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On Chaotification and Stabilization of Zeraoulia-Sprott Map

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Abstract: This paper explores the stabilization of chaotic dynamics in the Zeraoulia-Sprott (Z-S) map using both feedback and non-feedback control strategies. By leveraging the stability theorem for discrete systems and the principle that lower energy correlates with greater stability, we systematically investigate the transition from chaos to periodic behavior. The study compares the effectiveness of feedback-based stabilization techniques with a minimum energy approach, analyzing their impact across different periodic orbits and dimensions. Numerical simulations validate the proposed methods, demonstrating their feasibility and efficiency in controlling chaos within the Z-S system. These findings contribute to a broader understanding of chaos suppression in discrete nonlinear systems and offer practical insights into their control applications.

Keywords: *discrete dynamical systems; Zeraoulia-Sprott map; chaos control; chaotification; stabilization; feedback control; minimum energy control.*

Mathematics Subject Classification (2020): 39A33, 39A05, 93-03, 39A70.

1 Introduction

Nonlinear dynamical systems are essential in many scientific and engineering fields, where discrete-time models can exhibit chaotic behavior, with applications in secure communication, cryptography, neural networks, and control theory. The Zeraoulia–Sprott (Z-S) map, a simple yet complex 2D rational mapping, shows diverse dynamics—periodic, quasi-periodic, and chaotic—making it a valuable model for studying stability and control [1].

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Chaotic maps are key in nonlinear dynamics for understanding bifurcations, attractors, and stability in discrete systems. Recent studies on the Z-S map emphasize its high sensitivity to initial conditions and structural complexity [2], creating challenges for stability in practical uses. Control strategies, such as perturbation methods [3], stability criteria [4], and adaptive control [5,13,14,15,16], aim to regulate these behaviors, while chaotification has been explored for beneficial uses in coupled map lattices [6]. Despite applications like encryption [13], stabilizing chaotic maps, especially at p-periodic orbits, remain a crucial research focus [7].

The snap-back repeller theory is a common tool for analyzing discrete chaotic systems, offering criteria for chaos detection and control design [8–10]. Stabilization methods, both feedback and non-feedback, have proven effective for maps like the Hénon map [12]. This study applies these approaches to the chaotic Z-S map, targeting various periodic orbits. Using stability theorems and the principle linking lower energy states to greater stability, we evaluate and compare both methods through extensive numerical simulations, assessing their feasibility and potential applications.

The paper is organized as follows. After the Introduction, Section 2 examines the stability and instability regions of the system and studies how chaotification occurs within the stable region. Sections 3 and 4 focus on the unstable regions and develop a feedback control strategy to achieve stabilization. Section 5 presents numerical simulation for selected parameter values. Finally, Section 6 introduces a non-feedback (minimum energy) control approach for stabilization, supported by additional numerical results, followed by concluding remarks.

2 Problem and Formulation

The Z-S system is defined by the following recurrence:

$$\begin{cases} x_{k+1} = \dfrac{-ax_k}{1+y_k^2}, \\ y_{k+1} = x_k + by_k. \end{cases} \tag{1}$$

The system has several behaviors depending on the values of the parameters (a, b) according to G. Chen et al. [2].

Condition	$\lvert a\rvert$	$\lvert b\rvert$
1	< 1	< 1 : global asymptotic stability
2	< 1	> 1 : unbounded solutions
3	> 1	< 1 : nontrivial global attractor
4	> 1	> 1 : unbounded solutions

Table 1: Stability conditions based on parameters a and b.

In this paper, we will study the system behavior in the zone Ω defined by$\{(a, b)) \in \mathbb{R}^2 / \lvert a\rvert < 1, \lvert b\rvert < 1\}$ and $\{(a, b)) \in \mathbb{R}^2 / \lvert a\rvert > 1, \lvert b\rvert < 1\}$.

2.1 Chaotification of Z-S system

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{-ax}{1+y^2} \\ x + by \end{pmatrix}.$$

According to [2], the system is stable for $a \in]-1,1[$, $b \in]-1,1[$; we will chaotify it using the theorem from [10].

Theorem 2.1 *Consider the n-dimensional controlled system*

$$X_{k+1} = X_k + U_k \pmod 1, \qquad (*)$$

where the controller is given by $U_k = cX_k - F(0)$.

If F is continuously differentiable and $\|F(X)\| < m$ for $\|X\| < 1$ and some positive constant m, then by choosing $|a| > 2m + \sqrt{n}$ such that all eigenvalues of $(DF(0) + cE)'(DF(0) + cE)$ exceed 1, and $|a|$ is suitably large, the controlled system $(*)$ will be chaotic in the sense of Li–Yorke.

If F is smooth and bounded, and a and c are chosen to satisfy certain eigenvalue and magnitude conditions, the controlled n-dimensional system becomes Li–Yorke chaotic.

2.1.1 Application

- F is C^1 with $DF(X) = \begin{pmatrix} \dfrac{-a}{1+y^2} & \dfrac{2axy}{1+y^2} \\ 1 & b \end{pmatrix}$, $\quad \|F(X)\| \le \sqrt{5}$.

- Choosing $|c| = 6 > 2\sqrt{5} + \sqrt{2}$ gives $DF(0) + cE = \begin{pmatrix} -a+6 & 0 \\ 1 & b+6 \end{pmatrix}$ with the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\Big(d \pm \sqrt{f}\Big), \qquad \begin{aligned} d &= 12a - 12b + a^2 + b^2 + 73 \\ f &= (a^2 + 2ab + b^2 + 1)(a^2 - 2ab - 24a + b^2 + 24b + 145). \end{aligned}$$

- For $(a,b) \in (-1,1)^2$, $\lambda_1 > 1$ and $\lambda_2 > 1$, hence the system

$$\begin{cases} x(k+1) = \dfrac{-ax_k}{1+y_k^2} + 6x_k \pmod 1, \\ y(k+1) = x_k + by_k + 6y_k \pmod 1 \end{cases}$$

  is chaotic.

Numerical application. The system

$$\begin{cases} x_{k+1} = \dfrac{-0.5x_k}{1+y_k^2}, \\ y_{k+1} = x_k + 0.5y_k \end{cases} \qquad (2)$$

is stable and admits the origin as a stable fixed point.

According to the theorem above, the system

$$\begin{cases} x_{k+1} = \dfrac{-0.5x_k}{1+y_k^2} + 6x_k \pmod 1, \\ y_{k+1} = x_k + 0.5y_k + 6y_k \pmod 1 \end{cases} \qquad (3)$$

is chaotic.

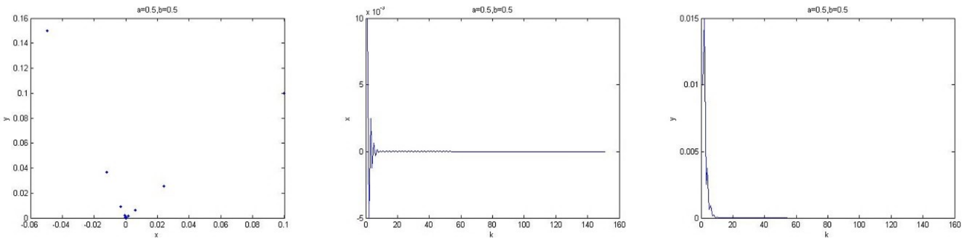

Figure 1: Phase spaces and evolution $(x, k), (y, k)$

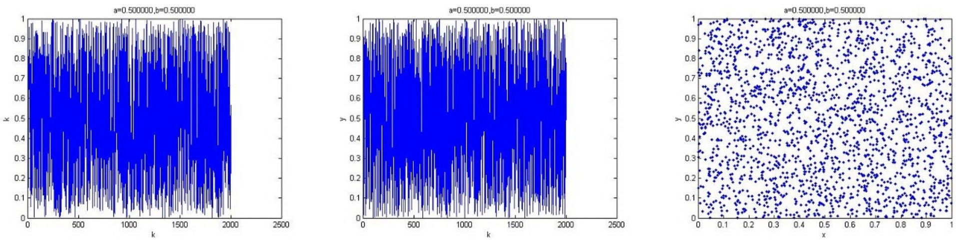

Figure 2: Chaotification.

3 Study of the Stability

Theorem 3.1 *If there exists at least one eigenvalue of the Jacobian matrix of the fixed point of one period greater than 1, the fixed point is unstable.*

The system has three fixed points:
$(0;0)\,,\left((1-b)\sqrt{-1-a};\sqrt{-1-a}\right),\left((b-1)\sqrt{-1-a};-\sqrt{-1-a}\right).$

3.1 Stability analysis

Fixed point $(0,0)$**:** For $a < -1$ and $-1 < b < 1$,

$$DF_{(0,0)} = \begin{pmatrix} -a & 0 \\ 1 & b \end{pmatrix}, \quad \lambda(DF) = (-a,\, b), \quad |-a| > 1.$$

Hence $(0,0)$ is unstable in Ω.

Fixed point $\left((1-b)\sqrt{-1-a},\ \sqrt{-1-a}\right)$:

$$DF_{(x^*,y^*)} = \begin{pmatrix} 1 & \frac{2(b-1)(a+1)}{a} \\ 1 & b \end{pmatrix}, \quad \lambda_{1,2} = \frac{1+b \pm \sqrt{\frac{(b-1)(7a+ab+8)}{a}}}{2}.$$

- If $7a+ab+8>0$: $\lambda_1 > 1$ for $a > -1$, $\lambda_2 > 1$ has no solution in $a < -1$.
- If $7a+ab+8<0$: Eigenvalues are complex conjugates; $|\lambda_{1,2}| > 1$ if $a < -2$.

4 Stabilizing Periodic Orbits of Z-S Map by Feedback Method

We consider the predictive control method with state feedback when the trajectory is near the fixed point. T. Ushio and S. Yamamoto [17] proposed, in this case, an alternative to the original method of Pyragas.

Supposing the system $X_{k+1} = f(X_k, U_k)$, Ushio and Yamamoto [17] propose a sequence in the form

$$U_k = K\left(f(X_k, o) - X^*\right) \diagup X^* = (x^*, y^*) \tag{4}$$

if it is desired that control be applied in an area such as

$$|X_k - X_{k-1}| < \varepsilon. \tag{5}$$

With ε being a positive enough number, the control will be determined as follows:

$$U_k = \begin{cases} K\left(f(X_k, o) - X^*\right) \text{if } |X_k - X_{k-1}| < \varepsilon, \\ o \quad \text{otherwise.} \end{cases} \tag{6}$$

4.1 Appliying the control law around fixed point $(0, 0)$

The control system is defined as follows: $f(X_k, 0) = f(X_k)$ with the vector

$$U_k = \begin{pmatrix} u_k^1 \\ u_k^2 \end{pmatrix} = \begin{pmatrix} K\left(\dfrac{-ax_k}{1+y_k^2} - 0\right) \\ 0\left(x_k + by_k - 0\right) \end{pmatrix}. \tag{7}$$

The system admits the origin as a fixed point, the gain K is chosen such that this fixed point becomes stable. We have

$$\begin{cases} x_{k+1} = \dfrac{-ax_k}{1+y_k^2} + u_k^1 \\ y_{k+1} = x_k + by_k + u_k^1 \end{cases} \Rightarrow \begin{cases} x_{k+1} = \dfrac{-ax_k}{1+y_k^2} + K\left(\dfrac{-ax_k}{1+y_k^2}\right) \\ y_{k+1} = x_k + by_k. \end{cases} \tag{8}$$

We linearize the system around the origin $Df\,(0,0)) = \begin{pmatrix} (1+K)\,(-a) & 0 \\ 1 & b \end{pmatrix}$ with the eigenvalues $-a - Ka, b,\ |b| \prec 1, |(1+K)\,(-a)| < 1$. The system is stable if $-1 < a(1+K) < 1$.

5 Numerical Simulation

Case $a = -7,\ b = -0.1$**.** All three fixed points are unstable; the bifurcation diagram and a positive Lyapunov exponent indicate chaos. Choose a scalar gain K so that $-1 < 7 + 7K < 1$ gives

$$K \in \left(-\tfrac{8}{7}, -\tfrac{6}{7}\right) \approx (-1.14, -0.85),$$

which places both closed-loop eigenvalues inside the unit disk, ensuring local stabilization of the targeted fixed point. For example, with $K = -1.1$ and control

$$U_k = \begin{pmatrix} -1.1\,\dfrac{-10\,x_k}{1+y_k^2} \\ 0 \end{pmatrix},$$

the uncontrolled system

$$\begin{cases} x_{k+1} = \dfrac{7x_k}{1+y_k^2}, \\ y_{k+1} = x_k - 0.1\, y_k \end{cases}$$

is stabilized under the chosen feedback.

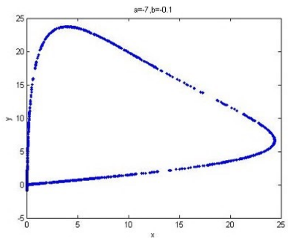

Figure 3: Zerraoulia-Sprotts attractor for $a = -7, b = -0.1$.

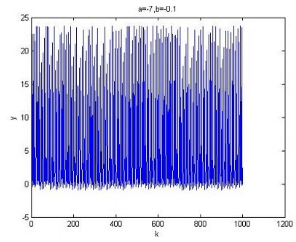

Figure 4: Synchronization (k, y).

The system under control is

$$\begin{cases} x_{k+1} = \dfrac{7x_k}{1+y_k^2} - 1.1\left(\dfrac{7x_k}{1+y_k^2}\right), \\ y_{k+1} = x_k - 0.1 y_k. \end{cases} \tag{9}$$

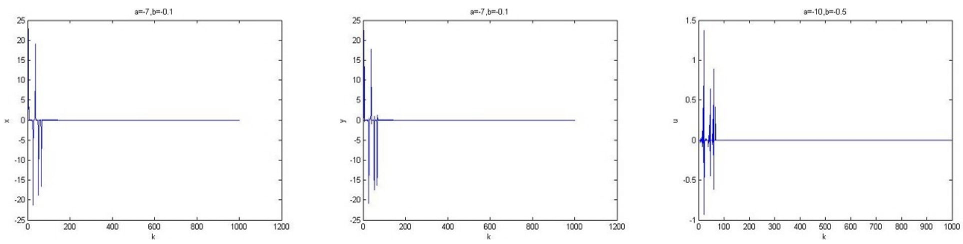

Figure 5: Stabilization (k,x), (k,y) and the control.

Around the fixed point $\left((1-b)\sqrt{-1-a}; \sqrt{-1-a}\right) = (x^*; y^*)$,

$$\begin{cases} x_{k+1} = \dfrac{-ax_k}{1+y_k^2} + k\left(\dfrac{-ax_k}{1+y_k^2} - x^*\right), \\ y_{k+1} = x_k + by_k + 0(x_k + by_k - y^*), \end{cases}$$

$$Df\left(x^*; y^*\right) = \begin{pmatrix} 1+k & \dfrac{2(1+k)(b-1)(1+a)}{a} \\ 1 & b \end{pmatrix}.$$

For $a = -7$, $b = -0.1$, the three fixed points are unstable. From the bifurcation diagram and the fact that one of Lyapunov's exponents is positive, the system is chaotic. We now calculate the gain K:

$$-1 < 7 + 7K < 1 \quad \Longrightarrow \quad -1.14 < K < -0.85.$$

When K belongs to this interval, the control system admits two eigenvalues less than 1, so the fixed point is surely stable. We take $K = -1.1$. The control is

$$U_k = \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} = \begin{pmatrix} -1.1\left(\dfrac{-10x_k}{1+y_k^2}\right) \\ 0 \end{pmatrix}.$$

The uncontrolled system is

$$\begin{cases} x_{k+1} = \dfrac{7x_k}{1+y_k^2}, \\ y_{k+1} = x_k - 0.1y_k. \end{cases}$$

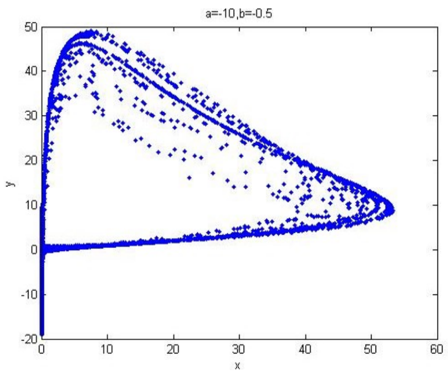

Figure 6: Zeraoulia–Sprott attractor.

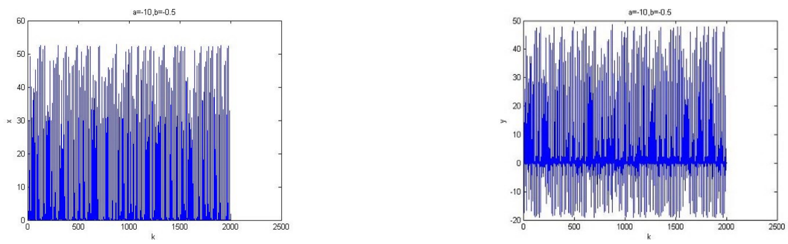

Figure 7: Zeraoulia-Sprott attractor for a=-10,b=-0.5 and the evolution (k,x) (k,y).

We take $K = -0.9$. The system under control becomes

$$\begin{cases} x_{k+1} = \dfrac{10x_k}{1+y_k^2} - 0.9\left(\dfrac{10x_k}{1+y_k^2} - 4.5\right), \\ y_{k+1} = x_k - 0.5y_k + 0(x_k - 0.5y_k - 3). \end{cases} \tag{10}$$

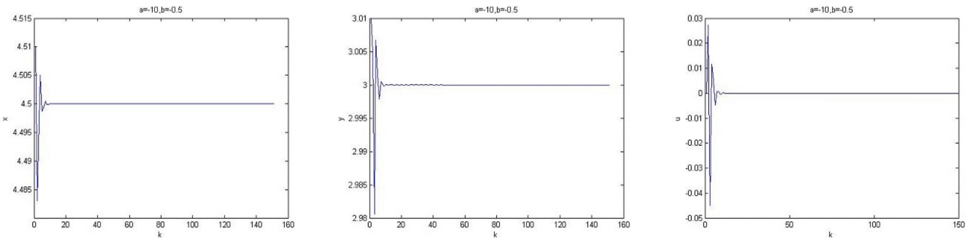

Figure 8: Stabilization (k,x), (k,y) and control.

6 Stabilizing Periodic Orbits of Z-S Map by Minimum Energy Method

For discrete chaotic systems, stabilizing high-periodic orbits with feedback is difficult due to the large gains required. Instead, we propose a non-feedback method for the Zeraoulia–Sprott map: by reducing the generalized energy (Lyapunov function) below a chosen threshold $E_m < E_{\text{av}}$, the system can be driven to periodic orbits.

6.1 Application of the method around unstable fixed point $(\mathbf{0}, \mathbf{0})$

The system is defined by the recurrence

$$\begin{cases} x_{k+1} = \dfrac{-ax}{1+y_k^2} \\ y_{k+1} = x_k + by_k \end{cases} \diagup (a,b) \in (\Omega)\,. \tag{11}$$

From the diagram below, we can notice the areas that can be chaotic.

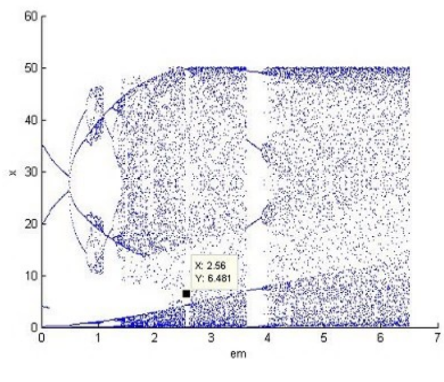

Figure 9: Bifurcation diagram (em, x).

We take as an example $a = -10, b = -0.1, x_0 = 0.1,\ y_0 = 0.1$. The chaotic behavior was confirmed by Lyapunov's exponent.

The Control.

We chose as a function of Lyapunov $E(x,y) = \dfrac{1}{a^2}\left(x^2 + y^2\right)$. The average value of E

for 100000 iterations is equal to 6.4485, now we will apply the control

$$\begin{cases} E_k = 0.01\left(x_k^2 + y_k^2\right), \\ x_{k+1} = \dfrac{10x_k}{1+y_k^2}, \\ y_{k+1} = \widetilde{y_k} \end{cases} \tag{12}$$

as

$$\widetilde{y_k} = \begin{cases} x_k - 0.1y_k, & E_k < E_m, \\ -\sqrt{E_m - x_k^2}, & E_k \geq E_m \;\; et \;\widetilde{y_k} < 0, \\ \sqrt{E_m - x_k^2}, & E_k \geq E_m \;\; et \widetilde{y_k} \geq 0. \end{cases}$$

According to the diagram, we distinguish the zone where the system has periodic unstable orbits, we will apply the method to make these orbits stable. We chose some samples: em=0.1 (7 periods), em=0.67 (14 periods), em=1.18 (14periodes), em=2.56 (22 periods), em=3.8 (15 periods).

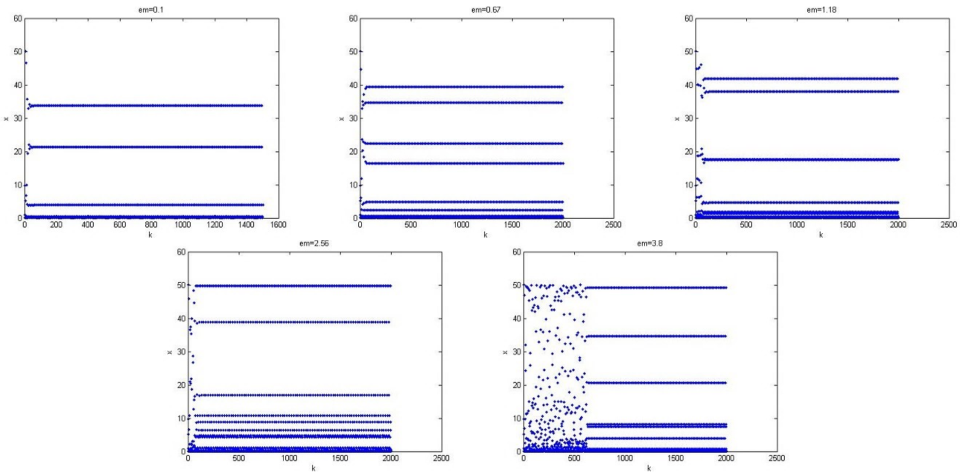

Figure 10: Different types of stabilisation.

Although $E \leq E_m$ is not always maintained, simulations show that the proposed method can stabilize the Z-S map at various p-periodic orbits by adjusting $E_m \in [0, 6.4485]$. Control starts at $n = 60$ and works for both low- and high-periodic orbits, with iteration costs ranging from ≈ 10 (1-periodic) to ≈ 30 (7-periodic).

7 Conclusion

This study examined the stabilization of the chaotic Zeraoulia-Sprott (Z-S) map using feedback and non-feedback control methods. By applying the stability theorem for discrete systems, we demonstrated the effectiveness of both approaches in directing the system toward periodic orbits. Numerical simulations confirmed the feasibility of these techniques, highlighting their potential for controlling chaos in nonlinear discrete systems. Future research may explore extensions to higher-dimensional maps and adaptive control strategies.

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