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Numerical Resolution of Transport-Diffusion Systems via Taylor Collocation Method

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Abstract: The primary objective of this paper is to develop a numerical approach for solving a system of transport-diffusion equations. The proposed method is based on Taylor polynomials, which are employed within a collocation method in the space $S_{p-1}^{(-1)}(\Pi_{N,M})$ to approximate the solution of the corresponding Volterra integro-differential equation. The convergence of the method is established, and numerical experiments are conducted to demonstrate its accuracy. This work contributes to the field of system dynamics by introducing a new computational approach to understanding and predicting the behavior of transport-diffusion systems. The Taylor collocation method enables precise numerical approximations of these equations, which is fundamental in modeling dynamic processes such as pollutant dispersion in a moving fluid and thermal diffusion in engineered systems.

Keywords: *system of transport-diffusion equations; Volterra integro-differential equation; collocation method; Taylor polynomials; error analysis.*

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1 Introduction

The transport-diffusion system is given by

$$\partial_t u(t,x) + v(t) \cdot \nabla u(t,x) - \kappa \Delta u(t,x) = g(t,x,u(t,x)) \tag{1}$$

for all $(t,x,u) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^d$. This system is classified as a parabolic partial differential equation (for a mathematical analysis, see [8], which provided essential theoretical foundations for the existence and uniqueness of the solution to this system). It is used to model the process of transport and diffusion of a substance contained in a fluid, or a property within a moving fluid.

Here, u represents the amount of the substance or the intensity of the property being transported and diffused, v is the velocity vector of transport, κ is the diffusion coefficient (a positive constant), and g is the source or reaction term. This equation has applications in various fields, including the analysis of dynamical systems and atmospheric problems such as air pollution. It can also model water pollution problems in seas and other aquatic systems.

In the study of this equation, two main cases arise depending on the value of g. The first case is when $g = 0$. Many researchers have studied this case numerically in the domain $0 < x < L$ under initial and boundary conditions, using different numerical methods such as the compact finite differences method of sixth order [7] and B-spline exponential collocation method [10]. The second case is when $g \neq 0$. Many researchers have investigated this scenario. In [1], Alhumaizi analyzed a convection-diffusion system with reaction using various standard reduction techniques. In [9], Liu presented a numerical analysis of a diffusion-migration (transport) model with a reaction describing the interaction between prey and predator in one-dimensional space under periodic boundary conditions.

The main goal of this paper is to resolve the system of transport-diffusion equations in a one-dimensional domain under initial conditions using an algorithm based on the Taylor collocation method. This method is known for its powerful performance in solving differential, integral, and integro-differential equations numerically (see, for example, [2,3,5]). Recently, a Taylor-based numerical framework was proposed for delay Volterra integral equations with mixed kernels and spatial variables, demonstrating high accuracy and reliable convergence properties [11]. Motivated by these results, the present work extends the Taylor collocation strategy to the case of proportional-delay Volterra equations.

The paper is organized as follows. In Section 2, we will pose the problem and convert it to a Volterra integro-differential equation of the second kind. In Section 3, we will approximate the solution of the Volterra integro-differential equation. We will also explore the convergence analysis and its order for the Volterra integro-differential equation. Section 4 will present a numerical example to illustrate the theoretical results. Finally, we will conclude with a summary of our research and prospects for further study.

2 Statement of the Problem

Consider the following system of transport-diffusion equations:

$$\frac{\partial u(t,x)}{\partial t} + v(t)\frac{\partial u(t,x)}{\partial x} - \kappa \frac{\partial^2 u(t,x)}{\partial x^2} = g(t,x,u(t,x)), \quad (t,x) \in D,$$

where $D = [0,T] \times [0,X] \subset \mathbb{R}^+ \times \mathbb{R}$ and $u = (u_1, u_2, \dots, u_d)^T \in \mathbb{R}^d$ (here, κ is a strictly positive constant), subject to the initial conditions

$$u(0,x) = u_0(x),\ u(t,0) = h(t),\ u_0(0) = h(0),\ \frac{\partial u(t,0)}{\partial x} = l(t).$$

Here, $h, l, g \in \mathbb{R}^d$ and $v = \mathrm{diag}(v_1, v_2, \dots, v_d)$ is a diagonal matrix.

We assume that $g(t,x,u(t,x))$ is affine in the third variable, i.e.,

$$g(t,x,u(t,x)) = a(t,x) \cdot u(t,x) + b(t,x),$$

such that a (reaction term) is a $d \times d$ matrix and $b = (b_1, b_2, \dots, b_d)^T$ represents the vector source term.

Integrating twice both sides of (1) from 0 to x with respect to the second variable yields a system of two-dimensional Volterra integro-differential equation of the form

$$\begin{aligned} u(t,x) = {} & \int_0^x \left[\frac{s-x}{\kappa} (a(t,s) \cdot u(t,s) + b(t,s)) \right] ds + \left(I_d - \frac{x}{\kappa} v(t) \right) h(t) + x l(t) \\ & + \frac{1}{\kappa} \int_0^x \left[(x-s) \frac{\partial u(t,s)}{\partial t} + v(t) u(t,s) \right] ds, \end{aligned}$$

then

$$u(t,x) = f(t,x) + \frac{1}{\kappa} \int_0^x \left[(x-s) \left(\frac{\partial u(t,s)}{\partial t} - a(t,s) \cdot u(t,s) \right) + v(t) u(t,s) \right] ds, \tag{2}$$

where

$$f(t,x) = \int_0^x \frac{s-x}{\kappa} b(t,s)\, ds + \left(I_d - \frac{x}{\kappa} v(t) \right) h(t) + x l(t).$$

The functions f, a, and v are smooth, with f and a defined on $D = [0,T] \times [0,X] \subset \mathbb{R}^2$, and v defined on $[0,T] \subset \mathbb{R}$.

3 Description of the Method

Let $\Pi_N = \{t_i \mid t_i = ih,\ i = 0,1,\dots,N\}$ and $\Pi_M = \{x_j \mid x_j = jk,\ j = 0,1,\dots,M\}$ denote the uniform partitions of the intervals $[0,T]$ and $[0,X]$, respectively, with step sizes given by $h = \frac{T}{N}$ and $k = \frac{X}{M}$. These partitions define a grid for D:

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(t_n, x_m),\ 0 \le n \le N,\ 0 \le m \le M\}.$$

We define the subintervals as follows:

$$\sigma_n = [t_n, t_{n+1}), \quad n = 0,1,\dots,N-2; \quad \sigma_{N-1} = [t_{N-1}, t_N],$$

$$\delta_m = [x_m, x_{m+1}), \quad m = 0,1,\dots,M-2; \quad \delta_{M-1} = [x_{M-1}, x_M],$$

and we define $D_{n,m} := \sigma_n \times \delta_m$ for all $n = 0,1,\dots,N-1$; $m = 0,1,\dots,M-1$. Moreover, let π_{p-1} represent the set of all real polynomials in $\mathbb{R}$ of degree not exceeding $p-1$ in t and x. We define the real polynomial spline space of degree $p-1$ as follows:

$$S_{p-1}^{(-1)}(\Pi_{N,M}) = \{\bar{u} : \bar{u}_{n,m} = \bar{u}|_{D_{n,m}} \in \pi_{p-1}^d,\ n = 0,\dots,N-1;\ m = 0,\dots,M-1\}.$$

Its dimension is $dNMp^2$, which corresponds to the total number of coefficients of the polynomials $\bar{u}_{n,m}$ for $n = 0, \ldots, N-1$; $m = 0, \ldots, M-1$. To determine these coefficients, we apply the Taylor polynomial on each rectangle.

First, we approximate u in the rectangles $D_{0,0}$ by the polynomials

$$\bar{u}_{0,0}(t,x) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} u(0,0)}{\partial t^i \partial x^j} t^i x^j \ ; \quad (t,x) \in D_{0,0}, \tag{3}$$

where $\dfrac{\partial^{i+j} u(0,0)}{\partial t^i \partial x^j}$ is the exact value of $\dfrac{\partial^{i+j} u}{\partial t^i \partial x^j}$ at the point $(0,0)$.

To obtain $\dfrac{\partial^{i+j} u}{\partial t^i \partial x^j}$, we differentiate equation (2) j times with respect to x and i times with respect to t, considering three distinct cases.

- If $j = 0$, then we have
$$\begin{aligned}\frac{\partial^i u(t,x)}{\partial t^i} &= \frac{\partial^i f(t,x)}{\partial t^i} + \frac{1}{\kappa} \int_0^x \frac{\partial^i (v(t) \cdot u(t,s))}{\partial t^i} ds \\ &\quad + \frac{1}{\kappa} \int_0^x (x-s) \left(\frac{\partial^{i+1} u(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot u(t,s))}{\partial t^i} \right) ds.\end{aligned}$$
- If $j = 1$, then we obtain
$$\begin{aligned}\frac{\partial^{i+1} u(t,x)}{\partial t^i \partial x} &= \frac{\partial^{i+1} f(t,x)}{\partial t^i \partial x} + \frac{1}{\kappa} \frac{\partial^i (v(t) u(t,x))}{\partial t^i} \\ &\quad + \frac{1}{\kappa} \int_0^x \left[\frac{\partial^{i+1} u(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot u(t,s))}{\partial t^i} \right] ds.\end{aligned}$$
- If $j \geq 2$, then we find
$$\begin{aligned}\frac{\partial^{i+j} u(t,x)}{\partial t^i \partial x^j} &= \frac{\partial^{i+j} f(t,x)}{\partial t^i \partial x^j} + \frac{1}{\kappa} \frac{\partial^i \left(v(t) \frac{\partial^{j-1} u(t,x)}{\partial x^{j-1}} \right)}{\partial t^i} \\ &\quad + \frac{1}{\kappa} \frac{\partial^{i+j-1} u(t,x)}{\partial t^{i+1} \partial x^{j-2}} - \frac{1}{\kappa} \frac{\partial^{i+j-2} (a(t,x) \cdot u(t,x))}{\partial t^i \partial x^{j-2}}.\end{aligned}$$

Second, we approximate u in the rectangles $D_{n,m}, n = 0, ..., N-1, \ m = 0, ..., M-1$ and $(n,m) \neq (0,0)$, by the polynomials

$$\bar{u}_{n,m}(t,x) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{u}_{n,m}(t_n, x_m)}{\partial t^i \partial x^j} (t-t_n)^i (x-x_m)^j \ ; \quad (t,x) \in D_{n,m}, \tag{4}$$

where $\hat{u}_{n,m}$ is the exact solution of the integral equation

$$\begin{aligned}\hat{u}_{n,m}(t,x) &= f(t,x) + \frac{1}{\kappa} \sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} (x-s) \left(\frac{\partial \bar{u}_{n,\rho}(t,s)}{\partial t} - a(t,s) \cdot \bar{u}_{n,\rho}(t,s) \right) ds \\ &\quad + \frac{1}{\kappa} \left(\sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} v(t) \bar{u}_{n,\rho}(t,s) ds + \int_{x_m}^{x} v(t) \hat{u}_{n,m}(t,s) ds \right) \\ &\quad + \frac{1}{\kappa} \int_{x_m}^{x} (x-s) \left(\frac{\partial \hat{u}_{n,m}(t,s)}{\partial t} - a(t,s) \cdot \hat{u}_{n,m}(t,s) \right) ds.\end{aligned} \tag{5}$$

To find $\frac{\partial^{i+j} u}{\partial t^i \partial x^j}$, we differentiate equation (5) j times with respect to x and i times with respect to t, considering three different cases.

- If $j = 0$, then we get

$$\begin{aligned}\frac{\partial^i \hat{u}_{n,m}(t,x)}{\partial t^i} &= \frac{\partial^i f(t,x)}{\partial t^i} + \int_{x_m}^{x} \frac{x-s}{\kappa} \left(\frac{\partial^{i+1} \hat{u}_{n,m}(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot \hat{u}_{n,m}(t,s))}{\partial t^i} \right) ds \\ &+ \frac{1}{\kappa} \sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} (x-s) \left(\frac{\partial^{i+1} \bar{u}_{n,\rho}(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot \bar{u}_{n,\rho}(t,s))}{\partial t^i} \right) ds \\ &+ \frac{1}{\kappa} \left(\sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} \frac{\partial^i \left(v(t) \bar{u}_{n,\rho}(t,s)\right)}{\partial t^i} ds + \int_{x_m}^{x} \frac{\partial^i \left(v(t) \hat{u}_{n,m}(t,s)\right)}{\partial t^i} ds \right).\end{aligned}$$

- If $j = 1$, then we get

$$\begin{aligned}\frac{\partial^{i+1} \hat{u}_{n,m}(t,x)}{\partial t^i \partial x} &= \frac{\partial^{i+1} f(t,x)}{\partial t^i \partial x} + \frac{1}{\kappa} \frac{\partial^i \left(v(t) \hat{u}_{n,m}(t,x)\right)}{\partial t^i} \\ &+ \frac{1}{\kappa} \sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} \left(\frac{\partial^{i+1} \bar{u}_{n,\rho}(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot \bar{u}_{n,\rho}(t,s))}{\partial t^i} \right) ds \\ &+ \frac{1}{\kappa} \int_{x_m}^{x} \left(\frac{\partial^{i+1} \hat{u}_{n,m}(t,s)}{\partial t^{i+1}} - \frac{\partial^i (a(t,s) \cdot \hat{u}_{n,m}(t,s))}{\partial t^i} \right) ds.\end{aligned}$$

- If $j \geq 2$, then we get

$$\begin{aligned}\frac{\partial^{i+j} \hat{u}_{n,m}(t,x)}{\partial t^i \partial x^j} &= \frac{\partial^{i+j} f(t,x)}{\partial t^i \partial x^j} + \frac{1}{\kappa} \frac{\partial^i \left(v(t) \frac{\partial^{j-1} \hat{u}_{n,m}(t,x)}{\partial x^{j-1}} \right)}{\partial t^i} \\ &+ \frac{1}{\kappa} \frac{\partial^{i+j-1} \hat{u}_{n,m}(t,x)}{\partial t^{i+1} \partial x^{j-2}} - \frac{1}{\kappa} \frac{\partial^{i+j-2} (a(t,x) \cdot \hat{u}_{n,m}(t,x))}{\partial t^i \partial x^{j-2}}.\end{aligned}$$

4 Study of the Convergence and the Error of the Numerical Method

This section will focus on studying the convergence of the method above and evaluating its error. The following lemmas will be important in this analysis.

Lemma 4.1 (*Taylor's theorem for functions of two independent variables*) *Let f be p times continuously differentiable on $D = [0,a] \times [0,b]$ and let $(x_0, y_0) \in D$. Then for all $(x,y) \in D$, we have*

$$\begin{aligned}f(x,y) &= \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j} (x-x_0)^i (y-y_0)^j \\ &+ \sum_{i+j=p} \frac{1}{i!j!} \frac{\partial^{i+j} f(x_1,y_1)}{\partial x^i \partial y^j} (x-x_0)^i (y-y_0)^j,\end{aligned}$$

where

$$\begin{cases} x_1 = \theta x + (1-\theta) x_0 \in [0,a], \\ y_1 = \theta y + (1-\theta) y_0 \in [0,b], \end{cases} \quad \theta \in (0,1).$$

Lemma 4.2 (*Discrete Gronwall-type inequality [6]*) *Let $\{k_j\}_{j=0}^{n}$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfy $\varepsilon_0 \leq p_0$ and*

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 1,$$

with $p_0 \geq 0$. Then ε_n can be bounded by $\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$

Lemma 4.3 *Let f and v be p-times continuously differentiable on their respective domains. Then, under the assumptions*

$$k < \min\Big\{\frac{\kappa}{A}, \frac{-V + \sqrt{V^2 + 2\kappa(1+A)}}{2(1+A)}\Big\},$$

where $V = \max\Big\{\binom{i}{r}\Big\|\frac{\partial^{i-r} v}{\partial t^{i-r}}\Big\|, \quad i = 0, ..., p, \quad r = 0, ..., i\Big\}$, $A = \max\Big\{\binom{i}{r}\Big\|\frac{\partial^{i-r} a}{\partial t^{i-r}}\Big\|, \quad i = 0, ..., p, \quad r = 0, ..., i\Big\}$, there exists a positive number $C(p)$ such that for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$ and $i + j = 0, 1, ..., p$, we have

$$\left\|\frac{\partial^{i+j} \hat{u}_{n,m}}{\partial t^i \partial x^j}\right\|_{L^\infty(D_{n,m})} \leq C(p),$$

where $\hat{u}_{0,0}(t,x) = u(t,x)$ for $(t,x) \in D_{0,0}$.

Proof. The proof follows from a more straightforward generalization of the techniques applied in Lemma 5 of [4]. □

The following theorem will give the convergence of the presented method.

Theorem 4.1 *Let f, a, and v be p times continuously differentiable on their respective domains. Assume that $\bar{u} \in S_{p-1}^{(-1)}(\Pi_{N,M})$ in equations (3) and (4) defines a unique approximate solution. Then the error function $e = u - \bar{u}$ satisfies the condition*

$$\|e\|_{L^\infty(D)} \leq C(h+k)^{p-1},$$

C is a finite constant independent of h and k.

Proof. Define the error $e(t,x)$ on $D_{n,m}$ by $e_{n,m}(t,x) = u(t,x) - \bar{u}_{n,m}(t,x)$ for all $n \in \{0, \ldots, N-1\}$ and $m \in \{0, \ldots, M-1\}$.

Claim 1. There exists a constant β_1 independent of h and k such that $\|e_{0,0}\|_{L^\infty(D_{0,0})} \leq \beta_1(h+k)^p$.

Let $(t,x) \in D_{0,0}$, by using Lemma 4.1, we obtain from (3) that

$$|e_{0,0}(t,x)| \leq \sum_{i+j=p} \frac{1}{i!j!}\left\|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}\right\| h^i k^j.$$

Hence, by Lemma 4.3, we have

$$|e_{0,0}(t,x)| \leq \underbrace{\frac{C(p)}{p!}}_{\beta_1}(h+k)^p.$$

Claim 2. There exists a constant β_2 independent of h and k such that $\|e_{n,m}\|_{L^\infty(D_{n,m})} \leq \beta_2(h+k)^{p-1}$ for all $n = 0, \dots, N-1$ and $m = 1, \dots, M-1$.

Let $(t,x) \in D_{n,m}$, we have from (5) that

$$|u(t,x) - \hat{u}_{n,m}(t,x)| \leq c_1 k \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|) + c_2 \int_{x_m}^{x} \|\partial_t u(t,s) - \partial_t \hat{u}_{n,m}(t,s)\| ds \\ + c_2 \int_{x_m}^{x} \|u(t,s) - \hat{u}_{n,m}(t,s)\| ds, \tag{6}$$

where c_1 and c_2 are positive numbers.

On the other hand, by differentiating (5) with respect to t, we have, for $x \neq x_m$,

$$\int_{x_m}^{x} (x-s)(\partial_t u(t,s) - \partial_t \hat{u}_{n,m}(t,s)) ds = \kappa(u(t,x) - \hat{u}_{n,m}(t,x)) \\ + \int_{x_m}^{x} ((x-s)a(t,s) - v(t))(u(t,s) - \hat{u}_{n,m}(t,s)) ds \\ - \sum_{\rho=0}^{m-1} \int_{x_\rho}^{x_{\rho+1}} (x-s)(\partial_t u(t,s) - \partial_t \bar{u}_{n,\rho}(t,s)) + v(t)(u(t,s) - \bar{u}_{n,\rho}(t,s)) ds.$$

Hence

$$c_3 \|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq \kappa \|u - \hat{u}_{n,m}\| + k c_4 \|u - \hat{u}_{n,m}\| + k c_5 \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|),$$

then

$$\|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq \frac{\kappa + k c_4}{c_3} \|u - \hat{u}_{n,m}\| + \frac{k c_5}{c_3} \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|). \tag{7}$$

If $x = x_m$: by differentiating (5) with respect to t, we obtain

$$\|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq \frac{k c_6}{\kappa} \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|), \tag{8}$$

using (7) and (8), we get

$$\|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq \Big(\frac{\kappa + k c_4}{c_3}\Big) \|u - \hat{u}_{n,m}\| + k c_7 \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|), \tag{9}$$

where c_3, c_4, c_5 and c_6 are positive numbers and $c_7 = k(\frac{c_5}{c_3} + \frac{c_6}{\kappa})$.

Then, from (6) and (9), we have

$$\|u - \hat{u}_{n,m}\| + \|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq k(c_1 + c_7) \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|) \\ + c_2 X \|\partial_t u - \partial_t \hat{u}_{n,m}\| + \Big(c_2 X + \frac{1}{c_3} + \frac{X c_4}{\kappa c_3}\Big) \|u - \hat{u}_{n,m}\|,$$

denote by $c_8 = \max\{c_2X, c_2X + \frac{1}{c_3} + \frac{Xc_4}{\kappa c_3}\}$, then

$$\begin{aligned}\|u - \hat{u}_{n,m}\| + \|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq k(c_1 + c_7) \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|) \\ + c_8(\|u - \hat{u}_{n,m}\| + \|\partial_t u - \partial_t \hat{u}_{n,m}\|).\end{aligned}$$

Hence

$$\|u - \hat{u}_{n,m}\| + \|\partial_t u - \partial_t \hat{u}_{n,m}\| \leq \underbrace{\frac{c_1 + c_7}{1 - c_8}}_{c_9} k \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|),$$

and by Lemma 4.3, we deduce that

$$\begin{aligned}\|\partial_t e_{n,m}\| + \|e_{n,m}\| &\leq \|\partial_t u - \partial_t \hat{u}_{n,m}\| + \|\partial_t \hat{u}_{n,m} - \partial_t \bar{u}\| + \|u - \hat{u}_{n,m}\| + \|\hat{u}_{n,m} - \bar{u}\| \\ &\leq c_9 k \sum_{\rho=0}^{m-1} (\|\partial_t e_{n,\rho}\| + \|e_{n,\rho}\|) + \frac{C(p)}{(p-1)!} \left(\frac{T + X}{p} + 1 \right) (h + k)^{p-1}.\end{aligned}$$

Using Lemma 4.2, we obtain

$$\|\partial_t e_{n,m}\| + \|e_{n,m}\| \leq \underbrace{\frac{C(p)}{(p-1)!} \left(\frac{T + X}{p} + 1 \right) \exp(c_9 X)}_{\beta_2} (h + k)^{p-1}.$$

This implies that for all $n = 0, ..., N-1$ and $m = 0, ..., M-1$,

$$\|e_{n,m}\| \leq \beta (h + k)^{p-1},$$

where $\beta = \max\{\beta_1, \beta_2\}$.

Thus, the proof of Theorem 4.1 is completed. □

5 Numerical Example

In this section, we will examine the practical application of the method discussed in detail in the previous sections to solve a transport-diffusion system. The calculations were performed using Maple 17 running under Windows 7 on a computer equipped with an Intel Core i7-2630QM CPU @2.00 GHz and 8,00 Go of RAM.

Example 5.1 Consider the linear two-dimensional Volterra integro-differential system of two equations

$$\begin{cases} u_1(t,x) & = f_1(t,x) + \frac{1}{3} \int_0^x (x - s) \left(\dfrac{\partial u_1(t,s)}{\partial t} + 3t^2 s u_1(t,s) - \ln(2) t^2 u_2(t,s) \right) + \\ & +e^{-t^2} u_1(t,s) ds \\ u_2(t,x) & = f_2(t,x) + \frac{1}{3} \int_0^x (x - s) \left(\dfrac{\partial u_2(t,s)}{\partial t} - t^3 e^{-t} u_1(t,s) - (t + s) u_2(t,s) \right) + \\ & +t^3 u_2(t,s) ds \end{cases}$$

for $t, x \in [0,1]$, where $f_1(t,x)$ and $f_2(t,x)$ are chosen so that the exact solution is $(u_1(t,x), u_2(t,x)) = (t^4 \sin(-x^2), -t^3 \cos(x))$. We can easily show that this system of equations is equivalent to the following transport-diffusion system:

$$\begin{cases} \frac{\partial u_1(t,x)}{\partial t} + e^{-t^2} \frac{\partial u_1(t,x)}{\partial x} - 3\frac{\partial^2 u_1(t,x)}{\partial x^2} & = -3t^2 x u_1(t,x) + \ln(2) t^2 u_2(t,x) - 3\frac{\partial^2 f_1(t,x)}{\partial x^2} \\ \frac{\partial u_2(t,x)}{\partial t} + t^3 \frac{\partial u_2(t,x)}{\partial x} - 3\frac{\partial^2 u_2(t,x)}{\partial x^2} & = t^3 e^{-t} u_1(t,x) + (t+x) u_2(t,x) - 3\frac{\partial^2 f_2(t,x)}{\partial x^2} \end{cases}$$

with the initial conditions $u_1(0,x) = u_1(t,0) = \frac{\partial u_1(t,0)}{\partial x} = 0$, $u_2(0,x) = \frac{\partial u_2(t,0)}{\partial x} = 0$, and $u_2(t,0) = -t^3$. The numerical results for $p = 3$ and $(N, M) = (5,5), (10,10)$ of the Taylor collocation method are shown in Table 1.

	$N = M = 5$		$N = M = 10$	
(t, x)	$e1$	$e2$	$e1$	$e2$
$(0, 0)$	0	0	0	0
$(0.1, 0.1)$	$9.99e-07$	$9.95e-04$	$1.13e-08$	$3.67e-08$
$(0.2, 0.2)$	$1.48e-06$	$1.96e-06$	$5.84e-07$	$7.80e-07$
$(0.3, 0.3)$	$2.39e-04$	$5.54e-04$	$5.30e-06$	$3.70e-06$
$(0.4, 0.4)$	$7.98e-05$	$3.15e-05$	$2.53e-05$	$9.52e-06$
$(0.5, 0.5)$	$1.64e-03$	$4.22e-04$	$8.48e-05$	$1.39e-05$
$(0.6, 0.6)$	$7.31e-04$	$3.09e-05$	$2.26e-04$	$7.26e-06$
$(0.7, 0.7)$	$4.90e-03$	$6.83e-04$	$5.08e-04$	$1.45e-04$
$(0.8, 0.8)$	$3.29e-03$	$1.23e-03$	$9.87e-04$	$6.94e-04$
$(0.9, 0.9)$	$8.14e-03$	$4.99e-03$	$1.66e-03$	$2.49e-03$
CPU time/sec	20.43 s		654.18s	

Table 1: Comparison of the absolute errors in Example 5.1.

6 Conclusion

This work presents a method for solving the transport-diffusion equation using the Taylor collocation method. This approach effectively approximates solutions to Volterra integro-differential equations, and a thorough analysis of the proposed method's convergence properties has been conducted.

The numerical experiments demonstrate the efficacy of the Taylor collocation method in solving the transport-diffusion system, even in the presence of reaction terms. These results confirm the method's utility in this context and highlight its potential for applications in other fields involving the spread of pollutants in a moving fluid or thermal diffusion in engineered systems in modeling dynamic processes, transport, and diffusion processes, such as environmental and atmospheric modeling.

Future research will aim to extend this approach to more complex systems and higher-dimensional domains. Additionally, the effect of various boundary conditions on the convergence and accuracy of the numerical solutions will be investigated.

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