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Fractional-Order 5D Hyperchaotic System: Stability and Modified Projective Synchronization

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Abstract: This paper introduces a novel five-dimensional fractional-order hyperchaotic system derived as a modification of the Lorenz model. The proposed system exhibits richer dynamics than its integer-order counterpart, including multiple coexisting attractors and complex bifurcation structures. The stability conditions of equilibrium points are derived using an extended fractional Routh–Hurwitz criterion, and Lyapunov exponent analysis confirms the existence of hyperchaotic behavior across a broad parameter range. To further explore its practical relevance, a Modified Projective Synchronization (MPS) strategy is applied to both identical and non-identical systems. Numerical simulations validate the theoretical analysis and highlight the potential of the proposed system for secure communication applications.

Keywords: *stability, 5D fractional-order system, Routh–Hurwitz criteria, modified projective synchronization.*

Mathematics Subject Classification (2020): 34A08, 34D06, 37D45, 70K05, 93C41, 93D05.

1 Introduction

Fractional calculus, whose origins go back more than three centuries, has gained much attention in recent studies because it can model complicated physical, chemical, and engineering systems with greater precision [7]. Fractional-order methods differ from conventional integer-order approaches in that they embrace memory and hereditary properties, enabling them to describe real-world phenomena more accurately. There have

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been profound theoretical advances in this domain, particularly in relation to the formalization of stability criteria for fractional-order differential equations [3], thereby reinforcing the foundational theoretical underpinnings of the discipline. In nonlinear dynamics, chaos synchronization of chaotic systems, where two or more chaotic systems synchronize their behavior through coupling or external drive, has emerged as a research topic of paramount interest. The groundbreaking study by Pecora and Carroll in 1990, established that two identical chaotic systems could achieve synchronization, thus catalyzing a vast array of applications, with secure communication emerging as one of the most pivotal. Following this breakthrough, numerous synchronization approaches have been proposed, including complete synchronization (CS) [6], phase synchronization (PS), lag synchronization (LS) [9], generalized synchronization (GS) [14], and projective synchronization [10]. Building on these foundational approaches, recent work has further expanded this repertoire: Hannachi and Amira [5] demonstrated full-state hybrid projective (FSHP) synchronization for 3D chaotic systems with three nonlinearities, while Ghettout et al. [4] investigated the adaptive synchronization of 4D hyperchaotic systems with infinite equilibria. Each of these paradigms offers a distinct perspective on chaos theory, contributing unique theoretical insights and practical advantages within its respective domain.

The present study is intended to add to the theoretical foundation of chaos research by conducting an in-depth investigation of stability and synchronization of fractional-order hyperchaotic systems. Specifically, it introduces a novel five-dimensional fractional-order hyperchaotic system and, in detail, investigates its stability with different parameter settings. Moreover, an improved MPS scheme is applied to the system and prove to be efficient in synchronizing fractional-order hyperchaotic system dynamics. These findings not only enhance current discussions in chaos theory but also provide a foundation for additional research on high-dimensional fractional-order synchronization methods. Table 1 presents a comparative analysis to emphasize the uniqueness and benefits of the suggested system. Important characteristics of a number of representative fractional-order chaotic and hyperchaotic systems from the literature are compiled in this table, including their dimensionality, equilibrium structure, dynamical behavior, and synchronization strategies. It offers a framework that illustrates how the current work surpasses previous contributions, especially with regard to dimensional extension complexity and synchronization robustness.

The originality of this work in the context of nonlinear dynamics and systems theory is manifested through several key aspects. Unlike fractional-order hyperchaotic models, the system proposed in this paper extends the generalized Lorenz framework to five dimensions by introducing a quadratic self-coupling term $x_4(x_4+1)$ in the first equation, which fundamentally alters the phase space topology and enables richer nonlinear interactions not present in the conventional Lorenz-type extensions. This unique structure produces both second– and third–order hyperchaos within the same parameter configuration. Unlike systems exhibiting only one type of hyperchaotic behavior, our system demonstrates controllable transitions between order-2 and order-3 hyperchaos through dual parameter control (fractional order q and system parameter d). This property is rare in the literature and provides enhanced flexibility for applications requiring tunable complexity. Furthermore, the modified projective synchronization (MPS) scheme developed here achieves guaranteed convergence for both identical and non-identical high-dimensional systems, thereby generalizing existing projective and adaptive synchronization approaches. These innovations place the present study at the intersection of nonlinear dynamics and control theory, offering new tools for secure communication, encryption, and the coordination of complex dynamical networks.

To highlight advances in fractional-order chaotic and hyperchaotic systems between 2020 and 2025, Table 1 summarizes representative models, their system properties, and synchronization methods.

system	dim.	fo	equilibrium	chaos	synchronization	ref.
wang et al.	n/a	yes	not defined	hyperchaotic	improved projective	[12]
chen et al.	4d	yes	single point	hyperchaotic	adaptive tracking	[1]
shao et al.	4d	yes	single point	hyperchaotic	sliding mode	[8]
eshaghi et al.	4d	yes	single point	hyperchaotic	chaos control in laser system	[2]
yaghoubi et al.	4d	yes	stable origin	hyperchaotic	robust adaptive synchronization (satellite system)	[13]
proposed	5d	yes	unstable point	hyperchaotic	mps	this work

Table 1: Comparison of recent fractional-order hyperchaotic systems (2020–2025).

1.1 Key contributions

The main contributions of this study are outlined as follows:

- The proposed novel five-dimensional fractional-order hyperchaotic system is derived from a Lorenz-type structure, exhibiting rich and complex dynamics.
- The system exhibits rich dynamical phenomena, including both chaotic and hyperchaotic attractors of orders 2 and 3, adjustable using fractional order and system parameters q.
- A rigorous theoretical analysis is conducted using the fractional Routh–Hurwitz criterion and eigenvalue-based stability conditions to determine local equilibrium behavior.
- A modified projective synchronization (MPS) scheme is successfully developed and applied to both identical and non-identical systems, achieving convergence through analytically derived control laws.
- The analytical framework is validated by high-resolution numerical simulations using the Adams–Bashforth–Moulton method, confirming robust synchronization and dynamic transitions.

These contributions advance the theoretical and applied study of fractional-order hyperchaotic systems and provide practical tools for control and synchronization in complex dynamical networks.

2 Fundamental Definitions and Preliminary Concepts

2.1 Fractional calculus

Fractional calculus generalizes classical differential calculus to arbitrary non-integer orders, represented by the fundamental operator, enabling a more precise characterization of complex dynamics ${}_aD_q^t$, where a and t are the bounds of the operation and $q \in \mathbf{R}$. The continuous differential integration operator is expressed as

$$ {}_aD_q^t = \begin{cases} \frac{d^q}{dt^q}, & q > 0, \\ 1, & q = 0, \\ \int_a^t (d\tau)^q, & q < 0. \end{cases} $$

Within the present analysis, the Caputo fractional derivative is adopted and mathematically defined as

$$ {}_aD_q^t f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad \text{for } n-1 < q < n. $$

Numerical schemes resolve fractional equations while ensuring stability. The predictor-corrector method excels in chaos simulations, leveraging the Adams-Bashforth-Moulton framework.

2.2 Analytical examination of stability in fractional-order dynamics

Examine a nonlinear fractional-order autonomous system expressed as

$$ \begin{cases} D^{q_1} x_1(t) = g_1(x_1, x_2, \dots, x_n), \\ D^{q_2} x_2(t) = g_2(x_1, x_2, \dots, x_n), \\ \vdots \\ D^{q_n} x_n(t) = g_n(x_1, x_2, \dots, x_n), \end{cases} \tag{1} $$

where $0 < q_i \leq 1$ for $i = 1, 2, \dots, n$. If $q_1 = q_2 = \dots = q_n = q$, the system (1) is designated as a commensurate-order system; otherwise, it is categorized as an incommensurate-order system.

Definition 2.1 An equilibrium point $(x_1^{eq}, x_2^{eq}, \dots, x_n^{eq})$ of the fractional dynamic system (1) is defined as

$$ g_i(x_1^{eq}, x_2^{eq}, \dots, x_n^{eq}) = 0, \quad \forall i = 1, 2, \dots, n. $$

Theorem 2.1 *Within the framework of the commensurate nonlinear fractional-order system* (1) *with* $0 < q_1 = q_2 = \dots = q_n = q \leq 1$, *the equilibrium states of* (1), *denoted as* $x^{eq} = (x_1^{eq}, x_2^{eq}, \dots, x_n^{eq})$, *are locally asymptotically stable if every eigenvalue* λ_i *of the Jacobian matrix* J *evaluated at the equilibrium points has the following condition:*

$$ J = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, $$

where $a_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_{x^{eq}}$ *for* $i, j = 1, 2, \dots, n$, *and we say stability is ensured:*

$$ |\arg(\lambda_i)| > \frac{q\pi}{2}. $$

These criteria define the basic stability conditions for partially ordered systems and create a sound theoretical basis for investigating the dynamical properties in both partially ordered, chaotic, and highly chaotic systems.

2.3 Routh-Hurwitz criteria for fractional-order systems

Examine the following commensurate fractional-order system:

$$D^q x = f(x),$$

where $q \in \,]0, 1]$, $x \in \mathbf{R}^3$. Suppose x_{eq} represents an equilibrium point of this system. The associated characteristic equation can be expressed as

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

The discriminant of this equation is expressed as

$$D(P) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3(a_1)^3 - 4(a_2)^3 - 27(a_3)^2.$$

According to the Routh-Hurwitz fractional criteria, the following conditions provide the necessary and sufficient requirements for the equilibrium point x_{eq} and to achieve local asymptotic stability in a fractional system:

1. If $D(P) > 0$, the system achieves local asymptotic stability provided that $a_1 > 0$, $a_3 > 0$, and $a_1a_2 - a_3 > 0$.
2. If $D(P) < 0$, $a_1 \geq 0$, $a_2 \geq 0$, $a_3 > 0$, stability is ensured when $q < 2/3$. In contrast, if $D(P) < 0$, $a_1 < 0$, $a_2 < 0$, $q > 2/3$, the equilibrium point is unstable.
3. If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_1a_2 - a_3 = 0$, then local asymptotic stability is preserved for all $q \in]0, 1[$.
4. A fundamental requirement for the equilibrium point x_{eq} is that it must be locally asymptotically stable if $a_3 > 0$.

3 Formal Characterization and Analytical Examination of the Models

3.1 Formulation of the five-dimensional hyperchaotic system

Q. Yang et al. [15] presented an advanced five-dimensional hyperchaotic extension of the generalized Lorenz system

$$\begin{cases} D^q x_1 &= a(x_2 - x_1), \\ D^q x_2 &= cx_1 + dx_2 - x_1x_3 + x_5, \\ D^q x_3 &= -bx_3 + x_1^2, \\ D^q x_4 &= ex_2 + fx_4, \\ D^q x_5 &= -kx_5 - rx_1, \end{cases}$$

where $a > 0$, $b > 0$, and $d > -c$, with a, b, c, d, and f denoting system parameters, while e represents the coupling coefficient and r, k are control parameters.

Expanding upon this framework, a fractional-order counterpart of the system is formulated as follows:

$$\begin{cases} D^q x_1 &= a(x_2 - x_1) + x_4(x_4 + 1), \\ D^q x_2 &= bx_1 + dx_2 - x_1x_3, \\ D^q x_3 &= -cx_3 + x_1^2, \\ D^q x_4 &= x_2 - x_4, \\ D^q x_5 &= -kx_5 - rx_4, \end{cases} \tag{2}$$

where $q \in [0,1]$ denotes the fractional-order parameter. The constraints on the parameters remain consistent, where $a > 0$, $c > 0$, and a, b, c, d serve as system coefficients, while r and k are control parameters that influence system stability and chaotic dynamics.

4 Investigation of Dynamical Properties in a Fractional-Order Hyperchaotic Framework

This section examines the dynamic characteristics of fractional-order hyperchaotic systems.

4.1 Fractional-order stability

The equilibrium points of the system described in (2) are identified by solving the equation $g(x_1, x_2, x_3, x_4, x_5) = 0$. After a simple calculation, it is found that this system has one trivial equilibrium point $E_0 \equiv (0,0,0,0,0)$, which always exists.

The stability of the equilibrium point E_0 relies on the Jacobian matrix, evaluated at this point and expressed as follows:

$$J = \begin{pmatrix} -a & a & 0 & 1 & 0 \\ b & d & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -r & -k \end{pmatrix}.$$

Hence, its characteristic polynomial is

$$P(\lambda) = -(c+\lambda)(k+\lambda)\left(\lambda^3 + (a-d+1)\lambda^2 + (-ab-ad+a-d)\lambda - ab - ad - b \quad\right). \tag{3}$$

This polynomial yields the roots $\lambda_1 = -c$, $\lambda_2 = -k$, and

$$P_2(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{4}$$

where $a_1 = a - d + 1$, $a_2 = -ab - ad + a - d$, $a_3 = -ab - ad - b$.

Following the Routh–Hurwitz criterion, the conditions under which all the eigenvalues of (4) lie within the angular sector $|\arg(\lambda_i)| > q\frac{\pi}{2}$, ensuring the stability of the fractional-order system if $D(p) > 0$, are as follows: $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$. These conditions, along with the constraints $c > 0, k > 0$, lead to the following system of inequalities:

$$\begin{cases} a > d - 1, \\ ab + ad + b < 0, \\ a + b - d - a^2b + ad^2 - a^2d - 2ad + a^2 + d^2 + abd > 0. \end{cases} \tag{5}$$

Consequently, $E_0(0,0,0,0,0)$ exhibits local asymptotic stability for all $q \in]0,1[$.

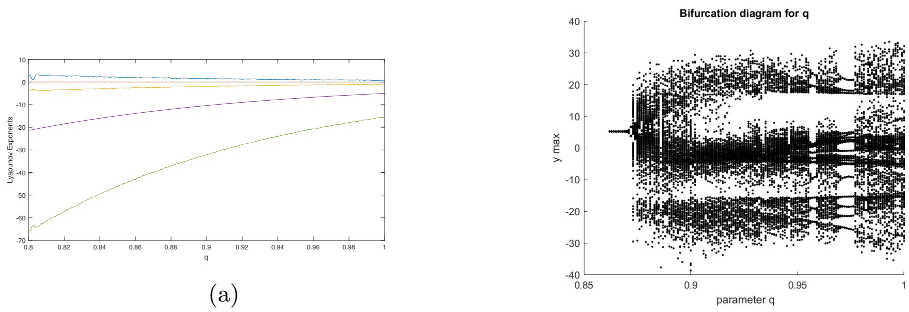
When the parameters (a,c,b,d,r,k) are set to $(10, 28, \frac{8}{3}, -2, 5, 0.05)$, the condition $ab + ad + b = 9.33 > 0$ violates the second requirement of equation (5). Solving the characteristic equation yields four negative real eigenvalues and one positive real eigenvalue: $\lambda_1 = -28$, $\lambda_2 = 0.05$, $\lambda_3 = -12.51$, $\lambda_4 = -1.14$, and $\lambda_5 = 0.65$. This indicates that the equilibrium point $E_0(0,0,0,0,0)$ is a saddle of index one and hence unstable. Additionally, the condition $\min|\arg(\lambda_i)| > q\frac{\pi}{2}$ for $i = 1,2,3,4$ is satisfied. As a result, it is possible to assess the potential for chaos within the system described by equation (2) when the order $q > 0$, in line with the five necessary conditions for the presence of a chaotic attractor in fractional-order systems. However, it remains impossible to ascertain the minimum order at which chaos manifests in the fractional-order system with the specified parameters.

4.2 Fixed system parameters and variable fractional order q

In this subsection, the system parameters are kept constant at $(a, c, b, d, r, k) = \left(10, 28, \frac{8}{3}, -2, 5, 0.05\right)$, while the fractional order q is varied within the range $[0.8, 1]$. The initial values of the state variables are set to $(3, 5, 9, 13, 0.1)$.

Fig.1(a) depicts the Lyapunov exponents of system (2) for different values of the fractional order q. The bifurcation diagram of system (2) as a function of q is shown in Fig.1(b). Table 2 presents the calculated Lyapunov exponents of system (2) and their corresponding dynamic behaviors for various values of the fractional order q.

Additionally, Fig.2 illustrates the projections of the order-2 hyper-chaotic attractor of system (2) when $q = 0.99$.



(a)

(b)

Figure 1: (a) Lyapunov exponents of fractional system (2) with $(a, c, b, d, r, k) = (10, 28, \frac{8}{3}, -2, 5, 0.05)$ and $q \in [0.8, 1]$; (b) Bifurcation diagram of system (2) versus q

q	LE_1	LE_2	LE_3	LE_4	LE_5	Dynamics
0.810	2.8522	-0.0024	-3.6231	-20.0310	-61.8718	Chaotic
0.813	3.1708	0.0006	-3.6179	-19.6049	-60.8646	Hyperchaotic (order 2)
0.856	2.0145	0.0022	-2.5577	-14.3831	-44.4397	Hyperchaotic (order 2)
0.993	0.8303	0.0038	0.9752	-5.4538	-16.4516	Hyperchaotic (order 3)

Table 2: Lyapunov exponents of fractional systems with $(a, c, b, d, r, k) = (10, 28, \frac{8}{3}, -2, 5, 0.05)$.

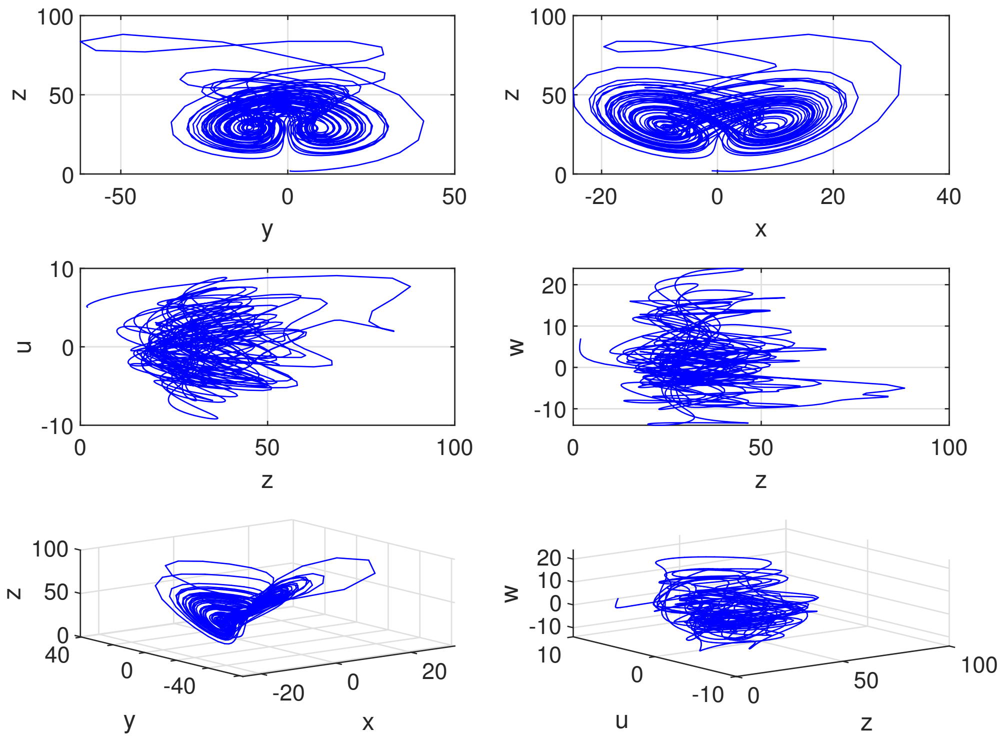
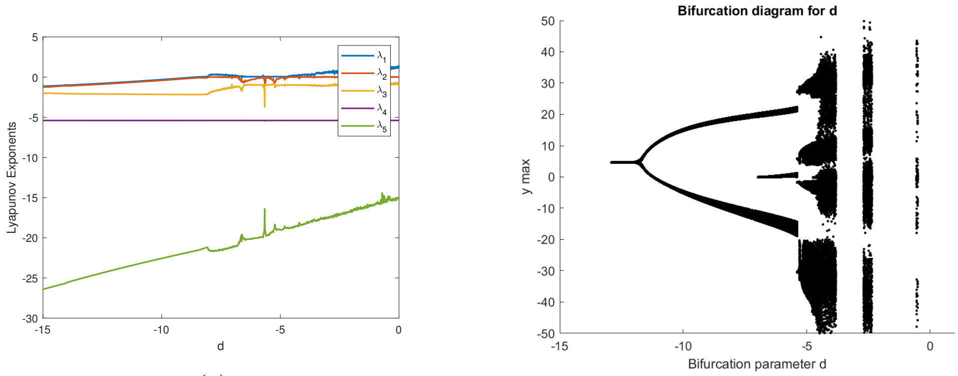


Figure 2: Hyperchaotic attractor of system (2) for $(a, c, b, d, r, k) = (10, 28, \frac{8}{3}, -2, 5, 0.05)$ and $q = 0.99$.

4.3 Fixed fractional order and variable system parameter d

In this subsection, the system parameters are $(a, c, b, r, k) = (10, 28, \frac{8}{3}, 5, 0.05)$, with $q = 0.99$, while exploring the impact of varying $d \in [-15, 0]$. The initial values for state variables are set to (3 5 9 13 0.1). The lyapunov exponents of equation (2) with respect to d are illustrated in Fig.3(a). The bifurcation diagram for the system described in equation (2) with $d \in [-15, 1]$ is depicted in Fig.3(b), using $(a, c, b, r, k) = (10, 28, \frac{8}{3}, 5, 0.05)$ and $q = 0.99$.

Table 3 presents the Lyapunov exponents of system (2) for various values of d. The corresponding dynamic behaviors for different d values are described. Additionally, Fig.4 displays the projections of the second-order hyperchaotic attractor for system (2) with $d = -6$ and $q = 0.99$.



(a) (b)

Figure 3: (a) Lyapunov exponents of fractional system (2) with $(a, c, b, r, k) = (10, 28, \frac{8}{3}, 5, 0.05)$ and $q = 0.99$, while $d \in [-15, 0]$; (b) Bifurcation diagram of system (2) versus d with $q = 0.99$.

d	LE_1	LE_2	LE_3	LE_4	LE_5	Dynamics
-6	0.2550	-0.0007	-0.9901	-1.1339	-18.2360	Chaotic
-4	0.6434	0.0002	-1.0325	-1.1256	-17.4954	Hyperchaotic (order 2)
-2	0.7419	0.0140	0.0525	-5.4538	-16.5421	Hyperchaotic (order 3)

Table 3: lyapunov exponents for fractional systems with $(a, c, b, r, k) = (10, 28, \frac{8}{3}, 5, 0.05)$, $q = 0.99$, and varying d.

5 Modified Projective Synchronization Framework

5.1 Synchronization of identical hyperchaotic systems via modified projective approach

Two identical hyperchaotic frameworks evolve, with drive dynamics defined as follows:

$$\begin{cases} D^q x_1 &= a(x_2 - x_1) + x_4(x_4 + 1), \\ D^q x_2 &= bx_1 + dx_2 - x_1 x_3, \\ D^q x_3 &= -cx_3 + x_1^2, \\ D^q x_4 &= x_2 - x_4, \\ D^q x_5 &= -kx_5 - rx_4. \end{cases} \tag{6}$$

The corresponding response system is formulated as

$$\begin{cases} D^q y_1 &= a(y_2 - y_1) + y_4(y_4 + 1) + \mu_1, \\ D^q y_2 &= by_1 + dy_2 - y_1 y_3 + \mu_2, \\ D^q y_3 &= -cy_3 + y_1^2 + \mu_3, \\ D^q y_4 &= y_2 - y_4 + \mu_4, \\ D^q y_5 &= -ky_5 - ry_4 + \mu_5. \end{cases} \tag{7}$$

Five control nonlinear functions, denoted μ_i $(i = 1, 2, 3, 4, 5)$, were introduced in (7) to synchronize the two identical systems in the sense of MPS.

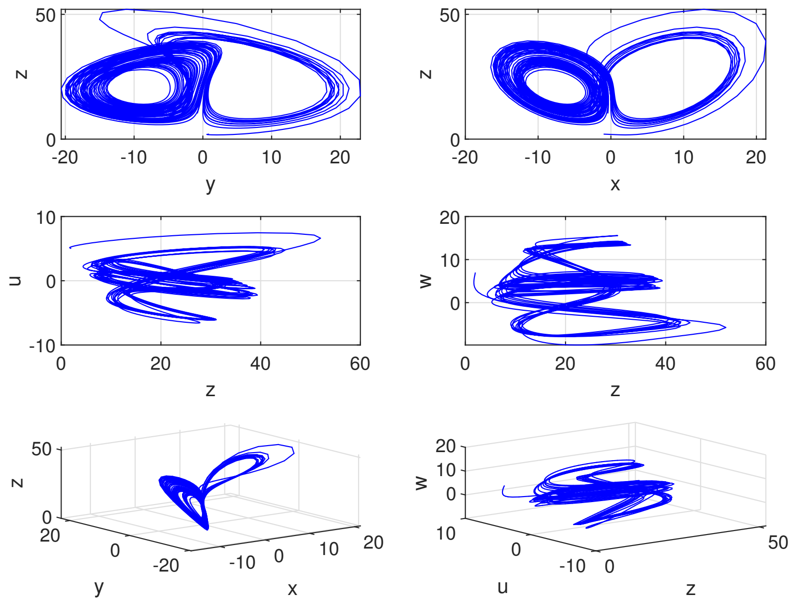


Figure 4: Hyperchaotic attractor of system (2) for $(a, c, b, r, k) = (10, 28, \frac{8}{3}, 5, 0.05)$, $d = -6$, and $q = 0.99$.

Defining the synchronization error as

$$e_i = y_i - \alpha_i x_i \quad \text{for} \quad i = 1, 2, 3, 4, 5,$$

the error dynamics can be derived by subtracting the response system from the drive system:

$$\begin{cases} D^q e_1 = a(e_2 - e_1) + e_4 + a(\alpha_2 - \alpha_1)x_2 + y_4^2 + (\alpha_4 - \alpha_1)x_4 + \mu_1, \\ D^q e_2 = de_2 + by_1 - y_1 y_3 - b\alpha_2 x_1 + \alpha_2 x_1 x_3 + \mu_2, \\ D^q e_3 = -ce_3 + y_1^2 + \alpha_3 x_1^2 + \mu_3, \\ D^q e_4 = -e_4 + y_2 - \alpha_4 x_2 - \alpha_4 x_5 + \mu_4, \\ D^q e_5 = -ry_4 - ke_5 + r\alpha_5 x_4 + \mu_5. \end{cases} \tag{8}$$

Construct appropriate control laws

$$\begin{aligned} \mu_1 &= -a(\alpha_2 - \alpha_1)x_2 - y_4^2 - (\alpha_4 - \alpha_1)x_4 - e_4, \\ \mu_2 &= -by_1 + y_1 y_3 + b\alpha_2 x_1 - \alpha_2 x_1 x_3, \\ \mu_3 &= -y_1^2 - \alpha_3 x_1^2, \\ \mu_4 &= -y_2 + \alpha_4 x_2 + \alpha_4 x_5, \\ \mu_5 &= ry_4 - r\alpha_5 x_4. \end{aligned} \tag{9}$$

Substituting these into the error dynamics yields a system of decoupled linear differential equations:

$$\begin{cases} D^q e_1 = a(e_2 - e_1), \\ D^q e_2 = de_2, \\ D^q e_3 = -ce_3, \\ D^q e_4 = -e_4, \\ D^q e_5 = -ke_5. \end{cases} \tag{10}$$

Taking the Laplace transformation on both sides of (10) gives

$$E_i(s) = \mathcal{L}(e_i(t)).$$

Given

$$\mathcal{L}\left(\frac{d^q e_i(t)}{dt^\alpha}\right) = s^q E_i(s) - s^{q-1} e_i(0) \quad i = 1, 2, 3, 4, 5,$$

results in the following system of equations:

$$\begin{aligned} s^q E_1(s) - s^{q-1} e_1(0) &= -aE_1(s) + aE_2(s), \\ s^q E_2(s) - s^{q-1} e_2(0) &= dE_2(s), \\ s^q E_3(s) - s^{q-1} e_3(0) &= -cE_3(s), \\ s^q E_4(s) - s^{q-1} e_4(0) &= -E_4(s), \\ s^q E_5(s) - s^{q-1} e_5(0) &= -kE_5(s). \end{aligned} \tag{11}$$

From 11, it follows that

$$\begin{cases} E_1(s) &= \frac{s^{q-1} e_1(0) + aE_2(s)}{s^q + a}, \\ E_2(s) &= \frac{s^{q-1} e_2(0)}{s^q - d}, \\ E_3(s) &= \frac{s^{q-1} e_3(0)}{s^q + c}, \\ E_4(s) &= \frac{s^{q-1} e_4(0)}{s^q + 1}, \\ E_5(s) &= \frac{s^{q-1} e_5(0)}{s^q + k}. \end{cases}$$

By applying the Laplace transformation and utilizing the final value theorem, it follows that

$$\begin{cases} \lim_{t\to\infty} e_2(t) &= \lim_{s\to 0^+} sE_2(s) = \lim_{s\to 0^+} \frac{s^q e_2(0)}{s^q - d} = 0, \\ \lim_{t\to\infty} e_1(t) &= \lim_{s\to 0^+} sE_1(s) = \lim_{s\to 0^+} \frac{s^q e_1(0) + saE_2(s)}{s^q + a} = 0, \\ \lim_{t\to\infty} e_3(t) &= \lim_{s\to 0^+} sE_3(s) = \lim_{s\to 0^+} \frac{s^q e_3(0)}{s^q + c} = 0, \\ \lim_{t\to\infty} e_4(t) &= \lim_{s\to 0^+} sE_4(s) = \lim_{s\to 0^+} \frac{s^q e_4(0)}{s^q + 1} = 0, \\ \lim_{t\to\infty} e_5(t) &= \lim_{s\to 0^+} sE_5(s) = \lim_{s\to 0^+} \frac{s^q e_5(0)}{s^q + k} = 0. \end{cases}$$

Therefore, both the response system (6) and the driving system (7) have attained MPS.

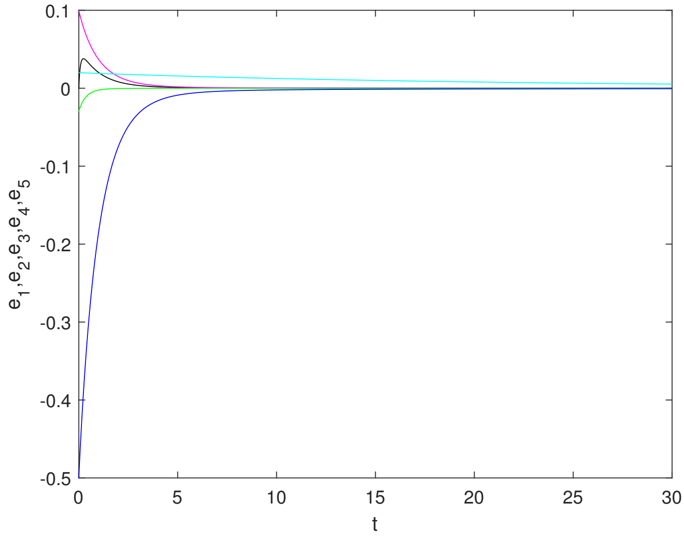


Figure 5: Synchronization errors between identical systems with $q = 0.95$.

5.2 Synchronization of non-identical hyperchaotic systems via modified projective approach

Suppose the system is the slave system [11], and its control is formulated as

$$\begin{cases} D^q y_1 = a_1(y_2 - y_1) + y_4 + \mu_1, \\ D^q y_2 = c_1 y_1 - y_1 y_3 + y_5 + \mu_2, \\ D^q y_3 = -b_1 y_3 + y_1 y_2 + \mu_3, \\ D^q y_4 = -h_1 y_4 - y_1 y_3 + \mu_4, \\ D^q y_5 = -k_1 y_1 - k_2 y_2 + \mu_5, \end{cases} \tag{12}$$

where μ_i, $i = 1, 2, 3, 4, 5$, are nonlinear control functions. The master system is given by

$$\begin{cases} D^q x_1 &= a(x_2 - x_1) + x_4(x_4 + 1), \\ D^q x_2 &= bx_1 + dx_2 - x_1 x_3, \\ D^q x_3 &= -cx_3 + x_1^2, \\ D^q x_4 &= x_2 - x_4, \\ D^q x_5 &= -kx_5 - rx_4. \end{cases} \tag{13}$$

Define the error variables as $e_i = y_i - l_i x_i$, where $i = 1, 2, 3, 4, 5$. Then the error dynamical system is given by

$$\begin{cases} D^q e_1 = a_1(e_2 - e_1) + l_1(a - a_1)x_1 + (l_1 a_1 - l_1 a)x_2 + e_4 + (l_1 + l_4)x_4 + l_1 x_4^2 + \mu_1, \\ D^q e_2 = c_1 e_2 - (l_2 b + c_1 l_1)x_1 - y_1 y_3 + y_5 + l_2 x_1 x_3 + d l_2 x_2 + \mu_2, \\ D^q e_3 = -b_1 e_3 + y_1 y_2 - l_3 x_1^2 + l_3(c + b_1)x_3 + \mu_3, \\ D^q e_4 = -h_1 e_4 - y_1 y_3 - l_4(h_1 - 1)x_4 - l_4 x_2 + \mu_4, \\ D^q e_5 = -k e_5 - k_1 y_1 - k_2 y_2 + k l_5 x_5 + r l_5 x_4 - k y_5 + \mu_5. \end{cases} \tag{14}$$

Then the active control μ_i $(i = 1, 2, 3, 4, 5)$ is defined as

$$\begin{aligned} \mu_1 &= -l_1 (a - a_1) x_1 - (l_2 a_1 - l_1 a) x_2 - (l_1 + l_4)x_4 - l_1 x_4^2, \\ \mu_2 &= -(l_2 b + c_1 l_1)x_1 + y_1 y_3 - y_5 - l_2 x_1 x_3 - d l_2 x_2, \\ \mu_3 &= -y_1 y_2 + l_3 x_1^2 - l_3(c + b_1)x_3, \\ \mu_4 &= y_1 y_3 + l_4(h_1 - 1)x_4 + l_4 x_2, \\ \mu_5 &= k_1 y_1 + k_2 y_2 - k l_5 x_5 - r l_5 x_4 + k y_5. \end{aligned} \tag{15}$$

Substituting (15) into (14) leads to

$$\begin{cases} D^q e_1 &= a_1 (e_2 - e_1) + e_4, \\ D^q e_2 &= c_1 e_2, \\ D^q e_3 &= -b_1 e_3, \\ D^q e_4 &= -h_1 e_4, \\ D^q e_5 &= -k e_5. \end{cases} \tag{16}$$

Take the Laplace transformation on both sides of (16) in a way similar to that in Subsection 5.1. It is found that $\lim\limits_{t \longrightarrow +\infty} e_i(t) = 0$ for $i = 1, 2, 3, 4, 5$. Therefore, the MPS of the systems (12) and (13) is achieved under the control law (15).

Therefore, both the response system (13) and the driving system (12) have attained MPS.

5.3 Numerical simulations

This section presents the findings of the simulation that illustrate the efficacy of the suggested synchronization scheme. The ABM method is used to solve the systems. For these numerical simulations, the parameters for the slave and master systems are set as $\left(10, \frac{8}{3}, 28, -1, 5, 0.05\right)$.

The experiments are carried out with a fixed fractional-order value of $q = 0.95$. The starting conditions for the master and slave systems are given as $(2, 6, 4, -1.5, 7.5)$ and $(8, 5.5, 6, 4.5, 1)$, respectively. The results confirm successful synchronization, as illustrated in Fig. 5, where the errors e_i, for $i = 1, 2, ..., 5$, converge to zero.

Additional simulations further validate the scheme. The parameters of the master system are $(35, 7, 35, -5, 5, 0.05)$, while the parameters of the slave system are $\left(10, 28, \frac{8}{3}, -2, 0, 12\right)$. With $q = 0.98$, the initial conditions are $(3, 5, 4, 2, 6)$ and $(8, 10, 6, 8, 1)$. As shown in Fig.6, synchronization is achieved with the errors converging to zero.

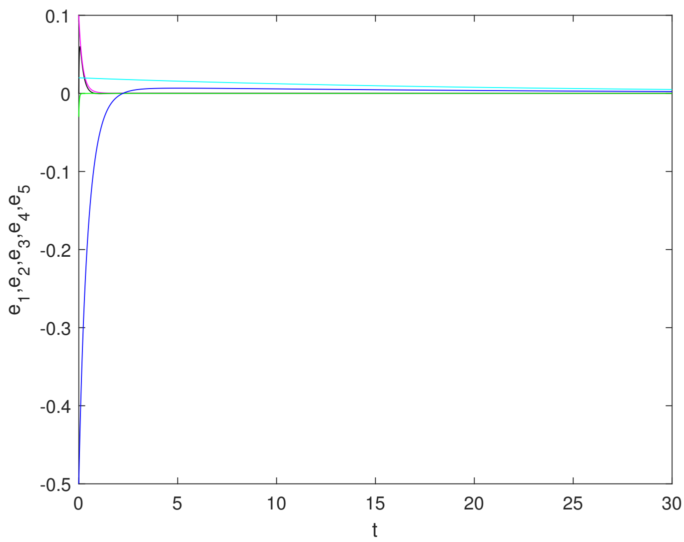


Figure 6: Synchronization errors between nonidentical systems with $q = 0.98$.

6 Discussion

The stability regions and synchronization results obtained demonstrate that the proposed five-dimensional hyperchaotic system with fractional order provides a flexible platform for investigating high-dimensional nonlinear dynamics. In particular, its ability to create second and third-order hyperchaos and synchronize identical and non-identical systems in the MPS distinguishes it from existing models. It increases its applicability in secure communication and control of complex networks.

7 Conclusion

This paper presents a five-dimensional fractional-order hyperchaotic system derived from a modified generalized Lorenz framework. Unlike previously reported fractional-order models, the proposed system introduces a nonlinear cross-coupling term together with tunable fractional orders, enabling both second- and third-order hyperchaotic attractors within a single parameter configuration.

Using the fractional Routh–Hurwitz criterion, we derived explicit stability conditions that map the parameter regions of chaotic and hyperchaotic behavior. Building on this analysis, we designed a modified projective synchronization (MPS) scheme and proved analytically that it synchronizes both identical and non-identical fractional-order systems. High-resolution numerical simulations support the theory and show robust convergence of synchronization errors.

Overall, the results extend the understanding of high-dimensional fractional-order nonlinear systems and offer practical tools for secure communications, encryption, and control of complex dynamical networks. The proposed system and synchronization approach also open promising directions for future research on high-dimensional fractional-order chaotic and hyperchaotic dynamics.

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