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Analysis and Numerical Simulations of Fractional Order Model of Insect-Pest Dynamics

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Abstract: In this paper, we study a fractional-order version of the dynamics of insect-pests. We establish the existence and uniqueness of non-negative solutions and using the Laplace transform technique, we prove the boundedness of the solutions of the fractional model. Using Lyapunov's indirect method and the extension of LaSalle's invariance principle, we study the local and global stability of the equilibrium points of the fractional model. Moreover, we illustrate our theoretical results with numerical simulations of fractional model using the Adams-Bashforth-Moulton scheme.

Keywords: *Caputo fractional derivative; Riemann-Liouville fractional integral; insect-pest; numerical simulation.*

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1 Introduction

Insect-pests pose a threat worldwide, mainly in Africa, the economic and social burden of the damage caused by insect-pests such as fruit flies is increasing at both the producer and industry levels. Crop losses in these countries total billions of dollars and have a harmful impact on their economic situation and nutritional level [11]. In view of this threat, integrated pest management has been proposed to producers to combat this problem for the crops.

The massive use of pesticides as a means of controlling these insects leads to environmental pollution and economical waste. In addition, pesticides are generally harmful products and must be handled with care. Unfortunately, the precautions to be taken when handling these chemicals are unknown or ignored by producers. In this regard, the

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integrated control approach against large insect invasions for the control of these fruit flies is to be favored. Thus, in order to propose an effective and inexpensive control program to reduce the proliferation of insect-pests in crops, a better knowledge of the dynamics of insect-pests is essential.

With the view of studying the impact of mating disruption control by using pheromone traps to distract males from females in order to reduce the proliferation of insect-pests, R. Anguelov *et al.* [3] introduced a generic model for insect-pests control using ordinary differential equations. In order to apply appropriate control methods against insect-pests, M. Djoukwe *et al.* [8] developed a delay model by applying strategies to control cocoa pests such as Sahlbergella singularis. In modeling, birth or growth processes are considered to be instantaneous phenomena. However, in real life, there is a time lag between the appearance of a new individual and its reproduction or activity. Therefore, to better reproduce as faithfully as possible by taking into account the total memory of past states in the dynamics of a given population, the singular or non-singular fractional differential operators are used in mathematical modeling [2,5,6]. Models based on ordinary differential equations are certainly of interest, but they do not take into account the historical life traits of pests, for example, an immature insect will go through different stages before becoming an adult capable of reproducing in turn. Fractional calculations, by allowing integrals and derivatives of integers orders as well as non-integers orders, represent a powerful tool in applied mathematics. Recently, particular attention has been focused on fractional derivatives as a tool for modeling certain real-life phenomena because of the hereditary nature and the memory effect of events. Systems described by fractional order models, using fractional differential equations based on the non-integer derivative, have attracted the interest of the scientific community. For this purpose, in [7], K. Diethelm studied the infection rate of dengue fever using a fractional model with the fractional derivative in the sense of Caputo and then compared with great success the numerical results obtained and the real data from the Cape Verde Islands.

Motivated by the applications of fractional calculus in modeling [5, 7], in this paper, we study a Caputo-type fractional model of the dynamics of insect-pests inspired by a classical model [3]. The goal of this work is to carry out the formulation and mathematical analysis of the established fractional model as well as numerical simulations to illustrate our theoretical results.

The work is organized as follows. We started by giving some useful basic notions for the analysis of fractional differential equations in the sense of Caputo in Section 2. In Section 3, we are interested in the formulation of the fractional model. Section 4 is devoted to the study of some properties of the solutions of the fractional model. In Section 5, we make numerical simulations and provide comments. We end this work by a conclusion.

2 Preliminaries on the Fractional Calculus

In this section, we recall some notions that will be used in the next sections. Let a and b be two real numbers such that $a < b$. Throughout this paper, $\mathcal{C}([a;b])$ denotes the space of continuous functions defined on $[a;b]$, $\mathcal{C}^n([a;b])$ stands for the class of all real-valued functions which are defined on $[a;b]$ and have continuous n-th order derivatives, and $L^1(a;b)$ is the usual Lebesgue space. Furthermore, we consider the sets $\mathbb{R}^n_+ := \{(x_1, x_2, \dots, x_n), x_i \geq 0, i = 1, 2, \dots, n\}$ and $\mathbb{R}^n_{++} := \{(x_1, x_2, \dots, x_n), x_i > 0, i = 1, 2, \dots, n\}$.

Definition 2.1 (Riemann-Liouville fractional integral [6, 10]) Let $\alpha \geq 0$, $f \in \mathcal{C}([a;b])$. The Riemann-Liouville fractional integral of order α of f is the function ${}^{RL}\mathcal{J}_a^{\alpha}[f]$ defined on $[a;b]$ by

$$ {}^{RL}\mathcal{J}_a^{\alpha}[f](t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi) d\xi, & \text{if } \alpha > 0, \\ f(t), & \text{if } \alpha = 0, \end{cases} $$

where $\Gamma(\cdot)$ is Euler's Gamma function defined on $\mathbb{R}_{++}$ by

$$ \Gamma(\alpha) := \int_0^{+\infty} r^{\alpha-1} e^{-r} dr. $$

Definition 2.2 (Caputo fractional derivative [6, 10]) Let $\alpha \geq 0$, $f \in \mathcal{C}^{\lceil \alpha \rceil}([a;b])$. The Caputo fractional derivative of f of order α is the function ${}^{C}\mathcal{D}_a^{\alpha}[f]$ defined on $[a;\, b]$ by

$$ {}^{C}\mathcal{D}_a^{\alpha}[f](t) := {}^{RL}\mathcal{J}_a^{\lceil \alpha \rceil - \alpha}\left[f^{(\lceil \alpha \rceil)} \right](t), $$

where $\lceil \cdot \rceil$ is a ceiling function defined by $\lceil \alpha \rceil := \min\{n \in \mathbb{Z} : \alpha \leq n\}$ and $f^{(\lceil \alpha \rceil)}$ denotes the $\lceil \alpha \rceil$-th derivative of f.

Let us recall the definition of the Mittag-Leffler function.

Definition 2.3 ([6]) Let $\alpha,\, \nu > 0$. Then the two-parameter Mittag-Leffler function $E_{\alpha,\nu}(\cdot)$ is defined by the series expansion

$$ E_{\alpha,\,\nu}(q) = \sum_{n=0}^{\infty} \frac{q^n}{\Gamma(\nu + \alpha n)}, \quad q \in \mathbb{R}. $$

We simply denote $E_{\alpha,\,1}(\cdot)$ by $E_{\alpha}(\cdot)$.

Proposition 2.1 ([6,10]) *Let $\alpha, \nu > 0$ and $\lambda \in \mathbb{R}$. Then*

$$ \mathcal{L}\{t^{\nu-1} E_{\alpha,\nu}(\mp \lambda t^{\alpha})\}(s) = \frac{s^{\alpha-\nu}}{s^{\alpha} \pm \lambda} \quad \left(\Re(s) > 0;\, |s| > |\lambda|^{\frac{1}{\alpha}} \right), \tag{1} $$

where $\mathcal{L}\{\cdot\}$ stands for the Laplace transform and $\Re(s)$ is the real part of s.

Proposition 2.2 ([6]) *Let $\alpha,\, \nu > 0$ and $q \in \mathbb{R}$, we have*

$$ E_{\alpha,\,\nu}(q) = q E_{\alpha,\,\alpha+\nu}(q) + \frac{1}{\Gamma(\nu)}. $$

In order to prove the properties of the fractional model solutions, we need the following useful lemmas.

Lemma 2.1 *Let $\alpha \in (0\,;\, 1]$, $f \in \mathcal{C}([a;b])$ and ${}^{C}\mathcal{D}_a^{\alpha}[f] \in \mathcal{C}((a;b])$. Then for all $t \in (a;b]$, there exists $\xi \in (a;\, t)$ for which*

$$ f(t) = f(a) + {}^{C}\mathcal{D}_a^{\alpha}[f](\xi) \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}. $$

Proof. Using Definition 2.1, we can write

$$^{RL}\mathcal{J}_a^{\alpha}\left[{}^{C}\mathcal{D}_a^{\alpha}[f]\right](t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}\,{}^{C}\mathcal{D}_a^{\alpha}[f](\tau)d\tau.$$

By the generalized mean value theorem for integrals, there exists $\xi \in (a;\, t)$ such that

$$\begin{aligned} ^{RL}\mathcal{J}_a^{\alpha}\left[{}^{C}\mathcal{D}_a^{\alpha}[f]\right](t) &= {}^{C}\mathcal{D}_a^{\alpha}[f](\xi)\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}d\tau \\ &= {}^{C}\mathcal{D}_a^{\alpha}[f](\xi)\frac{(t-a)^{\alpha}}{\alpha\Gamma(\alpha)} = {}^{C}\mathcal{D}_a^{\alpha}[f](\xi)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}. \end{aligned} \tag{2}$$

On the other hand, it follows from the Taylor expansion for Caputo derivatives (see [6], Corollary 3.9) that for all $\alpha \in (0;1]$ and $t \in (a;\, b]$, we have

$$^{RL}\mathcal{J}_a^{\alpha}\left[{}^{C}\mathcal{D}_a^{\alpha}[f]\right](t) = f(t) - f(a). \tag{3}$$

From (2) and (3) we obtain $f(t) = f(a) + {}^{C}\mathcal{D}_a^{\alpha}[f](\xi)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}$. □

Remark 2.1 ([13]) Suppose that $f \in \mathcal{C}([a;b])$ and ${}^{C}\mathcal{D}_a^{\alpha}[f] \in \mathcal{C}((a;b])$, $\alpha \in (0;1]$. It follows from Lemma 2.1 that if ${}^{C}\mathcal{D}_a^{\alpha}[f](t) \geq 0$ for all $t \in (a;b)$, then f is nondecreasing on $[a;b]$ and if ${}^{C}\mathcal{D}_a^{\alpha}[f](t) \leq 0$ for all $t \in (a;b)$, then f is non-increasing on $[a;b]$.

Remark 2.2 Note that the result established in Lemma 2.1 was initially presented in the work of Z.M. Odibat *et al.* [13]. However, it contains calculation errors.

Lemma 2.2 ([6,10]) Let $\alpha > 0$ and $f \in \mathcal{C}^{\lceil\alpha\rceil}(\mathbb{R}_+)$. Moreover, suppose that $f^{(\lceil\alpha\rceil)} \in L^1(0;b)$ for any $b > 0$ is of exponential order, the Laplace transforms $\mathcal{L}[f]$ and $\mathcal{L}\left[f^{(\lceil\alpha\rceil)}\right]$ exist, and $\lim_{t\to\infty} f^{(k)}(t) = 0$ for $k = 0, 1, \ldots, \lceil\alpha\rceil - 1$. Then the Laplace transform of the fractional derivative of Caputo is

$$\mathcal{L}\left[{}^{C}\mathcal{D}_0^{\alpha}[f]\right](s) = s^{\alpha}\mathcal{L}[f](s) - \sum_{k=0}^{\lceil\alpha\rceil-1} s^{\alpha-k-1}f^{(k)}(0).$$

As a particular case, if $\alpha \in (0;\, 1]$, we have

$$\mathcal{L}\left[{}^{C}\mathcal{D}_0^{\alpha}[f]\right](s) = s^{\alpha}\mathcal{L}[f](s) - s^{\alpha-1}f(0). \tag{4}$$

Lemma 2.3 ([15]) *Let $x \in \mathcal{C}^1(\mathbb{R}_+)$ such that $x(t) > 0$ for all $t \in \mathbb{R}_+$. Then we have*

$${}^{C}\mathcal{D}_0^{\alpha}\left[x - x^* - x^*\ln\left(\frac{x}{x^*}\right)\right](t) \leq \left(1 - \frac{x^*}{x(t)}\right){}^{C}\mathcal{D}_0^{\alpha}[x](t),\ \forall\alpha \in (0;1),\ x^* \in \mathbb{R}_{++}.$$

3 Mathematical Formulation of the Model

In this section, we briefly describe the different steps in the formulation of the standard model of the dynamics of a population of insect pests such as fruit flies [3]. Thus, two main stages of development can be considered. The first stage, called the immature stage (I), which includes eggs, larvae and pupae, constitutes one class and the second stage called the adult stage is divided into three classes: the class of mating females (Y), the class of females inseminated (F) and the class of males (M). Table 1 describes the model parameters.

Parameters	**Descriptions**
b	intrinsic egg-laying rate
r	female to male ratio
K	carrying capacity
γ	females fertilized by a single male
μ_I	natural death rate of immature
μ_F	natural death rate of females
μ_M	natural death rate of males
ν_I	transfer rate from I to Y
ν_Y	transfer rate from Y to F
δ	transfer rate from F to Y

Table 1: Biological description of model parameters.

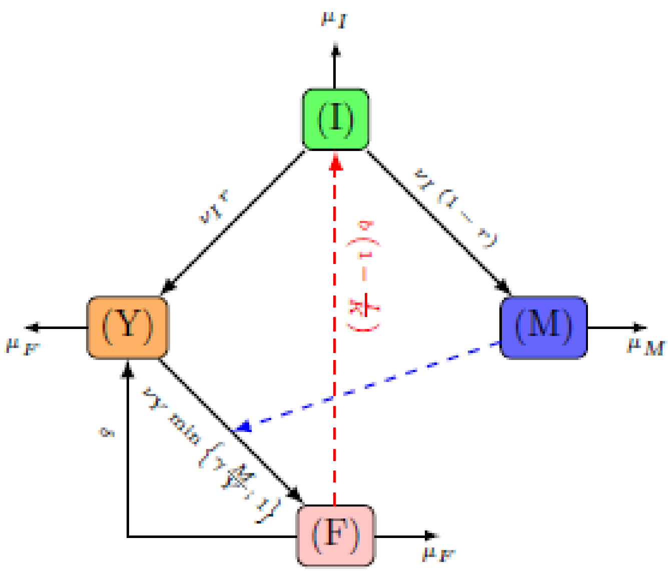


Figure 1: Compartmental representation of the different stages of insect-pest dynamics.

From Figure 1, we finally obtain the classical formulation of the model in the form of a system of the following ordinary differential equations:

$$\begin{cases} \frac{dI(t)}{dt} = b\left(1 - \frac{I}{K}\right)F - (\nu_I + \mu_I)\, I, \\ \frac{dY(t)}{dt} = r\nu_I I - \nu_Y \min\{\gamma \frac{M}{Y}, 1\}Y + \delta F - \mu_F Y, \\ \frac{dF(t)}{dt} = \nu_Y \min\{\gamma \frac{M}{Y}, 1\}Y - \delta F - \mu_F F, \\ \frac{dM(t)}{dt} = (1-r)\nu_I I - \mu_M M \end{cases} \tag{5}$$

with the following initial conditions:

$$I(0) = I_0 \geq 0, \quad Y(0) = Y_0 \geq 0, \quad F(0) = F_0 \geq 0, \quad M(0) = M_0 \geq 0. \tag{6}$$

The classical model (5) above was presented and studied by R. Angelov *et al.* in [3] as part of integrated pest management (IPM) programs. It is important to note that, in the classical model (integer-order model), the state of the system does not depend on its history. However, in real life, memory plays an essential role in the dynamics of a given population. The advantage of these fractional-order models is that they allow greater degrees of freedom and incorporate the memory effect into the model. With this in mind, we propose a fractional model derived from the classical model (5).
In this paper, it is assumed that there are enough male insects to mate with all the female insects available for mating. Therefore, the transfer rate from Y to F is ν_Y^α. Using the same approach as in K. Diethelm [7], the fractional model is obtained by replacing the classical derivative in the previous system by the fractional derivative in the sense of Caputo. Then taking into account the dimensions of model parameters (5), in this context, we obtain the following fractional model:

$$\begin{cases} {}^C\mathcal{D}_0^\alpha[I](t) = b^\alpha\left(1 - \frac{I}{K}\right)F - (\nu_I^\alpha + \mu_I^\alpha)\, I, \\ {}^C\mathcal{D}_0^\alpha[Y](t) = r\nu_I^\alpha I - (\nu_Y^\alpha + \mu_F^\alpha)Y + \delta^\alpha F, \\ {}^C\mathcal{D}_0^\alpha[F](t) = \nu_Y^\alpha Y - (\delta^\alpha + \mu_F^\alpha)F, \\ {}^C\mathcal{D}_0^\alpha[M](t) = (1-r)\nu_I^\alpha I - \mu_M^\alpha M, \end{cases} \tag{7}$$

where ${}^C\mathcal{D}_0^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0;1]$, under the initial conditions (6).

4 Mathematical Analysis of the Fractional Model

4.1 Existence, uniqueness and properties of solutions

In this subsection, we study the existence and uniqueness of non-negative solution of the fractional model (7) under the initial conditions (6).
The fractional differential system (7) can be put into the following appropriate form:

$$\begin{cases} {}^C\mathcal{D}_0^\alpha\,[\Phi]\,(t) = \mathcal{F}(\Phi(t)), \quad t \in \mathbb{R}_{++}, \\ \Phi(0) = \Phi_0 \geq 0 \end{cases} \tag{8}$$

with $\mathcal{F}(\cdot) := (\mathcal{F}_1(\cdot), \mathcal{F}_2(\cdot), \mathcal{F}_3(\cdot), \mathcal{F}_4(\cdot))^T$, $\Phi(t) := (I(t), Y(t), F(t), M(t))^T$ and $\Phi_0 := (I_0, Y_0, F_0, M_0)^T$, where $\mathcal{F}_1(I,Y,F,M) = b^\alpha\left(1 - \frac{I}{K}\right)F - (\nu_I^\alpha + \mu_I^\alpha)\, I$, $\mathcal{F}_2(I,Y,F,M) = r\nu_I^\alpha I - (\nu_Y^\alpha + \mu_F^\alpha)Y + \delta^\alpha F$, $\mathcal{F}_3(I,Y,F,M) = \nu_Y^\alpha Y - (\delta^\alpha + \mu_F^\alpha)F$ and $\mathcal{F}_4(I,Y,F,M) = (1-r)\nu_I^\alpha I - \mu_M^\alpha M$.

Let us move on to the essential results of this paragraph.

Theorem 4.1 *The fractional system (8) admits a maximal solution.*

Proof. The function $\mathcal{F}$ is continuous. Thus, the initial value problem (8) is equivalent to the following system of nonlinear Volterra integral equations of the second kind [10]: $\Phi(t) = \Phi_0 + {}^{RL}\mathcal{J}_0^{\alpha}\left[\mathcal{F}(\Phi(\cdot))\right](t)$. Moreover, $\mathcal{F}$ is of class $\mathcal{C}^1$ since each component of $\mathcal{F}$ is of class $\mathcal{C}^1$, so it is locally Lipschitz. Then we deduce the existence and uniqueness of the maximal solution Φ of the fractional system (8) [6,9]. □

Proposition 4.1 *The maximal solution Φ of (8) is non-negative and bounded.*

Proof. We will first show the non-negativity of the solution by proceeding to absurdity. From the uniqueness of solution, there are four possible cases.

Suppose that there exists $t_* > 0$ such that $I(t_*) = 0$, where $Y(t_*)$, $F(t_*)$, $M(t_*)$ are all non-negative and $I(t) < 0$ for all $t \in (t_*, t_1]$, where t_1 is sufficiently near t_*. Since $F(t_*) \geq 0$ and $I(t_*) = 0$, then from the second equation of the fractional system (7), we have

$$ {}^{C}\mathcal{D}_0^{\alpha}[I](t_*) = b^{\alpha} F(t_*) \geq 0. $$

Consequently, according to Lemma 2.1, we have ${}^{C}\mathcal{D}_0^{\alpha}[I](t) \geq 0$ for all $t \in [t_*,\, t_1]$. Therefore $I(t) \geq I(t_*)$ for all t sufficiently near t_*, which is a contradiction.
By the same way, we show that $Y(t) \geq 0$, $F(t) \geq 0$ and $M(t) \geq 0$.

We now prove the boundedness of the maximal solution.
From the first equation of (7), we get ${}^{C}\mathcal{D}_0^{\alpha}[I](t) \leq b^{\alpha}\left(1 - \frac{I(t)}{K}\right) F(t)$. So we have

$$ I(t) \leq K. \tag{9} $$

From the fourth equation of the system (7) and taking into account (9), we have

$$ {}^{C}\mathcal{D}_0^{\alpha}[M](t) = (1-r)\nu_I^{\alpha} I(t) - \mu_M^{\alpha} M(t) \leq (1-r)\nu_I^{\alpha} K - \mu_M^{\alpha} M(t). \tag{10} $$

Applying the Laplace transform to the both sides of (10) and using (4), we get

$$ \mathcal{L}\left[M\right](s) \leq \frac{(1-r)\nu_I^{\alpha} K s^{-1}}{s^{\alpha} + \mu_M^{\alpha}} + \frac{s^{\alpha-1} M_0}{s^{\alpha} + \mu_M^{\alpha}}. \tag{11} $$

Now applying the inverse Laplace transform to inequality (11) and taking into account (1), we obtain

$$ M(t) \leq \max\left\{ \frac{(1-r)\nu_I^{\alpha} K}{\mu_M^{\alpha}};\; M_0 \right\} =: K_M. \tag{12} $$

Under the assumption of male abundance in the region, i.e., $Y \leq \gamma M$, the third equation of the fractional system (7) gives

$$ {}^{C}\mathcal{D}_0^{\alpha}[F](t) \leq \nu_Y^{\alpha} \gamma M - (\delta^{\alpha} + \mu_F^{\alpha}) F \leq \nu_Y^{\alpha} \gamma K_M - (\delta^{\alpha} + \mu_F^{\alpha}) F. \tag{13} $$

Applying the Laplace transform to inequality (13), we get

$$ \mathcal{L}\left[F\right](s) \leq \frac{\nu_Y^{\alpha} \gamma K_M s^{-1}}{s^{\alpha} + \delta^{\alpha} + \mu_F^{\alpha}} + \frac{s^{\alpha-1} F_0}{s^{\alpha} + \delta^{\alpha} + \mu_F^{\alpha}}. \tag{14} $$

Let us now apply the inverse Laplace transform to inequality (14), we have

$$ F(t) \leq \max\left\{ \frac{\nu_Y^{\alpha} \gamma (1-r)\nu_I^{\alpha} K}{\mu_M^{\alpha}(\delta^{\alpha} + \mu_F^{\alpha})};\; \frac{\nu_Y^{\alpha} \gamma M_0}{\delta^{\alpha} + \mu_M^{\alpha}};\; F_0 \right\} =: K_F. \tag{15} $$

Similarly, from the second equation of system (7), we also have

$$Y(t) \leq \max\left\{\frac{r\nu_I^\alpha K + \delta^\alpha K_F}{\nu_Y^\alpha + \mu_F^\alpha}; Y_0\right\} =: K_Y. \tag{16}$$

□

Remark 4.1 It follows from Proposition 4.1 that the compact set Δ given by

$$\Delta := [0; K] \times [0; K_Y] \times [0; K_F] \times [0; K_M]$$

is positively invariant for the fractional system (8), where K_Y, K_F and K_M are defined by (16), (15) and (12), respectively.

Corollary 4.1 *The maximal solution Φ of (8) is a global solution.*

Proof. It follows from Proposition 4.1, that the maximal solution Φ of the fractional system (8) is bounded, then it is global [4]. □

4.2 Stability of the equilibrium points of the fractional system

In this part, we look for the equilibria of the fractional system (7) and we study their stability.

Remark 4.2 The equilibrium point without insect-pests of (7) is given by $\Phi_0^* = (0,0,0,0)^T$.

Proposition 4.2 *The basic offspring number $\mathcal{N}_0$ of (7) is given as follows:*

$$\mathcal{N}_0 = \frac{b^\alpha r\nu_I^\alpha \nu_Y^\alpha}{\mu_F^\alpha(\mu_I^\alpha + \nu_I^\alpha)(\delta^\alpha + \mu_F^\alpha + \nu_Y^\alpha)}. \tag{17}$$

Proof. We will use the Van Den Driessche and Watmough method [14]. The transfer functions $\mathfrak{V}$ and $\mathfrak{F}$ are defined by

$$\mathfrak{V}(\Phi) = \begin{pmatrix} (\nu_I^\alpha + \mu_I^\alpha) I \\ -r\nu_I^\alpha I + (\nu_Y^\alpha + \mu_F^\alpha)Y - \delta^\alpha F \\ -\nu_Y^\alpha Y + (\delta^\alpha - \mu_F^\alpha)F \\ -(1-r)\nu_I^\alpha I + \mu_M^\alpha M \end{pmatrix} \text{ and } \mathfrak{F}(\Phi) = \begin{pmatrix} b^\alpha\left(1 - \frac{I}{K}\right)F \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By calculating the Jacobian matrices of the functions $\mathfrak{V}$ and $\mathfrak{F}$ evaluated at the equilibrium point without insect-pests Φ_0^*, we obtain, respectively,

$$V = \begin{pmatrix} \nu_I^\alpha + \mu_I^\alpha & 0 & 0 & 0 \\ -r\nu_I^\alpha & \nu_Y^\alpha + \mu_F^\alpha & -\delta^\alpha & 0 \\ 0 & -\nu_Y^\alpha & \delta^\alpha + \mu_F^\alpha & 0 \\ -(1-r)\nu_I^\alpha & 0 & 0 & \mu_M^\alpha \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & 0 & b^\alpha & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So, we have

$$FV^{-1} = \begin{pmatrix} \frac{b^\alpha r\nu_I^\alpha\nu_Y^\alpha}{\mu_F^\alpha(\mu_I^\alpha+\nu_I^\alpha)(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)} & \frac{b^\alpha\nu_Y^\alpha}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)} & \frac{b^\alpha(\nu_Y^\alpha+\mu_F^\alpha)}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

As a result, we arrive at $\mathcal{N}_0 := \rho(FV^{-1}) = \frac{b^\alpha r\nu_I^\alpha \nu_Y^\alpha}{\mu_F^\alpha(\mu_I^\alpha+\nu_I^\alpha)(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)}$, where $\rho(FV^{-1})$ denotes the spectral radius of the next generation matrix FV^{-1}. □

Remark 4.3 Note that $\mathcal{N}_0$ presented here in (17) stands for the average number of offspring produced by a female insect during its life provided that an abundant resource is available. In the case of fractional models, this average number is of a hereditary nature. It depends on the history of the dynamics of the insect population unlike the classical case. Note that, when taking the limit $\alpha \to 1$, the basic offspring number $\mathcal{N}_0$ associated with the fractional model (7) converges to $\mathcal{N}_0$ of the classical model [3].

The process of finding the equilibrium points of systems of fractional differential equations in Caputo's sense is analogous to classical ordinary differential equations. We can now state the following result.

Proposition 4.3 *(i) If $\mathcal{N}_0 \le 1$, Φ_0^* is the unique equilibrium point of (7).*

(ii) If $\mathcal{N}_0 > 1$, the fractional model (7) admits a unique non-trivial equilibrium in the positive quadrant $\Phi^* = (I^*, Y^*, F^*, M^*)^T$ with $I^* = \left(1 - \frac{1}{\mathcal{N}_0}\right)K$, $Y^* = \frac{r\nu_I^\alpha(\delta^\alpha+\mu_F^\alpha)}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)}I^*$, $F^* = \frac{r\nu_I^\alpha\nu_Y^\alpha}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)}I^*$ and $M^* = \frac{(1-r)\nu_I^\alpha}{\mu_M^\alpha}I^*$, where $\mathcal{N}_0$ is given in (17).

Proof. We obtain the equilibrium points of (7) by setting its second member equal to zero. It follows that $M^* = \frac{(1-r)\nu_I^\alpha I^*}{\mu_M^\alpha}$, $Y^* = \frac{r\nu_I^\alpha(\delta^\alpha+\mu_F^\alpha)I^*}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)}$, $F^* = \frac{\nu_Y^\alpha r\nu_I^\alpha I^*}{\mu_F^\alpha(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)}$ and $I^*\left[K\left(\nu_I^\alpha\nu_Y^\alpha r b^\alpha - \mu_F^\alpha(\mu_I^\alpha+\nu_I^\alpha)(\delta^\alpha+\mu_F^\alpha+\nu_Y^\alpha)\right) - \nu_I^\alpha\nu_Y^\alpha r b^\alpha I^*\right] = 0$. Consequently, $I^* = 0$ or $I^* = \left(1 - \frac{1}{\mathcal{N}_0}\right)K$.
Thus, if $\mathcal{N}_0 \le 1$, $I^* = \left(1 - \frac{1}{\mathcal{N}_0}\right)K \le 0$, therefore $I^* = 0$, and in this case, Φ_0^* is the unique equilibrium point.
However, if $\mathcal{N}_0 > 1$, $I^* = \left(1 - \frac{1}{\mathcal{N}_0}\right)K > 0$, then we obtain that the non-trivial equilibrium point is Φ^*. □

Theorem 4.2 *The trivial equilibrium Φ_0^* of (7) is locally asymptotically stable if $\mathcal{N}_0 < 1$.*

Proof. The local stability of equilibria is obtained by using the Jacobian matrix $\mathbb{J}\mathcal{F}(\Phi)$ of the fractional system (7) defined as follows:

$$\mathbb{J}\mathcal{F}(\Phi) = \begin{pmatrix} -\frac{b^\alpha F}{K} - (\nu_I^\alpha + \mu_I^\alpha) & 0 & b^\alpha(1-\frac{I}{K}) & 0 \\ r\nu_I^\alpha & -(\nu_Y^\alpha+\mu_F^\alpha) & \delta^\alpha & 0 \\ 0 & \nu_Y^\alpha & -(\delta^\alpha+\mu_F^\alpha) & 0 \\ (1-r)\nu_I^\alpha & 0 & 0 & -\mu_M^\alpha \end{pmatrix}. \tag{18}$$

By evaluating the Jacobian matrix (18) at the equilibrium point Φ_0^*, we obtain then

$$\mathbb{J}\mathcal{F}(\Phi_0^*) = \begin{pmatrix} -\epsilon_1 & 0 & b^\alpha & 0 \\ r\nu_I^\alpha & -\epsilon_2 & \delta^\alpha & 0 \\ 0 & \nu_Y^\alpha & -\epsilon_3 & 0 \\ (1-r)\nu_I^\alpha & 0 & 0 & -\mu_M^\alpha \end{pmatrix}, \tag{19}$$

where $\epsilon_1 = \nu_I^\alpha + \mu_I^\alpha$, $\epsilon_2 = \nu_Y^\alpha + \mu_F^\alpha$ and $\epsilon_3 = \delta^\alpha + \mu_F^\alpha$.
The eigenvalues of the matrix $\mathbb{J}\mathcal{F}(\Phi_0^*)$ are the roots of its characteristic polynomial

$$\mathcal{P}(\lambda) = (\lambda + \mu_M^\alpha)\left(\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3\right)$$

with $A_1 = \epsilon_1 + \epsilon_2 + \epsilon_3$, $A_2 = \epsilon_1(\epsilon_2 + \epsilon_3) + \mu_F^\alpha(\epsilon_2 + \delta^\alpha)$ and $A_3 = \epsilon_1\mu_F^\alpha(\epsilon_2 + \delta^\alpha)(1 - \mathcal{N}_0)$.
It is easy to note that $\lambda_1 = -\mu_M^\alpha < 0$ is an eigenvalue of the matrix defined in (19), so it satisfies the condition $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ for $\alpha \in (0; 1]$. The other eigenvalues are the roots of the polynomial $U(\lambda) := \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3$. By applying the Routh-Hurwitz criterion on the polynomial $U(\lambda)$, we have the following sufficient conditions: $A_1 > 0$, $A_1A_2 - A_3 > 0$ and $A_3 > 0$. It is obvious that $A_1 > 0$, $A_1A_2 - A_3 > 0$ and so $\mathcal{N}_0 < 1$, $A_3 > 0$. We conclude that the other eigenvalues satisfies the condition $|\arg(\lambda)| > \frac{\alpha\pi}{2}$, when $\mathcal{N}_0 < 1$. Therefore, the trivial equilibrium point Φ_0^* of the fractional system (7) is locally asymptotically stable if $\mathcal{N}_0 < 1$. □
Note that the discriminant $\mathcal{D}(U)$ of the polynomial $U(\lambda)$ is defined (see [1]) by

$$\mathcal{D}(U) = 18A_1A_2A_3 + (A_1A_2)^2 - 4A_3A_1^3 - 4A_2^3 - 27A_3^2.$$

Using the fractional Routh-Hurwitz criterion established in [1], we deduce additional necessary and sufficient conditions for the local stability of the trivial equilibrium point.

Corollary 4.2 *The trivial equilibrium point* Φ_0^* *of* (7) *is locally asymptotically stable if and only if one of the following conditions holds:*

(*i*) $\mathcal{D}(U) > 0$, $A_1 > 0$, $A_3 > 0$ *and* $A_1A_2 > A_3$.

(*ii*) $\mathcal{D}(U) < 0$, $A_1 \geq 0$, $A_2 \geq 0$, $A_3 > 0$ *and* $\alpha < \frac{2}{3}$.

(*iii*) $\mathcal{D}(U) < 0$, $A_1 > 0$, $A_2 > 0$, $A_1A_2 = A_3$ *and* $\alpha \in (0, 1)$.

Remark 4.4 In the case when $\mathcal{D}(U) < 0$, $A_1 < 0$, $A_2 < 0$ and $\alpha > \frac{2}{3}$, the fractional system (7) is unstable. These instability conditions will not be treated in this model since biologically, A_1 and A_2 are always positive.

Theorem 4.3 *The trivial equilibrium point* Φ_0^* *of (7) is globally asymptotically stable in* Δ *if* $\mathcal{N}_0 < 1$.

Proof. We consider the Lyapunov candidate function $\mathcal{W}_1 : \mathbb{R}_+^4 \to \mathbb{R}$ defined by

$$\mathcal{W}_1(\Phi) = I + \frac{\nu_Y^\alpha b^\alpha}{\epsilon_2\epsilon_3 - \nu_Y^\alpha\delta^\alpha}Y + \frac{\epsilon_2 b^\alpha}{\epsilon_2\epsilon_3 - \nu_Y^\alpha\delta^\alpha}F + \frac{\epsilon_1(1 - \mathcal{N}_0)}{2(1-r)\nu_I^\alpha}M.$$

Note that $\mathcal{W}_1(\Phi_0^*) = 0$ and $\mathcal{W}_1(\Phi) > 0$, $\forall\Phi \in \mathbb{R}_+^4 \setminus \{\Phi_0^*\}$.
By differentiating $\mathcal{W}_1$ with respect to time in the sense of Caputo and taking into account the linearity of the differential operator ${}^C\mathcal{D}_0^\alpha$, we get

$${}^C\mathcal{D}_0^\alpha[\mathcal{W}_1(\Phi)](t)|_{(7)} = -\frac{b^\alpha IF}{K} - \frac{\epsilon_1\mu_M^\alpha(1 - \mathcal{N}_0)}{2(1-r)\nu_I^\alpha}M - \frac{\epsilon_1(1 - \mathcal{N}_0)}{2}I.$$

Consequently, as $\mathcal{N}_0 < 1$, it follows that

$${}^C\mathcal{D}_0^\alpha[\mathcal{W}_1(\Phi)](t)|_{(7)} = -\frac{b^\alpha IF}{K} - \frac{\epsilon_1\mu_M^\alpha(1 - \mathcal{N}_0)}{2(1-r)\nu_I^\alpha}M - \frac{\epsilon_1(1 - \mathcal{N}_0)}{2}I \leq 0.$$

According to the extension of the LaSalle invariance principle [12], we conclude that the trivial equilibrium point Φ_0^* is globally asymptotically stable if $\mathcal{N}_0 < 1$. □

Theorem 4.4 *The non-trivial equilibrium Φ^* of (7) is locally asymptotically stable if $\mathcal{N}_0 > 1$.*

Proof. The local stability of the non-trivial equilibrium Φ^* is obtained by using the Jacobian matrix $\mathbb{J}\mathcal{F}(\Phi^*)$ of the fractional system (7) defined as follows:

$$\mathbb{J}\mathcal{F}(\Phi^*) = \begin{pmatrix} -\frac{b^\alpha F^*}{K} - \epsilon_1 & 0 & b^\alpha(1-\frac{I^*}{K}) & 0 \\ r\nu_I^\alpha & -\epsilon_2 & \delta^\alpha & 0 \\ 0 & \nu_Y^\alpha & -\epsilon_3 & 0 \\ (1-r)\nu_I^\alpha & 0 & 0 & -\mu_M^\alpha \end{pmatrix}. \tag{20}$$

The characteristic polynomial of $\mathbb{J}\mathcal{F}(\Phi^*)$ is

$$\mathcal{Q}(\lambda) = (\lambda + \mu_M^\alpha)\left(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3\right),$$

where $B_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 + \frac{b^\alpha F^*}{K}$, $B_2 = (\epsilon_2 + \epsilon_3)\left(\epsilon_1 + \frac{b^\alpha F^*}{K}\right) + \mu_F^\alpha(\epsilon_2 + \delta^\alpha)$ and $B_3 = \frac{\mu_F^\alpha b^\alpha F^*}{K}(\epsilon_2 + \delta^\alpha)$. It is clear that $\lambda_1 = -\mu_M^\alpha < 0$ is an eigenvalue of the matrix defined in (20), so it satisfies the condition $|\arg(\lambda_1)| = \pi > \frac{\alpha\pi}{2}$ for $\alpha \in (0;1]$. The other eigenvalues are the roots of the polynomial $V(\lambda) := \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3$. By applying the Routh-Hurwitz criterion to the polynomial $V(\lambda)$, we get the following sufficient conditions: $B_1 > 0$, $B_1B_2 - B_3 > 0$ and $B_3 > 0$. It is easy to see that $B_1 > 0$, $B_1B_2 - B_3 > 0$ and $B_3 > 0$. We conclude that the other eigenvalues satisfy the condition $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for $\alpha \in (0;1]$. Since Φ^* exists if $\mathcal{N}_0 > 1$, we deduce that the non-trivial equilibrium Φ^* of (7) is locally asymptotically stable when $\mathcal{N}_0 > 1$. □

Remark 4.5 For the non-trivial equilibrium Φ^*, we do not need additional conditions for local stability because the existence condition guarantees the local stability.

Theorem 4.5 *The non-trivial equilibrium point Φ^* of (7) is globally asymptotically stable if $\mathcal{N}_0 > 1$.*

Proof. Consider the following candidate Lyapunov function $\mathcal{W}_2 : \mathbb{R}^4_{++} \to \mathbb{R}$ given by

$$\mathcal{W}_2(\Phi) = c_1\int_{I^*}^{I}\frac{\tau - I^*}{\tau}d\tau + c_2\int_{Y^*}^{Y}\frac{\tau - Y^*}{\tau}d\tau + c_3\int_{F^*}^{F}\frac{\tau - F^*}{\tau}d\tau + c_4\int_{M^*}^{M}\frac{\tau - M^*}{\tau}d\tau,$$

where $c_1 = \frac{K}{b^\alpha}$, $c_2 = \frac{K\epsilon_1}{rb^\alpha\nu_I^\alpha}$, $c_3 = \frac{K\epsilon_1 I^*}{b^\alpha\nu_Y^\alpha Y^*} + \frac{K\epsilon_1\delta^\alpha F^*}{b^\alpha r\nu_I^\alpha\nu_Y^\alpha Y^*}$ and $c_4 = \frac{I^*}{(1-r)\nu_I^\alpha}$.
We notice that $\mathcal{W}_2(\Phi^*) = 0$ and $\mathcal{W}_2(\Phi) > 0$, $\forall\Phi \in \mathbb{R}^4_{++}\setminus\{\Phi^*\}$.
By differentiating $\mathcal{W}_2$ with respect to time in the sense of Caputo and taking into account the linearity of the fractional derivative ${}^C\mathcal{D}_0^\alpha$ and using Lemma 2.3, we have

$$\begin{aligned} {}^C\mathcal{D}_0^\alpha[\mathcal{W}_2(\Phi)](t)|_{(7)} \;\le\;& c_1\left(1 - \frac{I^*}{I(t)}\right){}^C\mathcal{D}_0^\alpha[I](t) + c_2\left(1 - \frac{Y^*}{Y(t)}\right){}^C\mathcal{D}_0^\alpha[Y](t) \\ +\;& c_3\left(1 - \frac{F^*}{F(t)}\right){}^C\mathcal{D}_0^\alpha[F](t) + c_4\left(1 - \frac{M^*}{M(t)}\right){}^C\mathcal{D}_0^\alpha[M](t). \end{aligned}$$

Using the following non-trivial equilibrium conditions: $K = \frac{K\epsilon_1 I^*}{b^\alpha F^*} + I^*$, $\delta^\alpha F^* = \epsilon_2 Y^* - r\nu_I^\alpha I^*$, $\nu_Y^\alpha Y^* = \epsilon_3 F^*$ and $\mu_M^\alpha M^* = (1-r)\nu_I^\alpha I^*$, and after some calculations, we obtain

$$\begin{aligned} {}^C\mathcal{D}_0^\alpha[\mathcal{W}_2(\Phi)](t)|_{(7)} \;\le\;& \frac{K\epsilon_1 I^*}{b^\alpha}\left[3 - \frac{I^*F}{F^*I} - \frac{Y^*I}{I^*Y} - \frac{F^*Y}{Y^*F}\right] - IF\left(1 - \frac{I^*}{I}\right)^2 \\ +\;& \frac{K\epsilon_1\delta^\alpha F^*}{b^\alpha r\nu_I^\alpha}\left[2 - \frac{Y^*F}{F^*Y} - \frac{F^*Y}{Y^*F}\right] + I^*\left[1 - \frac{M}{M^*} - \frac{M^*I}{I^*M} + \frac{I}{I^*}\right]. \end{aligned}$$

By subtracting then adding $\frac{I^*}{I}+2$ to the expression $1-\frac{M}{M^*}-\frac{M^*I}{I^*M}+\frac{I}{I^*}$, we obtain

$$\begin{aligned} {}^C\mathcal{D}_0^\alpha[\mathcal{W}_2(\Phi)](t)|_{(7)} &\leq \frac{K\epsilon_1 I^*}{b^\alpha}\left[3-\frac{I^*F}{F^*I}-\frac{Y^*I}{I^*Y}-\frac{F^*Y}{Y^*F}\right]+I\left(1-F\right)\left(1-\frac{I^*}{I}\right)^2 \\ &+ \frac{K\epsilon_1\delta^\alpha F^*}{b^\alpha r\nu_I^\alpha}\left[2-\frac{Y^*F}{F^*Y}-\frac{F^*Y}{Y^*F}\right]+I^*\left[3-\frac{M}{M^*}-\frac{M^*I}{I^*M}-\frac{I^*}{I}\right]. \end{aligned}$$

By rewriting the above expression with the Volterra type function, it turns out that

$$\begin{aligned} {}^C\mathcal{D}_0^\alpha[\mathcal{W}_2(\Phi)](t)|_{(7)} &\leq -\frac{K\epsilon_1 I^*}{b^\alpha}\left\{\psi\left(\frac{I^*F}{F^*I}\right)+\psi\left(\frac{Y^*I}{I^*Y}\right)\right\}-I\left(F-1\right)\left(1-\frac{I^*}{I}\right)^2 \\ &- \frac{K\epsilon_1 I^*}{b^\alpha}\psi\left(\frac{F^*Y}{Y^*F}\right)-\frac{K\epsilon_1\delta^\alpha F^*}{b^\alpha r\nu_I^\alpha}\left\{\psi\left(\frac{Y^*F}{F^*Y}\right)+\psi\left(\frac{F^*Y}{Y^*F}\right)\right\} \\ &- I^*\left\{\psi\left(\frac{M}{M^*}\right)+\psi\left(\frac{M^*I}{I^*M}\right)\right\}-I^*\psi\left(\frac{I^*}{I}\right), \end{aligned}$$

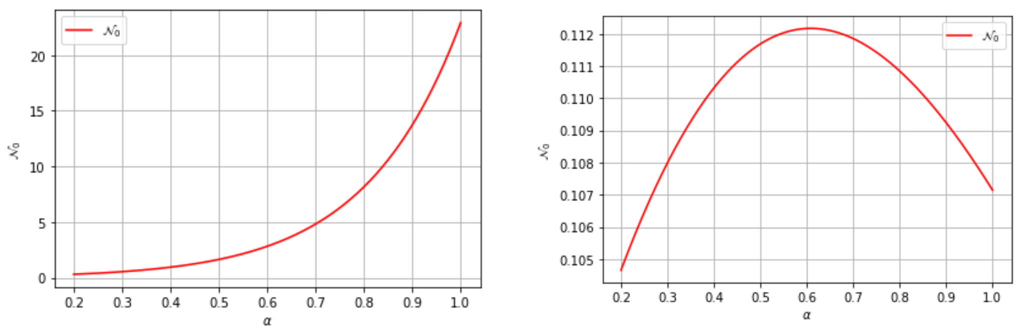
where $\psi(x)=x-1-\ln(x)$ for all $x\in\mathbb{R}_{++}$. Therefore, we have ${}^C\mathcal{D}_0^\alpha[\mathcal{W}_2(\Phi)](t)|_{(7)}\leq 0$. Then we can apply LaSalle's fractional invariance principle [12] to the limit set, where each solution is contained in the largest invariant set $\mathcal{E}$ defined by $\mathcal{E}=\left\{(I,Y,F,M)\in\Delta : {}^C\mathcal{D}_0^\alpha\mathcal{W}_2(\Phi)|_{(7)}=0\right\}$. It is easy to see that the largest invariant set in $\mathcal{E}$ is just the singleton $\{\Phi^*\}$. It is concluded that the non-trivial equilibrium point Φ^* is globally asymptotically stable in the interior of Δ when $\mathcal{N}_0>1$. □

5 Numerical Simulations and Interpretations

In this section, we perform some numerical simulations to illustrate the theoretical results obtained in the previous sections. For this, we use the Python software and the Adams-Bashforth-Moulton numerical scheme (see [6]). The graphs resulting from the numerical simulation give the dynamics of the different compartments of the fractional model with different values of fractional derivation order $\alpha\in(0;1]$ and the basic offspring number $\mathcal{N}_0$. Furthermore, the numerical values used are summarized in Table 2.

Parameters	**Values**		**References**		**Dimensions**
	$\mathcal{N}_0<1$	$\mathcal{N}_0>1$	$\mathcal{N}_0<1$	$\mathcal{N}_0>1$	
b	1	3.28	estimated	estimated	day^{-1}
r	0.57	0.57	[3]	[3]	–
K	1000	1000	[3]	[3]	–
γ	1	4	estimated	[3]	–
μ_I	0.01	0.01	estimated	estimated	day^{-1}
μ_F	0.08	0.08	estimated	estimated	day^{-1}
μ_M	0.001	0.001	estimated	estimated	day^{-1}
ν_I	1/250	1/25	estimated	[3]	day^{-1}
ν_Y	1/100	0.5	estimated	[3]	day^{-1}
δ	0.1	0.1	[3]	[3]	day^{-1}

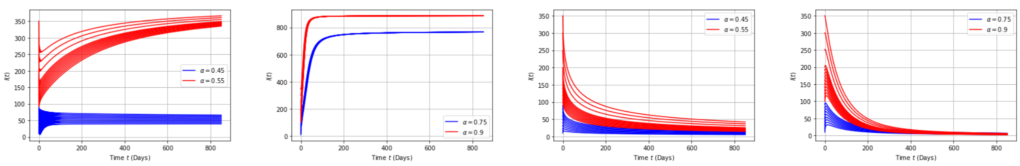
Table 2: The numerical values of the model parameters.



(a) (b)

Figure 2: Graph of offspring number $\mathcal{N}_0$ of the fractional model (7).

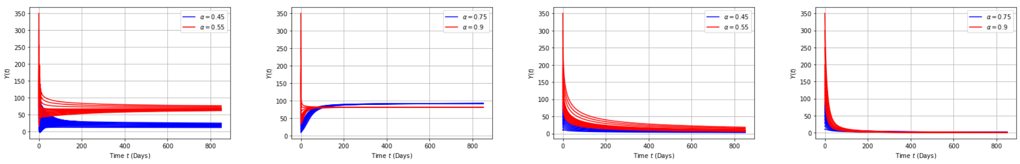
Figure 2 shows the influence of the derivation order on the offspring number with the data in Table 2, when $\mathcal{N}_0 > 1$ (Figure 2 (a)) and $\mathcal{N}_0 < 1$ (Figure 2 (b)).



(a) (b) (c) (d)

Figure 3: Simulation of the global asymptomatic behavior of the class of immature insects of the fractional model (7).

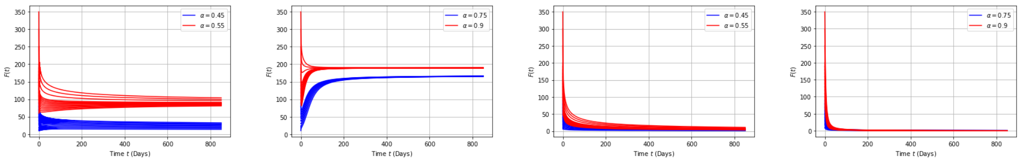
We observe that if $\mathcal{N}_0 > 1$ (Figure 3 (a) and (b)), we have a persistence of the population of immature insects. And if $\mathcal{N}_0 < 1$ (Figure 3 (c) and (d)), we see an extinction of the population of immature insects.



(a) (b) (c) (d)

Figure 4: Simulation of the global asymptomatic behavior of the class of female insects ready for mating of the fractional model (7).

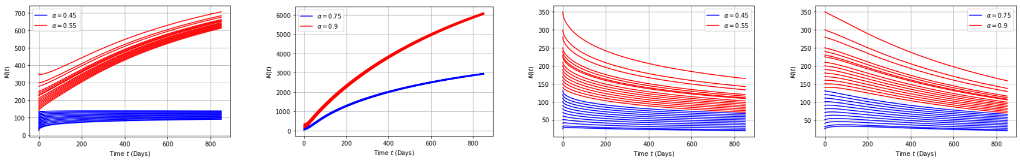
We notice that if $\mathcal{N}_0 > 1$ (Figure 4 (a) and (b)), we have a persistence of the population of female insects ready for mating. And if $\mathcal{N}_0 < 1$ (Figure 4 (c) and (d)), we see an extinction of the population of female insects ready for mating of the fractional model (7).



(a) (b) (c) (d)

Figure 5: Simulation of the global asymptomatic behavior of the class of females insects inseminated of the fractional model (7).

We observe that if $\mathcal{N}_0 > 1$ (Figure 5 (a) and (b)), we have a persistence of the population of females insects inseminated and if $\mathcal{N}_0 < 1$, (Figure 5 (c) and (d)), we see an extinction of the population of females insects inseminated.



(a) (b) (c) (d)

Figure 6: Simulation of the global asymptomatic behavior of the class of male insects of (7).

We notice that if $\mathcal{N}_0 > 1$ (Figure 6 (a) and (b)), we observe a persistence of the population of male insects and $\mathcal{N}_0 < 1$ (Figure 6 (c) and (d)), we see an extinction of the population of male insects of the fractional order model (7).

6 Conclusion

In this paper, we have proposed and studied an extension of a model of the nonlinear dynamics of insect pests which is governed by Caputo-type fractional order differential equations. We have proved the existence and uniqueness of non-negative solutions of the fractional model. Then, using the Laplace transform technique, we have also shown the boundedness of the solutions. On one hand, by applying the classical Routh-Hurwitz criterion, we have established that the trivial and non-trivial equilibrium points are locally asymptotically stable, when $\mathcal{N}_0 < 1$ and $\mathcal{N}_0 > 1$, respectively. On the other hand, the use of fractional Routh-Hurwitz criteria has allowed us to determine additional conditions ensuring the asymptotic stability of the trivial equilibrium point of the fractional-order model. Using LaSalle's fractional invariance principle, we have shown that the trivial and non-trivial equilibrium points are globally asymptotically stable respectively, when $\mathcal{N}_0 < 1$ and $\mathcal{N}_0 > 1$. We have performed numerical simulations to illustrate our theoretical results. These numerical results are obtained with the values of the fractional order of derivative $\alpha \in \{0.9, 0.75, 0.55, 0.45\}$. The fractional model can capture richer and more realistic dynamics, thus improving predictions and the management of infestations.

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