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Study of a Delay Viscoelastic Problem Involving a Generalized Fractional Proportional Derivative

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Abstract: In this paper, we study a nonlinear fractional viscoelastic problem with multiple delays. We consider the fractional model of Voigt in terms of generalized fractional proportional derivative. Using the Banach contraction principle, we prove the existence and uniqueness of the solution under some assumptions, then we confirm the dependence of the latter upon the initial data. The Hyers-Ulam-Rassias stability is established and the results are illustrated by a numerical example.

Keywords: *fractional calculus; Ulam stability; Banach contraction; fractional proportional derivative; rheological models.*

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Introduction

In recent years, fractional calculus has proven to be a powerful and versatile tool for modeling many physical and mechanical phenomena, despite the existence of many definitions and formulas for fractional derivatives, see [1–5]. The fractional derivative is an integral operator that has a memory term in the kernel, which is its advantage in modeling rheological phenomena. It can accurately describe the behavior of viscoelastic materials and well define the stress-strain relationships.

When we apply stress to an elastic material, the stress causes deformation, and the material will instantly return to its initial shape when this stress is removed. We say that the elastic deformation here is instantaneous and recoverable and the work is stored in the form of elastic energy.

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Hooke's law describes the behavior of a purely elastic ideal solid as follows:

$$\sigma = E\gamma, \tag{1}$$

where σ is the stress applied, γ is the strain and E is the elastic modulus.

By definition, the viscosity is the measure of fluid's resistance to flow. It also reflects the rate of dissipation of the material's strain energy in the flow. Once there is a dissipation of energy and the stress stops, the deformation becomes permanent and that behavior can be expressed by the following equation:

$$\sigma = \mu \frac{d\gamma}{dt}, \tag{2}$$

where the stress is proportional to the strain velocity and μ is the coefficient of viscosity.

Elastic materials that have the ability to dissipate mechanical energy due to viscous effects are called viscoelastic materials. There are many models that can describe viscoelastic phenomena, for example the Maxwell and Kelvin-Voigt models or combinations of them. The Kelvin-Voigt model is a purely viscous damper and a purely elastic spring connected in parallel, modelled by the following constitutive equation:

$$\mu \frac{d\gamma}{dt} + E\gamma = \sigma. \tag{3}$$

The above equation was studied by A. Chidouh et al. in [6]. They considered the generalized nonlinear equation (3) involving fractional derivative, where the existence results are established by means of the Guo-Krassnosl'ski theorem. In fact, there are many works and papers that focused on studying rheological models that make use of the fractional derivative due to the important role of the latter in studying viscoelastic phenomena. F. Mainardi and G. Spada in [7] have provided a general survey of the fractional viscoelastic models. Their analysis covered the creep, relaxation and viscosity properties of basic fractional models. The papers dealing with rheological models have usually relied on classical fractional derivatives such as the Rieman-Lioville, Caputo and Caputo-Fabrizio ones, see [1–3, 7, 8] and the references therein.

Recently, many definitions of the fractional derivative have appeared, similar to those of the conformable fractional derivative, and many have succeeded in generalizing the fractional derivative in a way that preserves its properties, see [9, 10] and the references therein. In 2017, F. Jarad et al. [11] generalized the fractional proportional integral and derivative, both containing the exponential function in their kernels. Those fractional operators are well-behaved and have several advantages over the classical derivatives, such as being an accurate generalization of the existing fractional derivatives and integrals of Caputo and Riemann-Liouville.

We point out that in rheological models, delay terms appear naturally in certain equations because the response of a viscoelastic material is not instantaneous, and here we consider materials that are not at rest for t belonging to $[-r, 0]$. We also refer to [12], one of the works that inspired us to study fractional differential equations involving delays, where the authors established the existence and stability of solutions for the following nonlinear fractional problem:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = \sum\limits_{j=1}^{n} a_j(t) f(t, x(t), x(t-\tau_j)); \quad t \in (0, T], 0 < \alpha < 1, \\ I_{0^+}^{1-\alpha} x(t)\big|_{t=0^+} = 0; \quad x(t) = \phi(t), t < 0 \text{ and } \lim\limits_{t \to 0} \phi(t) = 0. \end{cases}$$

Motivated by the above papers and also dealing with the nonlinear Kelvin-Voigt model containing the new generalized proportional Caputo derivative defined in [11], we consider the following problem:

$$ {}^{PC}D_{0^+}^{\alpha;\rho}\gamma(t)+\omega\gamma(t)=\sum_{j=1}^{n}a_j(t)\sigma_j(\gamma(t-\tau_j)),\ (t>0,\omega>0), \tag{4}$$

and

$$\gamma(t)=\varsigma(t),\ t\in[-r,0], \tag{5}$$

where ${}^{PC}D_{0^+}^{\alpha;\rho}$ denotes the generalized proportional Caputo derivative of order $0<\alpha<1$ and $0<\rho\leq 1$, taking into account that $0<r=\max_{j=1,\ldots,n}\{\tau_j\}\leq T$ and $\gamma(0)=\varsigma(0)=0$.

The content of this paper is arranged as follows. In the first section, we will give some basic definitions and some lemmas that will serve as an analysis tool in the rest of this paper. In Section 2, we rely mainly on two results of [11, Theorem 3.8 and Theorem 5.3] and transform our fractional problem into an equivalent integral equation in a very particular Banach space so that it allows us, under some assumptions, to study the existence and uniqueness of the solution to the above equations. The dependence of the solution on the initial data will also be discussed. In Section 3, we will use the approach of Ulam-Hyers-Rassias to study the stability. Finally, our results will be illustrated by a numerical example.

1 Method and Preliminaries

Here, we employ an effective method based on fixed-point theorems, along with some functional analysis tools. The selection of appropriate functional spaces and sets is crucial to facilitating the task. To achieve this, we first define a Banach space and a suitable subset of it, which forms the basis of this method. Then, we transform our fractional problem into an equivalent integral equation. Finally, from this derived equation, we construct a continuous operator from a compact subset to itself, and via the fixed-point theorems, prove the existence and uniqueness of the solution. All of this will be accomplished with the aid of some functional analysis tools, requiring the following concepts and definitions. We will begin with an important definition of a new fractional derivative which is called the generalized proportional fractional derivative, see [11].

Definition 1.1 For $\rho\in[0,1]$, let the functions $k_0,k_1:[0,1]\times[0,T]\to[0,\infty)$ be continuous such that for all $t\in[0,T]$, we have

$$\lim_{\rho\to 0^+}k_1(\rho,t)=1,\quad \lim_{\rho\to 1^-}k_1(\rho,t)=0,\quad \text{and } k_1(\rho,t)\neq 0,\rho\in[0,1),$$
$$\lim_{\rho\to 0^+}k_0(\rho,t)=0,\quad \lim_{\rho\to 1^-}k_1(\rho,t)=1,\quad \text{and } k_1(\rho,t)\neq 0,\rho\in[0,1).$$

Then the adjusted conformable derivative operator of order ρ is defined by

$$D^{\rho}u(t)=k_1(\rho,t)u(t)+k_0(\rho,t)u'(t). \tag{6}$$

The derivative mentioned in (6) is called a proportional derivative.

Here, consider $k_1(\rho,t)=1-\rho$ and $k_0(\rho,t)=\rho$. The equality (6) takes the form

$$D^{\rho}u(t)=(1-\rho)u(t)+\rho u'(t), \tag{7}$$

which will be the focus of our study regarding the proportional derivative in the rest of this paper. For more details, see [1, 8] and the references therein.

Definition 1.2 Let $\rho \in (0,1]$ and $\alpha > 0$. The generalized proportional fractional integral of the function u is defined by

$$(I_{0^+}^{\alpha,\rho}u)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}u(s)ds,\ t \in [0,T]. \tag{8}$$

Definition 1.3 Let $\rho, \alpha \in (0,1]$. The generalized proprotional Caputo derivative of the function u is defined by

$$({}^{PC}D_{0^+}^{\alpha,\rho})u(t) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{-\alpha}(D^\rho u)(s)ds,\ t \in [0,T].$$

Note that, if $\rho = 1$, we get

$$({}^{PC}D^{\alpha,1}u)(t) = ({}^{C}D^\alpha u)(t),$$

where ${}^{C}D^\alpha$ denotes the Caputo derivative of order α, which means that the latter is a particular case of the generalized proportional Caputo derivative.

Now, let us give important lemmas which play a key role in this work.

Lemma 1.1 *[11] If* $\rho \in (0,1], \beta > 0$ *and* $\alpha \in (0,1]$*. Then, for* u *being a real continuous function on* $[0,T]$*, we have the following statements:*

1.

$$I^{\alpha,\rho}(I_{0^+}^{\beta,\rho}u)(t) = I^{\beta,\rho}(I_{0^+}^{\alpha,\rho}u)(t) = (I_{0^+}^{\alpha+\beta,\rho}u)(t); \tag{9}$$

2.

$$I_{0^+}^{\alpha,\rho}\ \left({}^{PC}D_{0^+}^{\alpha,\rho}u\right)(t) = u(t) - u(0)\exp\left(\frac{\rho-1}{\rho}t\right). \tag{10}$$

Lemma 1.2 *[13] Let* $0 < \alpha < 1$ *and* $\beta \geq \alpha$*, then we have*

$$E_{\alpha,\beta}(-z) \leq \frac{1}{\Gamma(\beta)}; t > 0, \tag{11}$$

where $E_{\alpha,\beta}\ (z)$ *is the Mittag-Leffler function defined as follows:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}; (z, \alpha, \beta \in C; \operatorname{Re}\alpha > 0). \tag{12}$$

2 Existence Results

First of all, we consider the constitutive equation as a linear generalized proportional fractional problem involving the Voigt model

$$({}^{PC}D_{0^+}^{\alpha,\rho}\gamma)(t) + \omega\gamma(t) = \sigma(t), \gamma(0) = 0; (t > 0, \omega > 0). \tag{13}$$

Mechanical phenomena are usually described by differential equations, while we can also describe the phenomena by integral equations which are very suitable for theoretical work such as studying the stability of solutions or other tasks, where it is sufficient for these integral equations to be equivalent to the differential ones.

Now, we transform our equations (13) into an equivalent integral equation, where we work on highlighting the space to which the solution belongs, so that we ensure the equivalence between our two equations, which is what was not addressed in [11] as the author used the Laplace transform.

Theorem 2.1 *Let* $0 < \alpha < 1, \omega > 0$ *and* $\sigma \in C\left([0;T],\mathbb{R}\right)$. *Then the initial value problem (13) is equivalent to the following integral equation:*

$$\gamma(t) = \frac{1}{\rho^{\alpha}} \int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\sigma(s)ds \tag{14}$$

in the Banach space of continuous functions.

Proof. Apply the generalized proportional fractional integral to equations (13). Then, using the first statement (9) of Lemma (1.1), we get

$$\begin{aligned}\gamma(t) &= (I_{0^+}^{\alpha;\rho}\sigma)(t) - \omega(I_{0^+}^{\alpha;\rho}\gamma)(t) + \gamma(0)\exp\left(\frac{\rho-1}{\rho}t\right)\\ &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}\sigma(s)ds\\ &- \frac{\omega}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}\gamma(s)ds.\end{aligned}$$

We will use the successive approximations method to derive the solution, taking into account

$$\gamma_0(t) = (I_{0^+}^{\alpha;\rho}\sigma)(t) = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}\sigma(s)ds \tag{15}$$

and

$$\gamma_m(t) = \gamma_0(t) - \omega(I_{0^+}^{\alpha;\rho}\gamma_{m-1})(t). \tag{16}$$

By (10), the second statement of Lemma 1.1, we get

$$\begin{aligned}\gamma_1(t) &= \gamma_0(t) - \omega(I_{0^+}^{\alpha;\rho}\gamma_0)(t)\\ &= (I_{0^+}^{\alpha;\rho}\sigma)(t) - \omega(I_{0^+}^{2\alpha;\rho}\sigma)(t)\end{aligned}$$

and

$$\begin{aligned}\gamma_2(t) &= \gamma_0(t) - \omega(I_{0^+}^{\alpha;\rho}\gamma_1)(t)\\ &= (I_{0^+}^{\alpha;\rho}\sigma)(t) - \omega(I_{0^+}^{2\alpha;\rho}\sigma)(t) + \omega^2(I_{0^+}^{3\alpha;\rho}\sigma)(t).\end{aligned}$$

Continuing this process, we obtain

$$\gamma_m(t) = \frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}\sum_{k=0}^{m}\frac{(t-s)^{k\alpha}}{\rho^{k\alpha}\Gamma(k\alpha+\alpha)}(-\omega)^k\sigma(s)ds.$$

Taking the limit as $m \to \infty$, denote $\lim\limits_{m\to\infty} \gamma_m(t) = \gamma(t), t \in [0,T]$ (γ is the unique solution which belongs to the Banach space of real continous functions) and consider (12), we obtain

$$\gamma(t) = \frac{1}{\rho^{\alpha}} \int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\sigma(s)ds. \tag{17}$$

The proof is complete.

Now, we turn our attention to the nonlinear problem (13), where

$$\sigma(t) = \sum_{j=1}^{n} a_j(t)\sigma_j(\gamma(t-\tau_j))$$

and

$$\gamma(t) = \varsigma(t),\ t \in [-r,0],$$

taking into account that $0 < r = \max_{j=1,\dots,n}\{\tau_j\} \le T$ and $\gamma(0) = \varsigma(0) = 0$.

Let us introduce the following hypotheses.

(C1) $a_j : [0,T] \to \mathbb{R}$ are continuous functions with $A_j = \sup\limits_{t\in[0,T]} |a_j(t)|$.

(C2) $\sigma_j : C[-v,T] \to C[-v,T]$ are Lipschitz functions, i.e., there exists $B_j > 0$ such that

$$\|\sigma_j(\gamma_1) - \sigma_j(\gamma_2)\|_{C[-v,T]} \le B_j \|\gamma_1 - \gamma_2\|_{C[-v,T]};\gamma_1,\gamma_2 \in C[-v,T]. \tag{18}$$

We suppose that $\Sigma = \sum\limits_{j=1}^{n} A_j B_j$ and $\sigma_j(0) \neq 0$.

Theorem 2.2 *Assume that (C1) and (C2) are satisfied. If*

$$\frac{T^{\alpha}\Sigma}{\rho^{\alpha}\Gamma(\alpha+1)} \le 1, \tag{19}$$

then problem (4) and (5) has a unique solution $\gamma \in (C[-r,T],\mathbb{R})$.

Proof. Let us define the following space:

$$X = \left\{\gamma \in (C[-v,T],\mathbb{R}) : \gamma\big|_{[-r,0]} = \varsigma\right\},$$

which is a Banach space endowed with the sup-norm

$$\|\gamma\|_X = \sup_{t\in[-r,T]} |\gamma(t)|.$$

We consider the operator $P : X \to X$ defined by

$$(P\gamma)(t) = \begin{cases} \varsigma(t) & \text{for } t \in [-r,0], \\ \frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right) & \\ \qquad \times\sum\limits_{j=1}^{n} a_j(s)\sigma_j(\gamma(s-\tau_j))ds \text{ for } t \in [0,T]. & \end{cases}$$

From Theorem 2.1, finding a solution of (4) and (5) in $(C[-r,T],\mathbb{R})$ is equivalent to finding a fixed point of the operator P.

For any $\gamma_1,\gamma_2 \in (C[-v,T],\mathbb{R})$ and each $t\in[-r,T]$, we have

$$|(P\gamma_1)(t)-(P\gamma_2)(t)| \leq \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right) \times\left(\sum_{j=1}^n |a_j(s)|\,B_j\,|\gamma_1(s-\tau_j)-\gamma_2(s-\tau_j)|\right)ds.$$

Putting $z=(s-\tau_j)$, we get

$$\begin{aligned}|(P\gamma_1)(t)-(P\gamma_2)(t)| \leq{}& \sum_{j=1}^n\frac{1}{\rho^\alpha}\int_{-\tau_j}^0 \exp\left(\frac{\rho-1}{\rho}(t-z-\tau_j)\right)(t-z-\tau_j)^{\alpha-1}\\ &\times E_{\alpha,\alpha}\left(-\omega\left(\frac{t-z-\tau_j}{\rho}\right)^\alpha\right)|a_j(s)|\,B_j\,|\gamma_1(z)-\gamma_2(z)|\,dz\\ &+\sum_{j=1}^n\frac{1}{\rho^\alpha}\int_0^{t-\tau_j} \exp\left(\frac{\rho-1}{\rho}(t-z-\tau_j)\right)(t-z-\tau_j)^{\alpha-1}\\ &\times E_{\alpha,\alpha}\left(-\omega\left(\frac{t-z-\tau_j}{\rho}\right)^\alpha\right)|a_j(s)|\,B_j\,|\gamma_1(z)-\gamma_2(z)|\,dz.\end{aligned}$$

We have $\gamma_1(t)=\gamma_2(t)=\varsigma(t)$ for $t\in[-r,0]$. Taking into account

$$\exp\left(\frac{\rho-1}{\rho}(t-z-\tau_j)\right)\leq 1,\ \text{ for } \rho\in(0,1], \tag{20}$$

and (11), we get

$$|(P\gamma_1)(t)-(P\gamma_2)(t)| \leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\sum_{j=1}^n\int_0^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}|a_j(s)|\,B_j\,|\gamma_1(z)-\gamma_2(z)|\,dz.$$

Therefore,

$$\|(P\gamma_1)(t)-(P\gamma_2)(t)\|_X \leq \left(\frac{T^\alpha\Sigma}{\rho^\alpha\Gamma(\alpha+1)}\right)\|\gamma_1-\gamma_2\|_X\,.$$

In view of (19) and Banach's fixed point, we deduce that P is a contraction. Hence, the problem (4)-(5) has a unique solution on $(C[-r,T],\mathbb{R})$. This completes the proof.

Corollary 2.1 *Under the conditions (18), the unique solution of (4) and (5) depends continuously on function* $\varsigma(t)$.

Proof. Let γ_1 and γ_2 be solutions of problem (4)-(5), corresponding to the functions ς_1 and ς_2, respectively. Then we have

$$|\gamma_1(t)-\gamma_2(t)| = \left|\frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right.$$

$$\times\sum_{j=1}^{n} a_j(s)\left|\sigma_j(\gamma_1(s-\tau_j))-\sigma_j(\gamma_2(s-\tau_j))\right|\Bigg|.$$

Putting $z=(s-\tau_j)$ and taking into account (20) and (11), we get

$$\begin{aligned}
|\gamma_1(t)-\gamma_2(t)| &\le \frac{\Sigma}{\rho^\alpha\Gamma(\alpha)}\int_{-\tau_j}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}|\gamma_1(z)-\gamma_2(z)|\,dz\\
&\le \frac{\Sigma}{\rho^\alpha\Gamma(\alpha)}\int_{-\tau_j}^{0}(t-z-\tau_j)^{\alpha-1}|\gamma_1(z)-\gamma_2(z)|\,dz\\
&+\frac{\Sigma}{\rho^\alpha\Gamma(\alpha)}\int_{0}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}|\gamma_1(z)-\gamma_2(z)|\,dz\\
&\le \frac{\Sigma}{\rho^\alpha\Gamma(\alpha)}\sup_{z\in[-v,0]}|\gamma_1(z)-\gamma_2(z)|\int_{-\tau_j}^{0}(t-z-\tau_j)^{\alpha-1}dz\\
&+\frac{\Sigma}{\rho^\alpha\Gamma(\alpha)}\sup_{z\in[0,T]}|\gamma_1(z)-\gamma_2(z)|\int_{0}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}dz\\
&\le \frac{(t^\alpha-(t-\tau_j)^\alpha)\,\Sigma}{\rho^\alpha\Gamma(\alpha+1)}\|\varsigma_1-\varsigma_2\|_{C[-v,0]}\\
&+\frac{(t-\tau_j)^\alpha\Sigma}{\rho^\alpha\Gamma(\alpha+1)}\|\gamma_1-\gamma_2\|_{C[0,T]}.
\end{aligned}$$

Consequently, we have

$$\|\gamma_1-\gamma_2\|_{C[0,T]}\le\frac{T^\alpha\Sigma}{\rho^\alpha\Gamma(\alpha+1)}\|\gamma_1-\gamma_2\|_{C[0,T]}+\frac{T^\alpha\Sigma}{\rho^\alpha\Gamma(\alpha+1)}\|\varsigma_1-\varsigma_2\|_{C[-r,0]}.$$

Then, in view of (9), we get

$$\|\gamma_1-\gamma_2\|_{C[0,T]}\le\frac{T^\alpha\Sigma}{\rho^\alpha\Gamma(\alpha+1)-T^\alpha\Sigma}\|\varsigma_1-\varsigma_2\|_{C[-r,0]}.$$

This implies the dependence of $\gamma(t)$ on the initial data $\varsigma(t)$ and completes the proof.

3 Stability Results

Definition 3.1 The problem (4)-(5) has the Ulam-Hyers stability if for each $\varepsilon>0$ and each $y\in C([-v,T],\mathbb{R})$, there exists a solution of the following problem:

$$\begin{cases}\left|({}^{PC}D_{0^+}^{\alpha,\rho}y)(t)+\omega y(t)-\sum_{j=1}^{n}a_j(t)\sigma_j((t-\tau_j))\right|\le\varepsilon, t\in[0,T],\\ \quad y(t)=\varsigma(t), t\in[-r,0],\ \varsigma(0)=0.\end{cases}\tag{21}$$

There exist a solution $\gamma\in C([-v,T],\mathbb{R})$ of (4)-(5) and a real number $k>0$ such that

$$|y(t)-\gamma(t)|\le k\varepsilon.$$

Remark 3.1 A function $y \in C([-v,T],\mathbb{R})$ is a solution of the problem (21) if and only if there exists a function $h \in C([0,T],\mathbb{R})$ such that for every $t \in [0,T]$, we have $|h(t)| \leq \varepsilon$ and

$$\begin{cases} ({}^{PC}D_{0^+}^{\alpha;\rho}y)(t) + \omega y(t) = \displaystyle\sum_{j=1}^{n} a_j(t)\sigma_j(y(t-\tau_j)) + h(t), t \in [0,T], \\ y(t) = \varsigma(t), t \in [-r,0],\ \varsigma(0) = 0. \end{cases} \tag{22}$$

Lemma 3.1 *If* $y \in C([-v,T],\mathbb{R})$ *is a solution of the problem (21), then for* $t \in [0,T]$, y *satisfies the relation*

$$\begin{aligned} &\left| y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right. \\ &\left.\times\sum_{j=1}^{n} a_j(t)\sigma_j(y(s-\tau_j))ds\right| \\ &\leq \frac{T^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\varepsilon. \end{aligned} \tag{23}$$

Proof. For $y(t) = \varsigma(t),\ t \in [-r,0]$, the solution of (22) satisfies

$$\begin{aligned} y(t) = {}& \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right) \\ &\times\left(\sum_{j=1}^{n} a_j(t)\sigma_j(y(s-\tau_j)) + h(t)\right)ds. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left| y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right. \\ &\left.\times\sum_{j=1}^{n} a_j(t)\sigma_j(y(s-\tau_j))ds\right| \\ &\leq \left|\frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)h(t)ds\right| \\ &\leq \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)|h(t)|\,ds \\ &\leq \left(\frac{T^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right)\varepsilon. \end{aligned}$$

The proof is complete.

Theorem 3.1 *If the problem (4)-(5) has a unique solution, then it is Ulam-Hyers stable.*

Proof. From Theorem 2.1, the problem (4)-(5) has a unique solution γ in $C([-v,T],\mathbb{R})$. Let y be a solution of (21), then we obtain, for each $t \in [-r,T]$,

$$|y(t)-\gamma(t)| = \left| y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right.$$

$$\begin{aligned}
&\times\sum_{j=1}^{n} a_j(s)\sigma_j(\gamma(s-\tau_j))ds\Bigg|\\
&\le \left|y(t)-\frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\right.\\
&\left.\times\sum_{j=1}^{n} a_j(s)\sigma_j(y(s-\tau_j))ds\right|\\
&+\frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\\
&\times\sum_{j=1}^{n}\left|a_j(s)\right|\left|\sigma_j(y(s-\tau_j))-\sigma_j(\gamma(s-\tau_j))\right|ds.
\end{aligned}$$

By Lemma 3.1, we get

$$|y(t)-\gamma(t)|\le \frac{T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\varepsilon+\frac{\sum_{j=1}^{n} A_jB_j}{\rho^{\alpha}\Gamma(\alpha)}\int_{-\tau_j}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}\left|y(z)-\gamma(z)\right|dz$$

As $y(z)=x(z)=\varsigma(z)$ for $z\in[-\tau_j,0]$, we get

$$\begin{aligned}
|y(t)-x(t)| &\le \frac{T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\varepsilon+\frac{\Sigma}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}\left|y(z)-\gamma(z)\right|dz\\
&\le \frac{T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\varepsilon+\frac{T^{\alpha}\Sigma}{\rho^{\alpha}\Gamma(\alpha+1)}\left\|y-\gamma\right\|_{C[0,T]}.
\end{aligned}$$

Hence,

$$\left\|y-\gamma\right\|_{C[0,T]}\le \frac{T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)-T^{\alpha}\Sigma}\varepsilon$$

and (4)-(5) is Ulam-Hyers stable with the constant $k=\frac{T^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)-T^{\alpha}\Sigma}$.

Let us now give a generalized results on stability that is known as the Ulam-Hyers-Rassias stability.

Definition 3.2 The equation (4)-(5) has the Ulam-Hyers-Rassias stability with respect to $\phi(t)$ if there exists a real number $C>0$ such that for each $\varepsilon>0$ and for each $y\in C([-v,T],\mathbb{R})$, there is a solution of the following problem:

$$\begin{cases}\left|({}^{PC}D_{0^+}^{\alpha,\rho}y)(t)+\omega y(t)-\sum_{j=1}^{n} a_j(t)\sigma_j(y(t-\tau_j))\right|\le\varepsilon\phi(t),t\in[0,T],\\ y(t)=\varsigma(t),t\in[-r,0],\ \varsigma(0)=0.\end{cases}\tag{24}$$

There exists a solution $\gamma\in C([-v,T],\mathbb{R})$ of (4)-(5) such that

$$|y(t)-\gamma(t)|\le C\phi(t)\varepsilon.$$

Before we give the following lemma, we assume that for a continous function $\phi(t)$, there is a positive constant θ such that

$$\int_0^t (t-s)^{\alpha-1}\phi(s)ds \leq \theta\phi(t).$$

Lemma 3.2 *If $y \in C([-v,T],\mathbb{R})$ is a solution of the problem (24), then for $t \in [0,T]$, y satisfies the relation*

$$\begin{aligned}&\left|y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right.\\ &\left.\times\sum_{j=1}^n a_j(t)\sigma_j(y(s-\tau_j))ds\right|\\ &\leq \frac{\theta\varepsilon}{\rho^\alpha\Gamma(\alpha)}\phi(t).\end{aligned} \tag{25}$$

Proof. For $y(t) = \varsigma(t),\ t \in [-r,0]$, the solution of (24) satisfies

$$\begin{aligned}y(t) = {}& \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\\ &\times\left(\sum_{j=1}^n a_j(t)\sigma_j(y(s-\tau_j)) + h(t)\right)ds,\end{aligned}$$

where $|h(t)| \leq \varepsilon\phi(t)$. Then

$$\begin{aligned}&\left|y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right.\\ &\left.\times\sum_{j=1}^n a_j(t)\sigma_j(y(s-\tau_j))ds\right|\\ &\leq \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)|h(t)|\,ds\\ &\leq \frac{\varepsilon}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\phi(s)ds\\ &\leq \frac{\varepsilon\theta}{\rho^\alpha\Gamma(\alpha)}\phi(s).\end{aligned}$$

The proof is complete.

Theorem 3.2 *If problem (4)-(5) has a unique solution, then it is Ulam-Hyers-Rassias stable.*

Proof. From Theorem 2.1, the problem (4)-(5) has a unique solution γ in $C([-v,T],\mathbb{R})$. Let y be a solution of (24), then we obtain, for each $t \in [-v,T]$, that

$$|y(t)-\gamma(t)| = \left|y(t) - \frac{1}{\rho^\alpha}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^\alpha\right)\right.$$

$$
\begin{aligned}
&\times\sum_{j=1}^{n} a_j(s)\sigma_j(\gamma(s-\tau_j))ds\Bigg| \\
&\leq \left|y(t)-\frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right)\right. \\
&\left.\times\sum_{j=1}^{n} a_j(s)\sigma_j(y(s-\tau_j))ds\right| \\
&+\frac{1}{\rho^{\alpha}}\int_0^t \exp\left(\frac{\rho-1}{\rho}(t-s)\right)(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\omega\left(\frac{t-s}{\rho}\right)^{\alpha}\right) \\
&\times\sum_{j=1}^{n}\left|a_j(s)\right|\left|\sigma_j(y(s-\tau_j))-\sigma_j(\gamma(s-\tau_j))\right|ds.
\end{aligned}
$$

By Lemma 3.2, we get

$$
|y(t)-\gamma(t)| \leq \frac{\theta\varepsilon}{\rho^{\alpha}\Gamma(\alpha)}\phi(t)+\frac{\sum_{j=1}^{n}B_j l_j}{\rho^{\alpha}\Gamma(\alpha)}\int_{-\tau_j}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}\left|y(z)-\gamma(z)\right|dz.
$$

As $y(z)=\gamma(z)=\varsigma(z)$ for $z\in[-\tau_j,0]$, we get

$$
\begin{aligned}
|y(t)-\gamma(t)| &\leq \frac{\theta\varepsilon}{\rho^{\alpha}\Gamma(\alpha)}\phi(t)+\frac{\sum_{j=1}^{n}A_j B_j}{\rho^{\alpha}\Gamma(\alpha)}\int_{0}^{t-\tau_j}(t-z-\tau_j)^{\alpha-1}\left|y(z)-\gamma(z)\right|dz \\
&\leq \frac{\theta\varepsilon}{\rho^{\alpha}\Gamma(\alpha)}\phi(t)+\frac{T^{\alpha}\Sigma}{\rho^{\alpha}\Gamma(\alpha+1)}\left\|y-\gamma\right\|_{C[0,T]}.
\end{aligned}
$$

Hence,

$$
\left\|y-\gamma\right\|_{C[0,T]} \leq \frac{\alpha\theta}{\rho^{\alpha}\Gamma(\alpha+1)-T^{\alpha}\Sigma}\varepsilon\phi(t),
$$

and (4)-(5) is Ulam-Hyers-Rassias stabe with respect to $\phi\,(t)$, where

$$
C=\frac{\alpha\theta}{\rho^{\alpha}\Gamma(\alpha+1)-T^{\alpha}\Sigma}.
$$

Example

Consider the following delay fractional problem:

$$
\begin{cases}
{}^{PC}D_{0^+}^{\frac{1}{2},\frac{1}{2}}\gamma(t)+\pi\gamma(t)=\sum\limits_{j=1}^{2}\frac{1}{7j}\cos(jt)\sin(j\gamma(t-j)+1),\ t\in[0,\pi], \\
\gamma(t)=t,\ t\in[-2,0],
\end{cases}
\tag{26}
$$

where ${}^{PC}D_{0^+}^{\frac{1}{2},\frac{1}{2}}$ denotes the generalized proportional Caputo derivative of order $\alpha=\frac{1}{2}$ and $\rho=\frac{1}{2}$.

Here, $\sigma_j(x) = \frac{\sin(jx+1)}{7j}$, $j = \overline{1,2}$. Take into account that in this example, we have two stress functions $\sigma_1(x) = \frac{1}{7}\sin(x+1)$ and $\sigma_2(x) = \frac{1}{14}\sin(2x+1)$ which are Lipschitz functions such that $\sigma_j(0) \neq 0$ and the Lipschitz constant $B_j = \frac{1}{7}$ for $j = \overline{1,2}$.

On the other hand, $a_j(t) = \cos(jt)$ such that $\max_{0\le t\le\pi} |a_j(t)| = \max_{0\le t\le\pi} |\cos(jt)| = 1$ for $j = \overline{1,2}$.

Then we obtain that $\Sigma = \sum_{j=1}^{2} A_j B_j = \frac{2}{7}$. So, the assumptions (C1) and (C2) of Theorem 2.2 are fulfilled as well as condition (19):

$$\frac{T^\alpha \Sigma}{\rho^\alpha \Gamma(\alpha+1)} = \frac{8}{7\sqrt{2\pi}} = 0.46 < 1.$$

Therefore, from Theorem 2.2, we conclude that our problem (26) has a unique solution in $C([-2,\pi],\mathbb{R})$, which makes it Ulam-Hyers stable as follows from the above analysis.

4 Discussion

In this work, we studied a fractional differential equation involving a generalized fractional proportional derivative. Our equation is a generalization of the classical viscoelastic Kelvin-Voigt model with the addition of a delay term. Using a fractional-order derivative in the model gives us a more accurate representation of memory than integer order, while the delay term reflects the viscoelastic behavior observed in certain mechanical systems. The existence and uniqueness of the solutions were established using the fixed point technique. This technique is very effective when appropriate assumptions are made regarding the nonlinear terms and delay functions, thus ensuring the validity of the proposed fractional delayed model. So, our technique is not only a tool for solving many types of equations, but it is also a powerful means that facilitates the study of stability in a way that has never been done before. The stability of the solutions was analyzed according to the Ulam concept. These stability properties confirm that approximate solutions remain close to precise solutions, highlighting the reliability of the proposed model. The results obtained appear to be very strong and generalize many previous findings. These results were further confirmed by a numerical example that explains the method used in detail.

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