Nonlinear Dynamics and Systems Theory, **26** (2) (2026) 195–202

A Quintic Nonlinear Differential System with a Non-Algebraic Limit Cycle Around a Non-Elementary Singular Point

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Received: June 11, 2025; Revised: April 13, 2026

Abstract: In this paper, we consider a class of quintic planar polynomial differential systems with a non-elementary singular point. We establish sufficient conditions for the existence of a hyperbolic non-algebraic limit cycle surrounding this singularity. Moreover, the limit cycle is explicitly expressed in polar coordinates. To demonstrate the applicability of our results, a concrete example is provided along with its corresponding phase portrait.

Keywords: *polynomial differential system; non-algebraic limit cycle; first integral; Riccati differential equation.*

Mathematics Subject Classification (2020): 34A05, 70K05, 34C05, 34C07, 34C25.

1 Introduction

One of the main problems in the qualitative theory of differential equations is the study of limit cycles, that is, isolated periodic solutions among all periodic solutions of planar differential systems of the form

$$\begin{cases} \dot{x} = \frac{dx}{dt} = P(x,y), \\ \dot{y} = \frac{dy}{dt} = Q(x,y), \end{cases} \tag{1}$$

where P and Q are real polynomials in the variables x and y. The degree n of the polynomial system of differential equations is the maximum of the degrees of the polynomials

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P and Q. Limit cycles of plane vector fields were first proposed by French mathematician Poincaré in his very famous classical papers entitled "Integral curves defined by differential equations" (1881, 1882, 1885, 1886). Later on van der Pol [16] in 1926, Lienard in 1928, and Andronov [2] in 1929, showed that the periodic solution of a self-sustained oscillation of a circuit in a vacuum tube was a limit cycle in the sense defined by Poincaré.

Hilbert proposed a list of 23 problems to guide the advancement of mathematical science, sparking intensive research throughout the 20th century. Among these, only the Riemann hypothesis and the second part of Hilbert's 16th problem remain unsolved to this day. The second part of the 16th Hilbert problem consists of two components: it seeks an upper bound on the number of possible limit cycles and their positions for planar polynomial differential systems of a given degree [5]. The theory of limit cycles, serving as an indispensable mathematical tool, has found broad and significant applications in modern physics, chemistry, biology, and other disciplines. In turn, the progress in these fields continues to drive research on limit cycles.

Let Ω be a non-empty open and dense subset of $\mathbb{R}^2$. We say that a non–locally constant C^1 function $\varphi : \Omega \to \mathbb{R}$ is a first integral of the polynomial differential (2) in Ω if φ is constant on the trajectories of the polynomial differential system (2) contained in Ω, i.e., if

$$\frac{d\varphi(x,y)}{dt} - \frac{\partial\varphi(x,y)}{\partial x}P(x,y) - \frac{\partial\varphi(x,y)}{\partial y}Q(x,y) \equiv 0 \text{ at the points of } \Omega.$$

Let us take into consideration a not closed set $\digamma \subseteq \mathbb{R}^2$ and function V of class $C^1(\digamma)$ defined by $V(x,y) : \mathbb{R}^2 \supseteq \digamma \longrightarrow \mathbb{R}$. We say that the curve defined by means of the equation $V(x,y) = 0$ is an invariant curve of the system (2) under the condition that the following equation is satisfied:

$$P(x,y)\frac{\partial V}{\partial x}(x,y) + Q(x,y)\frac{\partial V}{\partial y}(x,y) = K\,(x,y)\,V(x,y)$$

with $K(x,y)$ being a polynomial in the variables x and y such as $\deg K(x,y) \leq q-1$. The polynomial $K(x,y)$ is known as the cofactor of the polynomial curve $V(x,y) = 0$.

The study of nonlinear differential systems is of central importance for understanding complex dynamical behaviors that cannot be captured by linear models. In particular, limit cycles play a key role in describing self-sustained oscillations in physical, biological, and engineering systems [4,8,9,12,15]. While linear systems provide foundational insight, they fail to account for the intricate behaviors introduced by nonlinearities. Therefore, investigating the existence and characteristics of limit cycles in nonlinear planar systems, especially those with higher-degree nonlinearities and non-elementary singularities, contributes significantly to the theoretical development and practical understanding of nonlinear dynamics.

In the qualitative theory of planar differential systems, an algebraic limit cycle of degree p is an oval of an irreducible invariant algebraic curve $V(x,y) = 0$ of degree p, which is a limit cycle of the system, otherwise it is called non-algebraic, see [11, 14]. In general, the orbits of a polynomial differential system (1) are contained in analytic curves that are not algebraic, see for example [1, 6, 10], an even more difficult problem is to give an explicit form of them, for example, the well-known limit cycle of the van der Pol differential system presented in 1926, was not proved until 1995, when Odani [13] showed that it was non-algebraic and the der Pol differential system can be written as a polynomial differential system (1) of degree 3, but its limit cycle is not known explicitly. The first examples of explicit non-algebraic limit cycles appeared in the works of A.Gasull, Giacomini and Torregrosa [6] and J.Giné and M.Grau [7], Al-Dossary [1] for $n = 5$, and Benterki Lilibre [3] for $n = 3$.

In this work, we consider a class of quintic planar polynomial differential systems that feature a non-elementary singular point. We establish sufficient conditions for the existence of a hyperbolic, non-algebraic limit cycle surrounding this singularity. Moreover, we derive an explicit expression for the limit cycle, contributing both to the theoretical understanding and practical identification of such cycles. To illustrate the applicability of our results, a specific example is constructed, and its phase portrait is presented.

2 Main Result

As our main result, we shall prove the following theorem.

Theorem 2.1 *Consider the quintic polynomial differential systems*

$$\left\{ \begin{array}{l} \dot{x} = 2hy\left(a^2y^2+x^2\right) + wax\left(-\left(x^2+y^2\right)^2 + h\left(x^2+y^2\right)\right), \\ \dot{y} = -2hx\left(a^2y^2+x^2\right) + way\left(-\left(x^2+y^2\right)^2 + h\left(x^2+y^2\right)\right), \end{array} \right. \tag{2}$$

then the following statements hold.

(1) If $a \in \mathbb{R}^*$, $w \in \mathbb{R}^*$ and $h > 0$, the system (2) has the first integral

$$I(x,y) = \frac{\exp \int_0^{\arctan \frac{y}{x}} \frac{2}{\sqrt{h}}\phi(s)ds}{\sqrt{x^2+y^2} - \sqrt{h}} + \int_0^{\arctan \frac{y}{x}} \left(\exp \int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right) \frac{1}{h}\phi(s)ds,$$

where $\phi(\theta) = \frac{wa}{2(a^2\sin^2\theta + \cos^2\theta)}$.

(2) If $wa < 0$ and $h > 0$, the system (2) possesses exactly one non-algebraic limit cycle around a non-elementary singular point, this limit cycle is explicitly given in polar coordinates (r, θ) by the expression

$$r(\theta, r_*) = \sqrt{h} + \frac{\left(\exp \int_0^{\theta} \frac{-2}{\sqrt{h}}\phi(s)ds\right)^{-1}}{\frac{1}{r_* - \sqrt{h}} - \int_0^{\theta} \left(\exp \int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right) \frac{1}{h}\phi(s)ds}$$

with

$$r_* = \frac{\left(\exp \int_0^{2\pi} \frac{2}{\sqrt{h}}\phi(s)ds\right)^{-1} - 1}{\int_0^{2\pi} \left(\exp \int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right) \frac{1}{h}\phi(s)ds} + \sqrt{h}.$$

Proof. Firstly, we prove that the origin is the unique equilibrium non-elementary point of the system (2).

We note that

$$\begin{aligned} x\dot{y} - y\dot{x} &= x\left(-2hx\left(a^2y^2+x^2\right) + way\left(-\left(x^2+y^2\right)^2 + h\left(x^2+y^2\right)\right)\right) \\ &\quad -y\left(2hy\left(a^2y^2+x^2\right) + wax\left(-\left(x^2+y^2\right)^2 + h\left(x^2+y^2\right)\right)\right) \\ &= -2h\left(a^2y^2+x^2\right)\left(x^2+y^2\right), \end{aligned}$$

thus, the equilibrium points of system (2) are present in the curve's equation

$$-2h\left(a^2y^2+x^2\right)\left(x^2+y^2\right)=0 \tag{3}$$

because $h\neq 0$ and $a\neq 0$. Then the origin of coordinates is the only solution to the equation (3), we conclude that the origin, also known as an equilibrium point, is a degenerate non-elementary singular point of the system (2), for the reason that the linear part of this system is identically zero.

The Jacobian matrix of the differential system (2) at the origin $O(0,0)$ is

$$D_j(0,0)=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So the origin $(0,0)$ is a non-elementary point.

(1) To prove statements (1) and (2), we write the polynomial differential system (2) in polar coordinates $(r;\theta)$, then these system becomes

$$\begin{cases} \dot{r}=war\left(-r^4+hr^2\right), \\ \dot{\theta}=-2hr\left(a^2r^2\sin^2\theta+r^2\cos^2\theta\right). \end{cases} \tag{4}$$

Taking θ as an independent variable, we can rewrite the system (4) as the Riccati differential equation as follows:

$$\frac{dr}{d\theta}=\frac{1}{h}\phi(\theta)r^2-\phi(\theta), \tag{5}$$

where $\phi(\theta)=\frac{wa}{2(a^2\sin^2\theta+\cos^2\theta)}$.

Note that since $h\in\mathbb{R}_+^*$, we have $\theta'=-2hr\left(a^2r^2\sin^2\theta+r^2\cos^2\theta\right)<0$ for all $\theta\in\mathbb{R}$. So, the orbits $r(\theta)$ of the differential equation (5) reverse their orientation with respect to the orbits $(r(t);\theta(t))$ or $((x(t);y(t))$ of the differential systems (4) and (2), respectively.

Fortunately, the equation (5) is integrable since it possesses the particular solution $r=\sqrt{h}$, the general solution of equation (5) is given by

$$r=\left(\sqrt{h}+\frac{1}{\rho}\right), \tag{6}$$

where ρ is a function of the variable θ. Indeed, substituting the solution $r=\left(\sqrt{h}+\frac{1}{\rho}\right)$ into the Riccatti equation (5), we obtain the linear equation

$$\frac{d\rho}{d\theta}=-\frac{2}{\sqrt{h}}\,\phi(\theta)\rho-\frac{1}{h}\phi(\theta). \tag{7}$$

The general solution of linear equation (7) is

$$\rho(\theta,c)=\left(\exp\int_0^\theta\frac{2}{\sqrt{h}}\phi(s)ds\right)^{-1}\left(c+\int_0^\theta\left(\exp\int_0^s\frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\left(\frac{-1}{h}\phi(s)\right)ds\right),$$

where $c\in\mathbb{R}$. Going back through the changes of variables (6), we obtain the general solution of (5)

$$r(\theta,c)=\sqrt{h}+\frac{\exp\int_0^\theta\frac{2}{\sqrt{h}}\phi(s)ds}{c-\int_0^\theta\left(\exp\int_0^s\frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds}, \tag{8}$$

where $\phi(\theta) = \frac{wa}{2(a^2\sin^2\theta+\cos^2\theta)}$. By passing to Cartesian coordinates, we deduce the first integral

$$I(x,y) = \frac{\exp\int_0^{\arctan\frac{y}{x}} \frac{2}{\sqrt{h}}\phi(s)ds}{\sqrt{x^2+y^2}-\sqrt{h}} + \int_0^{\arctan\frac{y}{x}} \left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds.$$

The trajectories of the system (2) are the level curves $I(x,y) = c$, $c \in \mathbb{R}$, and since these curves are obviously non-algebraic except for the curve (Γ) which corresponds to $c \to +\infty$, any other limit cycle, if it exists, should also be non-algebraic. Hence, statement (1) is proved.

(2) If we put $\theta = 0$ in the solution (8), we have

$$r(0,c) = \sqrt{h} + \frac{1}{c}.$$

Let $r_0 = \sqrt{h}+\frac{1}{c}$, so $r(0,r_0) = r_0 > 0$, corresponds to the value $c = \frac{1}{r_0-\sqrt{h}}$, which allows to rewrite the general solution of (8) as

$$r(\theta,r_0) = \sqrt{h} + \frac{\exp\int_0^\theta \frac{2}{\sqrt{h}}\phi(s)ds}{\frac{1}{r_0-\sqrt{h}} - \int_0^\theta\left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds},$$

where $r_0 = r_0(0)$ and $\phi(\theta) = \frac{wa}{2(a^2\sin^2\theta+\cos^2\theta)}$.

The periodic solutions of system (2) must satisfy the following condition:

$$r(2\pi, r_0) = r(0, r_0). \tag{9}$$

The condition (9) is equivalent to

$$r_0 = r_* = \frac{\exp\int_0^{2\pi} \frac{-2}{\sqrt{h}}\phi(s)ds - 1}{\int_0^{2\pi}\left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds} + \sqrt{h}, \tag{10}$$

r_* is the intersection of the periodic orbit with the positive $x-$semi-axis.

Let us show that the right-hand side of (10) is strictly positive.

Indeed, since $aw < 0$ and $h > 0$, then $\phi(\theta) = \frac{wa}{2(a^2\sin^2\theta+\cos^2\theta)} < 0$ and $\frac{1}{\exp\int_0^\theta \frac{2}{\sqrt{h}}\phi(s)ds} - 1 < 0$ for all $\theta \in \mathbb{R}$, so

$$\begin{aligned} r_* &= \frac{\exp\int_0^{2\pi} \frac{-2}{\sqrt{h}}\phi(s)ds - 1}{\int_0^{2\pi}\left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds} + \sqrt{h} \\ &= \frac{\frac{1}{\exp\int_0^\theta \frac{2}{\sqrt{h}}\phi(s)ds} - 1}{\int_0^{2\pi}\left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds} + \sqrt{h} > 0. \end{aligned}$$

Substituting the value of r_* into (2), we get the candidate solution

$$r(\theta,r_*) = \sqrt{h} + \left(\exp\int_0^\theta \frac{2}{\sqrt{h}}\phi(s)ds\right)\left(\frac{1}{r_*-\sqrt{h}} - \int_0^\theta\left(\exp\int_0^s \frac{2}{\sqrt{h}}\phi(\eta)d\eta\right)\frac{1}{h}\phi(s)ds\right)^{-1}.$$

Next, we prove that $r(\theta, r_*) > 0$. Indeed, since $aw < 0$ and $h > 0$, we have $\phi(\theta) = \frac{wa}{2(a^2 \sin^2\theta + \cos^2\theta)} < 0$ for all $\theta \in \mathbb{R}$, so

$$\begin{aligned} r(\theta, r_*) &= \sqrt{h} + \frac{\exp \int_0^\theta \frac{2}{\sqrt{h}} \phi(s) ds}{\frac{1}{r_* - \sqrt{h}} + \int_0^\theta \left(\exp \int_0^s \frac{2}{\sqrt{h}} \phi(\eta) d\eta \right) \left(\frac{-1}{h} \phi(s) \right) ds} \\ &= \sqrt{h} + \frac{\exp \int_0^\theta \frac{2}{\sqrt{h}} \phi(s) ds}{\frac{\left(\exp \int_0^{2\pi} \frac{2}{\sqrt{h}} \phi(s) ds \right)^{-1} - 1}{\int_0^{2\pi} \left(\exp \int_0^s \frac{2}{\sqrt{h}} \phi(\eta) d\eta \right) \frac{1}{h} \phi(s) ds} + \int_0^\theta \left(\exp \int_0^s \frac{2}{\sqrt{h}} \phi(\eta) d\eta \right) \left(\frac{-1}{h} \phi(s) \right) ds} > 0 \end{aligned}$$

for all $\theta \in \mathbb{R}$. To prove that the periodic orbit is a hyperbolic limit cycle, we consider (2) and introduce the Poincare return map $r_0 \mapsto P(r_0) = r(2\pi, r_0)$, see [14].

We compute

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r = r_*} = \exp \int_0^{2\pi} \frac{2}{\sqrt{h}} \phi(s) ds < 1.$$

Since $\phi(\theta) < 0$ for all $\theta \in \mathbb{R}$, we have $\int_0^{2\pi} \frac{2}{\sqrt{h}} \phi(s) ds < 0$,

therefore

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r = r_*} < 1.$$

For this reason, the limit cycle for the ordinary differential equation (5) is stable. Finally, system (2) has exactly one non-algebraic limit cycle which is the only existing limit cycle. Consequently, this is an unstable and hyperbolic limit cycle for the differential system (2). This completes the proof of statement (2) of Theorem 2.1.

Example 2.1 When $a = -2, h = 4, w = 1$, system (2) reads

$$\left\{ \begin{array}{l} \dot{x} = 8y\left(4y^2 + x^2\right) + 2x\left(x^4 + y^4\right) - 8x\left(x^2 + y^2\right) + 4x^3y^2, \\ \dot{y} = -8x\left(4y^2 + x^2\right) + 2y\left(x^4 + y^4\right) - 8y\left(x^2 + y^2\right) + 4x^2y^3, \end{array} \right. \tag{11}$$

the system (11) satisfies the conditions of the statement (2) of Theorem 2.1, hence the system (11) possesses one non-algebraic limit cycle which is an unstable and hyperbolic limit cycle for the differential system (11) as shown in Figure 1 and it is explicitly given in polar coordinates (r, θ) by the expression

$$r(\theta, r_*) = 2 + \left(\exp \int_0^\theta \phi(s) ds \right) \left(\frac{1}{r_* - 2} - \frac{1}{4} \int_0^\theta \left(\exp \int_0^s \phi(\eta) d\eta \right) \phi(s) ds \right)^{-1}$$

with $\phi(\theta) = \frac{-1}{4\sin^2\theta + \cos^2\theta}$, and

$$r_* = \frac{\left(\exp \int_0^{2\pi} \phi(s) ds \right)^{-1} - 1}{\int_0^{2\pi} \left(\exp \int_0^s \frac{2}{\sqrt{h}} \phi(\eta) d\eta \right) \frac{1}{h} \phi(s) ds} + 2 \simeq 2.25.$$

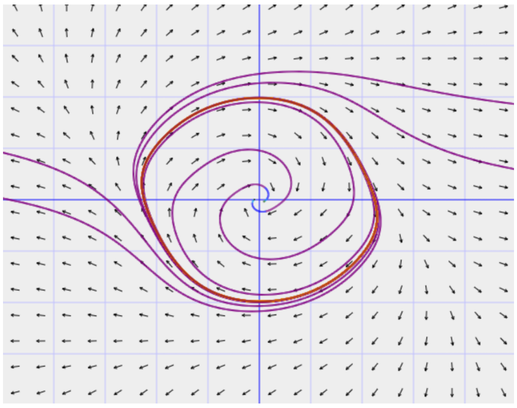

Figure 1: Limit cycle of the system (11) in the Poincare disc.

3 Conclusion

In this paper, we studied a class of quintic planar polynomial differential systems featuring a non-elementary singular point. We established sufficient conditions for the existence of a hyperbolic, non-algebraic limit cycle surrounding this singularity. Furthermore, we provided an explicit expression for the limit cycle in polar coordinates. To demonstrate the applicability of our theoretical findings, a concrete example was constructed, and its corresponding phase portrait was presented. These results contribute to the qualitative theory of nonlinear differential systems and offer valuable insights into the dynamics near non-elementary singularities.

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