Nonlinear Dynamics and Systems Theory, **26** (2) (2026) 184–194

A New Hybrid Conjugate Gradient Method Based on RMIL, LS and CD Methods

Samia Khelladi * and Youcef Elhamam Hemici

Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Setif 1 University Ferhat Abbas, 19000, Algeria.

Received: March 17, 2025; Revised: March 30, 2026

Abstract: We propose a new hybrid conjugate gradient method for unconstrained optimization problems. The method combines the RMIL, LS and CD formulas through a convex combination to build a more effective search direction. This approach aims to improve both convergence and numerical performance. The hybrid parameter β_k is updated to satisfy the conjugacy condition and global convergence is proven under the strong Wolfe line search. Numerical results on test problems show that the new method is better than other existing approaches in terms of iterations and CPU time.

Keywords: *unconstrained optimization; hybrid conjugate gradient method; descent direction; line search; global convergence.*

Mathematics Subject Classification (2020): 65K05, 90C26, 90C30, 93-10.

1 Introduction

Unconstrained optimization problems appear in many areas such as machine learning, data fitting, finance and control systems. These problems are generally written as

$$\begin{cases} \min \ f(x) \\ x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

When the dimension n is large, solving this problem efficiently becomes important. One of the most effective and memory-saving methods used is the nonlinear conjugate gradient (CG) method. These methods are known for their low storage requirements and

* Corresponding author: mailto:samia.boukaroura@univ-setif.dz

are widely used in practice due to their simplicity and strong performance. Moreover, the nonlinear CG method plays a crucial role in nonlinear dynamics [9, 11], where many problems such as trajectory optimization, equilibrium analysis and model calibration can be reformulated as large-scale unconstrained optimization problems. In such contexts, CG methods provide a computationally efficient framework to handle the high dimensions and complex structure typical of dynamical systems, without the need to compute or store second-order derivatives.

Starting from an initial point x_0, the conjugate gradient method updates the solution using the formula

$$x_{k+1} = x_k + \alpha_k d_k, \tag{2}$$

where $\alpha_k > 0$ is the step size, often found using a line search, and d_k is the search direction computed by

$$d_k = \begin{cases} -g_0, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k \geq 1. \end{cases} \tag{3}$$

Here, $g_k = \nabla f(x_k)$ is the gradient of the objective function at the k-th step, and β_k is a scalar used to generate the new search direction.

Several formulas for β_k exist in the literature. Notable examples include those by Hestenes and Stiefel (HS), Fletcher and Reeves (FR), Polak–Ribière–Polyak (PRP) and Conjugate Descent (CD) [1]. In addition, see Liu and Storey (LS) [10], Dai and Yuan (DY) [2], and Rivaie et al. (RMIL) [13]. According to Hager and Zhang [7], methods like FR, CD, and DY have solid convergence properties, while others, such as HS, PRP and LS, often give better numerical results.

Some classical formulas for β_k are given by

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{d_k^T y_k}, \quad \beta_k^{FR} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\|g_k\|^2}, \quad \beta_k^{CD} = -\frac{\|g_{k+1}\|^2}{g_k^T d_k},$$

$$\beta_k^{LS} = -\frac{g_{k+1}^T y_k}{g_k^T d_k}, \quad \beta_k^{DY} = \frac{\|g_{k+1}\|^2}{d_k^T y_k}, \quad \beta_k^{RMIL} = \frac{g_{k+1}^T y_k}{\|d_k\|^2},$$

where $y_k = g_{k+1} - g_k$ and $\| \cdot \|$ denotes the Euclidean norm.

To improve the performance and stability of CG methods, researchers have developed hybrid formulas by combining two or more β_k expressions. Some are convex combinations, while others are non-convex. Many such combinations have been proposed in the literature [3, 4, 8, 12–14].

In this paper, we propose a new hybrid method called RMILCDLS, which blends the RMIL, CD and LS formulas using a convex combination. The aim is to take advantage of the strong global convergence of RMIL and CD, while also using the numerical performance of the LS method.

The rest of this paper is organized as follows. Section 2 presents the new β_k formula and algorithm. Section 3 contains the convergence analysis under the strong Wolfe conditions. Section 4 gives numerical results and Section 5 concludes the paper.

2 New Formula of β_k and Description of the Corresponding Algorithm

In this section, we propose a new hybrid conjugate gradient method for solving (1). The method is based on a convex combination of the RMIL, CD and LS conjugate gradient formulas. The proposed parameter β_k^{New} is defined by

$$\beta_k^{\text{New}} = \lambda_k \beta_k^{\text{RMIL}} + \theta_k \beta_k^{\text{CD}} + (1 - \lambda_k - \theta_k)\beta_k^{\text{LS}}, \quad \lambda_k, \theta_k \in [0,1],\ \lambda_k + \theta_k \le 1. \tag{4}$$

Substituting the expressions for the individual CG parameters, we get

$$\beta_k^{\text{New}} = \lambda_k \frac{g_{k+1}^T y_k}{\|d_k\|^2} + \theta_k \frac{\|g_{k+1}\|^2}{-g_k^T d_k} + (1 - \lambda_k - \theta_k)\frac{g_{k+1}^T y_k}{-g_k^T d_k}, \tag{5}$$

where $y_k = g_{k+1} - g_k$.

The search direction is generated using the recurrence

$$d_0 = -g_0, \quad d_{k+1} = -g_{k+1} + \beta_k^{\text{New}} d_k. \tag{6}$$

To ensure global convergence and sufficient descent condition, we employ the strong Wolfe line search conditions

$$f(x_k + \alpha_k d_k) \le f(x_k) + \delta \alpha_k g_k^T d_k, \tag{7}$$

$$\sigma g_k^T d_k \le g_{k+1}^T d_k \le -\sigma g_k^T d_k, \tag{8}$$

where $0 < \delta \le \sigma < \frac{4}{5}$.

The parameters λ_k and θ_k are chosen to satisfy the conjugacy condition

$$y_k^T d_{k+1} = 0. \tag{9}$$

Substituting (6) into (9) and using (4), we get

$$\begin{aligned} y_k^T d_{k+1} &= -y_k^T g_{k+1} + \beta_k^{\text{New}} y_k^T d_k = 0, \\ \Rightarrow y_k^T g_{k+1} &= \beta_k^{\text{New}} y_k^T d_k. \end{aligned} \tag{10}$$

Replacing β_k^{New} by its expression from (4), we obtain

$$\begin{aligned} y_k^T g_{k+1} &= \left[\lambda_k \beta_k^{\text{RMIL}} + \theta_k \beta_k^{\text{CD}} + (1 - \lambda_k - \theta_k)\beta_k^{\text{LS}}\right] y_k^T d_k \\ &= \lambda_k(\beta_k^{\text{RMIL}} - \beta_k^{\text{LS}}) y_k^T d_k + \theta_k(\beta_k^{\text{CD}} - \beta_k^{\text{LS}}) y_k^T d_k + \beta_k^{\text{LS}} y_k^T d_k. \end{aligned} \tag{11}$$

From (11), we get

$$\lambda_k = \frac{y_k^T g_{k+1} - \theta_k(\beta_k^{\text{CD}} - \beta_k^{\text{LS}}) y_k^T d_k - \beta_k^{\text{LS}} y_k^T d_k}{(\beta_k^{\text{RMIL}} - \beta_k^{\text{LS}}) y_k^T d_k}. \tag{12}$$

To ensure feasibility of the parameters, we enforce the following bounds:

$$\lambda_k = \begin{cases} 0, & \text{if } \lambda_k < 0, \\ 1, & \text{if } \lambda_k > 1, \\ 1 - \theta_k, & \text{if } \lambda_k + \theta_k > 1. \end{cases}$$

Depending on the values of λ_k and θ_k, the parameter β_k^{New} reduces to the known methods

$$\beta_k^{\text{New}} = \begin{cases} \beta_k^{\text{RMIL}}, & \lambda_k = 1,\ \theta_k = 0, \\ \beta_k^{\text{CD}}, & \lambda_k = 0,\ \theta_k = 1, \\ \beta_k^{\text{LS}}, & \lambda_k = 0,\ \theta_k = 0, \\ \lambda_k \beta_k^{\text{RMIL}} + (1 - \lambda_k)\beta_k^{\text{LS}}, & \theta_k = 0,\ \lambda_k \in (0,1), \\ \lambda_k \beta_k^{\text{RMIL}} + (1 - \lambda_k)\beta_k^{\text{CD}}, & \theta_k = 1 - \lambda_k,\ \lambda_k \in (0,1), \\ \theta_k \beta_k^{\text{CD}} + (1 - \theta_k)\beta_k^{\text{LS}}, & \lambda_k = 0,\ \theta_k \in (0,1), \\ \lambda_k \beta_k^{\text{RMIL}} + \theta_k \beta_k^{\text{CD}} + (1 - \lambda_k - \theta_k)\beta_k^{\text{LS}}, & \lambda_k, \theta_k \in (0,1),\ \lambda_k + \theta_k < 1. \end{cases} \tag{13}$$

Now we give the corresponding algorithm for our β_k^{New}.

Algorithm 2.1 (RMILCDLS)
Begin algorithm
Step 0: *Given a starting point* $x_0 \in \mathbb{R}^n$ *and a parameter* $\varepsilon > 0$, *compute* $g_0 = \nabla f(x_0)$, *then set* $d_0 = -g_0$.
Step 1: *If* $\|g_k\| \leq \varepsilon$, ***Stop****; otherwise go to* ***Step 2****.*
Step 2: *Compute the step-size* α_k *using (7) and (8).*
Step 3: *Update* $x_{k+1} = x_k + \alpha_k d_k$.
Step 4: *Compute* $g_{k+1} = \nabla f(x_{k+1})$, $y_k = g_{k+1} - g_k$.
Step 5: *If* $g_{k+1}^T g_k = 0$, *then set* $\lambda_k = 0$;
otherwise compute λ_k *as in (12) with* $0 \leq \theta_k \leq 1$.
Step 6: *Compute* β_k^{New} *using formula (5).*
Step 7: *Set* $d_{k+1} = -g_{k+1} + \beta_k^{New} d_k$.
Step 8: *If* $\left|g_{k+1}^T g_k\right| \geq 0.2\|g_{k+1}\|^2$ *(restart criterion of Powell [4]),*
then set $d_{k+1} = -g_{k+1}$.
Step 9: *Let* $k = k+1$ *and go back to* ***Step 1****.*
End algorithm

3 Convergence Analysis

Throughout this section, we make the following assumptions.

Assumption (i): The level set $S = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ is bounded, i.e., there exists a constant $B > 0$ such that

$$\|x\| \leq B, \text{ for all } x \in S. \tag{14}$$

Assumption (ii): In a neighborhood N of S, the function f is continuously differentiable and its gradient $\nabla f(x)$ is Lipschitz continuous, i.e., there exists a constant $0 < L < \infty$ such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\,\|x - y\|\,, \text{ for all } x, y \in N. \tag{15}$$

Under Assumptions **(i)** and **(ii)** on f, there exists a constant $\mu \geq 0$ such that

$$\|\nabla f(x)\| \leq \mu, \text{ for all } x, y \in N. \tag{16}$$

To establish the sufficient descent condition, we introduce the following theorem.

Theorem 3.1 *Let* $\{d_k\}_{k\in\mathbb{N}}$ *be given by (6),* α_k *satisfy (7) and (8), then*

$$g_{k+1}^T d_{k+1} \leq -c\,\|g_{k+1}\|^2\,, \; k = 0, 1, .., \tag{17}$$

where $c > 0$ *and* $\sigma < \frac{4}{5}$.

Proof. Induction is used to show (17). Since $d_0 = -g_0$, we get $g_0^T d_0 = -\|g_0\|^2 < 0$. Consider that (17) holds for $k > 0$.

We have

$$\left|g_{k+1}^T g_k\right| \geq 0.2\,\|g_{k+1}\|^2\,. \tag{18}$$

If (18) holds, then $g_{k+1}^T d_{k+1}^{New} = -\|g_{k+1}\|^2 < 0$.

The search direction that meets the sufficient descent condition is achieved.
If (18) does not hold, then

$$\left|g_{k+1}^T g_k\right| < 0.2 \left\|g_{k+1}\right\|^2. \tag{19}$$

Using (8), we can get that

$$y_k^T d_k = (g_{k+1} - g_k)^T d_k \geq -(1-\sigma) g_k^T d_k. \tag{20}$$

And

$$\left|\frac{g_{k+1}^T d_k}{y_k^T d_k}\right| \leq \frac{\sigma}{(1-\sigma)}.$$

Multiplying both sides of (6) by g_{k+1}^T, we get

$$\begin{aligned} g_{k+1}^T d_{k+1}^{New} &= -\left\|g_{k+1}\right\|^2 + \lambda_k \beta_k^{RMIL} g_{k+1}^T d_k + \theta_k \beta_k^{CD} g_{k+1}^T d_k \\ &\quad + (1-\lambda_k-\theta_k)\beta_k^{LS} g_{k+1}^T d_k. \end{aligned}$$

We have seven cases.

Case 01: If $\lambda_k = 1$, $\theta_k = 0$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{RMIL}$. For the RMIL direction $d_{k+1}^{RMIL} = -g_{k+1} + \beta_k^{RMIL} d_k$, it is proved in [13] that

$$g_{k+1}^T d_{k+1}^{RMIL} \leq -a_1 \|g_{k+1}\|^2 \text{ for all } k,$$

where $a_1 > 0$.

Case 02: If $\lambda_k = 0$, $\theta_k = 1$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{CD}$. For the conjugate descent direction $d_{k+1}^{CD} = -g_{k+1} + \beta_k^{CD} d_k$, it is proved in [4] that

$$g_{k+1}^T d_{k+1}^{CD} \leq -a_2 \|g_{k+1}\|^2 \text{ for all } k,$$

where $a_2 > 0$.

Case 03: If $\lambda_k = 0$, $\theta_k = 0$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{LS}$. For the Liu-Storey direction $d_{k+1}^{LS} = -g_{k+1} + \beta_k^{LS} d_k$, it is proved in [4] that

$$g_{k+1}^T d_{k+1}^{LS} \leq -a_3 \|g_{k+1}\|^2 \text{ for all } k,$$

where $a_3 > 0$.

Case 04: If $\lambda_k \in]0,1[$, $\theta_k = 0$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{RMILLS}$.

We have

$$d_{k+1}^{New} = \lambda_k d_{k+1}^{RMIL} + \theta_k d_{k+1}^{CD} + (1-\lambda_k-\theta_k) d_{k+1}^{LS}.$$

Hence,

$$g_{k+1}^T d_{k+1}^{New} = \lambda_k g_{k+1}^T d_{k+1}^{RMIL} + \theta_k g_{k+1}^T d_{k+1}^{CD} + (1-\lambda_k-\theta_k) g_{k+1}^T d_{k+1}^{LS}. \tag{21}$$

With (21), we get

$$g_{k+1}^T d_{k+1}^{RMILLS} = \lambda_k g_{k+1}^T d_{k+1}^{RMIL} + (1-\lambda_k) g_{k+1}^T d_{k+1}^{LS}.$$

$\exists w_1, w_2 \in \mathbb{R} : 0 < w_1 < \lambda_k < w_2 < 1$, then

$$g_{k+1}^T d_{k+1}^{RMILLS} \leq -(w_1 a_1 + w_2 a_3) \left\|g_{k+1}\right\|^2 = -a_4 \left\|g_{k+1}\right\|^2,$$

where $a_4 > 0$.

Case 05: If $\theta_k = 1 - \lambda_k$ and $\lambda_k,\, \theta_k \in]0,1[$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{RMILCD}$.
With (21), we get

$$g_{k+1}^T d_{k+1}^{RMILCD} = \lambda_k g_{k+1}^T d_{k+1}^{RMIL} + (1-\lambda_k) g_{k+1}^T d_{k+1}^{CD}.$$

Clearly, the sufficient descent condition is satisfied, which means that

$$g_{k+1}^T d_{k+1}^{RMILCD} \leq -a_5 \left\| g_{k+1} \right\|^2,$$

where $a_5 > 0$.

Case 06: If $\lambda_k = 0$, $\theta_k \in]0,1[$, we have $g_{k+1}^T d_{k+1}^{New} = g_{k+1}^T d_{k+1}^{CDLS}$.
With (21), we get

$$g_{k+1}^T d_{k+1}^{CDLS} = \theta_k g_{k+1}^T d_{k+1}^{CD} + (1-\theta_k) g_{k+1}^T d_{k+1}^{LS}.$$

Case 06 is the same as **Case 04** and **Case 05**. So, the sufficient descent condition is satisfied, which means that

$$g_{k+1}^T d_{k+1}^{CDLS} \leq -a_6 \left\| g_{k+1} \right\|^2,$$

where $a_6 > 0$.

Case 07: If $\lambda_k,\, \theta_k \in]0,1[$ and $0 < \lambda_k + \theta_k < 1$, we have

$$g_{k+1}^T d_{k+1}^{New} = \lambda_k g_{k+1}^T d_{k+1}^{RMIL} + \theta_k g_{k+1}^T d_{k+1}^{CD} + (1-\lambda_k-\theta_k) g_{k+1}^T d_{k+1}^{LS}.$$

$\exists k_1, k_2, k_3, k_4 \in \mathbb{R} : 0 < k_1 < \lambda_k < k_2 < 1,\ 0 < k_3 < \theta_k < k_4 < 1$, then

$$\begin{aligned} g_{k+1}^T d_{k+1}^{New} &\leq -(k_1 a_1 + k_2 a_2 + (1-k_3-k_4) a_3) \left\| g_{k+1} \right\|^2 \\ &= -a_7 \left\| g_{k+1} \right\|^2, \end{aligned}$$

where $a_7 > 0$.

The proof is complete.

Lemma 3.1 *Suppose that Assumptions (i) and (ii) hold. Consider common iterate (2), where d_k is a descent direction that verifies (17) and α_k is determined by the strong Wolfe line search (7) and (8). Then the Zoutendijk condition*

$$\sum_{k \geq 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \tag{22}$$

holds.

Proof. The proof follows directly from [15].

Lemma 3.2 *Suppose that Assumptions (i) and (ii) hold and $\alpha_k > 0$ is determined by the strong Wolfe line search (7) and (8). Then there exists $\alpha^* > 0$ such that*

$$\alpha_k \geq \alpha_* > 0 \text{ for all } k.$$

Proof. The proof follows directly from [4].

Theorem 3.2 *Consider the RMILCDLS conjugate gradient method and suppose that Assumptions (i), (ii) and (8) hold. Then either $g_k = 0$ for some k or*

$$\lim_{k\to\infty} \inf \|g_k\| = 0. \tag{23}$$

Proof. Suppose that $g_k \neq 0$ for all k. Then we are going to prove (23).
Suppose by contradiction that (23) does not hold. Then there exists $t > 0$ such that

$$\|g_k\| \geq t \text{ for all } k. \tag{24}$$

Let D denote the diameter of the level set S, and define the step $s_k = x_{k+1} - x_k = \alpha_k d_k$. From equation (4), we have

$$\left|\beta_k^{\text{New}}\right| \leq \left|\beta_k^{\text{RMIL}}\right| + \left|\beta_k^{\text{CD}}\right| + \left|\beta_k^{\text{LS}}\right|. \tag{25}$$

We begin by analyzing each of the components on the right-hand side. By definition,

$$\left|\beta_k^{\text{RMIL}}\right| = \frac{\left|g_{k+1}^T y_k\right|}{\|d_k\|^2}.$$

From equations (24) and (17), we have

$$\|d_k\|^2 \geq \frac{-(1-\sigma)g_k^T d_k}{L\alpha_k} \geq \frac{(1-\sigma)c\|g_k\|^2}{L\alpha_k} \geq \frac{(1-\sigma)ct^2}{L\alpha_k}.$$

Thus, we obtain the bound

$$\frac{1}{\|d_k\|^2} \leq \frac{L\alpha_k}{(1-\sigma)ct^2}. \tag{26}$$

Next, for $\left|g_{k+1}^T y_k\right|$, we have

$$\begin{aligned} \left|g_{k+1}^T y_k\right| &\leq \|g_{k+1}\|\,\|y_k\| \\ &\leq \mu\,\|g_{k+1} - g_k\| \\ &\leq \mu L\,\|x_{k+1} - x_k\| \\ &\leq \mu L\,\|s_k\| \\ \left|g_{k+1}^T y_k\right| &\leq \mu L D. \end{aligned} \tag{27}$$

Combining equations (26) and (27), we get

$$\left|\beta_k^{\text{RMIL}}\right| \leq \frac{\mu L^2 D\alpha_k}{(1-\sigma)ct^2}. \tag{28}$$

For $\left|\beta_k^{\text{CD}}\right|$, we have

$$\left|\beta_k^{\text{CD}}\right| \leq \frac{\|g_{k+1}\|^2}{-g_k^T d_k} \leq \frac{\mu^2}{c\|g_k\|^2} \leq \frac{\mu^2}{ct^2}. \tag{29}$$

Finally, for $\left|\beta_k^{\text{LS}}\right|$, we have

$$\left|\beta_k^{\text{LS}}\right| = \left|\frac{g_{k+1}^T y_k}{-g_k^T d_k}\right| \leq \frac{\|g_{k+1}\|\,\|y_k\|}{-g_k^T d_k} \leq \frac{\mu L D}{ct^2}. \tag{30}$$

From (28), (29) and (30), we can write

$$\left|\beta_k^{New}\right| \leq \frac{\mu L^2 D\alpha_k}{(1-\sigma)ct^2} + \frac{\mu^2 + \mu LD}{ct^2}. \tag{31}$$

Now, we can write

$$\|d_{k+1}\| \leq \|g_{k+1}\| + \left|\beta_k^{New}\right| \|d_k\|. \tag{32}$$

We have $s_k = \alpha_k d_k$. So, we can write $d_k = \frac{s_k}{\alpha_k}$. Then, from (31) and (32), we get

$$\begin{aligned} \|d_{k+1}\| &\leq \mu + \left(\frac{\mu L^2 D\alpha_k}{(1-\sigma)ct^2} + \frac{\mu^2 + \mu LD}{ct^2}\right)\frac{\|s_k\|}{\alpha_k} \\ &\leq \mu + \left(\frac{\mu L^2 D\alpha_*}{(1-\sigma)ct^2} + \frac{\mu^2 + \mu LD}{ct^2}\right)\frac{D}{\alpha_*} = P. \end{aligned} \tag{33}$$

This implies that

$$\sum_{k\geq 1} \frac{1}{\|d_{k+1}\|^2} = \infty. \tag{34}$$

On the other hand, from (18), (24) and from the Zoutendijk condition (22), it follows that

$$c^2 t^4 \sum_{k\geq 0} \frac{1}{\|d_k\|^2} \leq \sum_{k\geq 0} \frac{c^2\|g_k\|^4}{\|d_k\|^2} \leq \sum_{k\geq 0} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty,$$

which contradicts (34). Therefore, (24) does not hold. Then $\lim_{k\to\infty} \inf \|g_k\| = 0$.

This completes the proof.

4 Numerical Experiments

In this section, we present several numerical tests to evaluate the performance of RMIL-CDLS algorithm. We select various test functions given in Table 4, taken from the CUTE library [1,6] for different dimensions, from $n = 10$ to $n = 800000$. At the same time, we present a numerical comparison with other conjugate gradient algorithms, namely RMIL, LS, CD and HSDYCD [8]. The stopping criterion is $\|g_k\| \leq \varepsilon$, where $\varepsilon = 10^{-6}$, or the iteration number > 4000. The step-size α_k is computed by the strong Wolfe line search with $\sigma = 0.01$ $\delta = 0.1$. In order to evaluate the data of iterations number, CPU time and for gradient evaluation and objective function evaluation, we use the performance profile of Dolan and Moré [5] presented in Figure 1.

4.1 Commentaries

The performance profile results, as depicted in Figures 1 (a), (b), (c) and (d), show that our proposed conjugate gradient method RMILCDLS, using β_k^{New}, consistently outperforms the other methods based on β_k^{RMIL}, β_k^{CD}, β_k^{LS} and β_k^{HSDYCD}.

Functions	Dimensions	Functions	Dimensions
Genrose	10000	Pen1	200, 1000
Osb2	10	Rosex	500, 1000
Eg2	100	Diagonal3	500, 2000
Freuroth	460	Singx	1000, 2000
Pen2	160	Diagonal1	800, 2000
Dixon3dq	150	Trid	500, 8000
Fletcbv3	100	Ie	500, 1500
Power1	150	Raydan1	500, 5000
Biggsb1	300	Lin	100, 1300
Dixmaani	360	Nonscomp	5000, 80000
Bv	2000, 20000	Dqrtic	5000, 150000
Liarwhd	6000, 30000	Dqdrtic	9000, 90000
Woods	150000, 200000	Dixmaana	6000, 90000
Quartc	80000, 50000	Dixmaanb	24000, 48000
Dixmaanc	2700, 27000	Dixmaand	12000, 90000
Dixmaane	2400, 48000	Dixmaanf	15000, 60000
Dixmaang	12000, 90000	Edensch	7000, 40000, 50000
Diagonal2	8000, 50000	Cosine	6000, 100000, 800000
Dixmaank	12000, 12000	Fletchcr	1000, 50000, 200000
Dixmaanl	2400, 24000	Exdenschnb	6000, 24000, 300000
Himmelbg	70000, 240000	Exdenschnf	90000, 280000, 600000
Dixmaanh	6000, 150000	Raydan2	2000, 20000, 500000
Dixmaanj	3000, 15000	Sine	100000, 250000, 500000

Table 1: List of test functions and their dimensions.

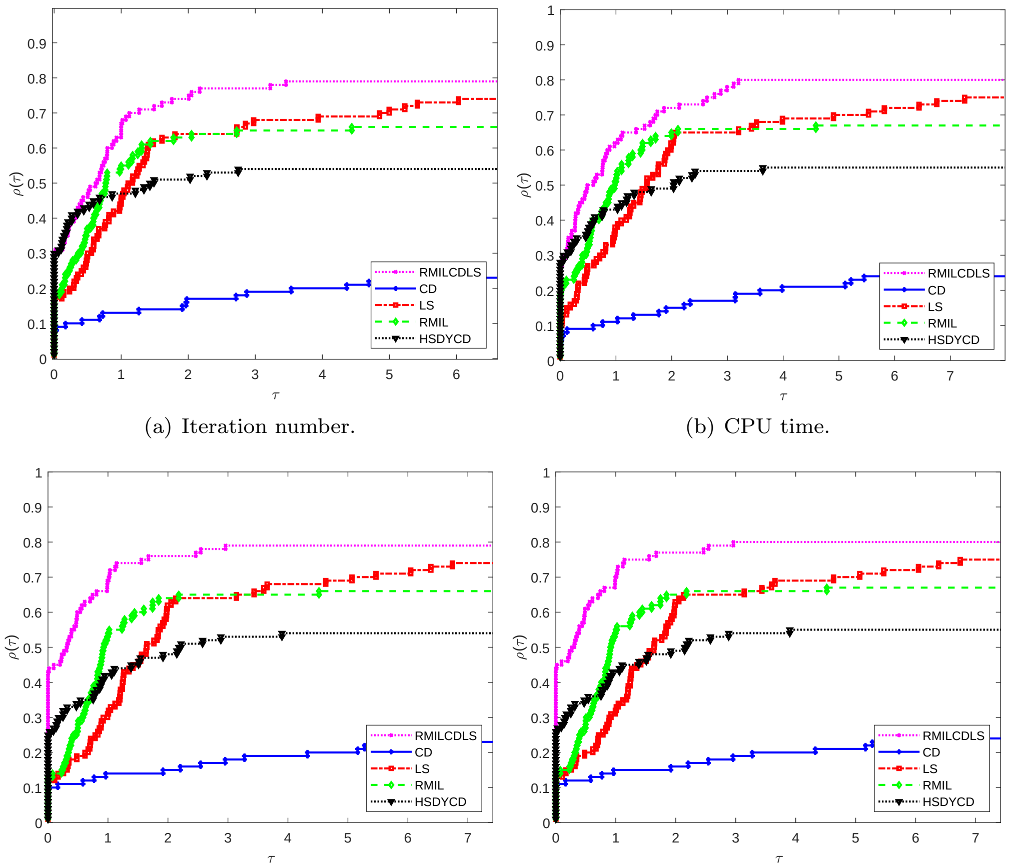
5 Conclusion

In this paper, we have proposed a new hybrid conjugate gradient method for solving unconstrained optimization problems, where the parameter β_k^{New} is a convex combination of β_k^{RMIL}, β_k^{CD} and β_k^{LS}.

We have provided the proof of the sufficient descent condition for the search direction and the global convergence of our proposed method for nonlinear functions using the strong Wolfe line search.

The numerical tests carried out confirm the effectiveness of our proposed RMILCDLS algorithm in terms of iterations number, computation time, gradient evaluation and objective function evaluation compared with other standard conjugate gradient methods, namely the RMIL, LS, CD and HSDYCD algorithms.

In particular, the convex combination strategy introduced in the construction of β_k^{New} offers a new balance between convergence reliability and computational efficiency. This makes the RMILCDLS algorithm a valuable tool for solving large-scale problems arising in various fields such as nonlinear dynamics, optimal control, machine learning, image processing and compressive sensing.



(a) Iteration number. (b) CPU time.

(c) Objective function evaluation. (d) Gradient evaluation.

Figure 1: Performance profile of RMILCDLS, RMIL, CD, LS and HSDYCD algorithms.

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