ z_1(0)] = [1,\ 1,\ 1]$, while the initial state of the slave system (12) is set to $[x_2(0),\ y_2(0),\ z_2(0)] = [2,\ 2,\ 2]$.

According to the definition of the synchronization error, the initial condition of the error system (14) is therefore $[e_1(0),\ e_2(0),\ e_3(0)] = [1,\ 1,\ 1]$.

Figure 4(a) illustrates the time evolution of the synchronization errors $e_1(t)$, $e_2(t)$, $e_3(t)$, which all converge to zero, indicating successful synchronization. Figure 4(b) depicts the trajectories of systems (11) and (12) in the 3D phase space, showing their convergence. Figure 4(c) presents the time responses of the state variables $x(t)$, $y(t)$, $z(t)$ for both systems, highlighting the agreement between their dynamics over time.

7   Conclusion

This study explores the chaotic behavior of the fractional-order Shapovalov model, an economic system developed to analyze the dynamic behavior of medium-sized enterprises. The incorporation of fractional calculus offers a powerful framework for modeling real-world economic phenomena more accurately, as it captures memory effects and hereditary characteristics often overlooked by classical integer-order models.
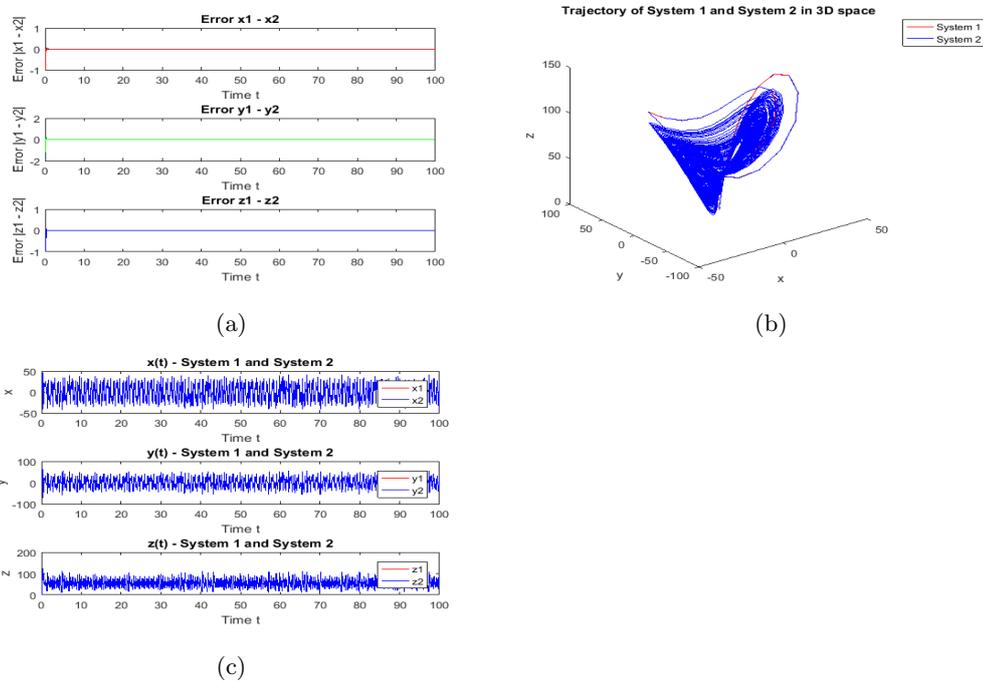
(a)

(b)

(c)

Figure 4: The complete synchronization between systems (11) and (12).

In this context, the suppression of chaos is addressed through the design of both linear and nonlinear feedback controllers. Additionally, the problem of complete synchronization is analyzed using the Lyapunov stability approach. Numerical simulations further validate the effectiveness of the proposed methods in stabilizing the chaotic dynamics of the system.

Stabilizing such economic models contributes meaningfully to the field by providing strategies for managing chaotic behavior and enhancing the reliability of economic forecasting and policy-making. Studying the dynamics of the Shapovalov model, along with the application of control and synchronization techniques, can support the forecasting processes in medium-sized enterprises. Moreover, it can assist in guiding these enterprises toward achieving their strategic goals by modifying and regulating their dynamic behavior.

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