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Asymptotic Behaviour of Feedback Controlled Systems and the Ubiquity of the Brockett-Krasnosel'skiĭ-Zabreĭko Property

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Abstract: A well-known topological barrier – the Brockett-Krasnosel'skiĭ-Zabreĭko necessary condition on the underlying vector field – to stability of equilibria (or stabilizability of equilibria by regular feedback) of ordinary differential equations (or controlled differential equations) is shown to persist in a wider context of differential inclusions (encompassing controlled differential equations with nonsmooth feedback) that exhibit attracting compacta.

Keywords: *Brockett-Krasnosel'skiĭ-Zabreĭko condition; feedback controlled system.*

Mathematics Subject Classification (2000): 34D05, 93D15.

1 Introduction

Let $f\colon \mathbb{R}^N \to \mathbb{R}^N$ be locally Lipschitz and consider the system

$$\dot{x} = f(x). \tag{1}$$

By [1, Theorem 52.1], if (1) has an asymptotically stable (that is, Lyapunov stable and attractive) equilibrium ξ, then the (isolated) zero ξ of $-f$ has index $\mathrm{ind}(-f,\xi) = 1$ and so, for all $\epsilon > 0$ sufficiently small, $\deg_{\mathrm{B}}(-f, \mathbb{B}_\epsilon(\xi), 0) = 1$, where $\deg_{\mathrm{B}}$ denotes Brouwer degree and $\mathbb{B}_\epsilon(\xi)$ denotes the open ball of radius ϵ centred at ξ. Therefore,

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by properties of Brouwer degree, $f(\mathbb{R}^N)$ contains an open neighbourhood of 0. Now let $f\colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be locally Lipschitz and consider the controlled system

$$\dot{x} = f(x, u). \tag{2}$$

If (2) is stabilizable in the sense that there exists a time-invariant locally Lipschitz feedback $u = k(x)$ that renders some point of $\mathbb{R}^N$ an asymptotically stable equilibrium of the feedback system $\dot{x} = f(x, k(x))$, then, by the above result, the image of f contains an open neighbourhood of 0. This is Brockett's necessary condition for stabilizability, originally proved in [2, Theorem 1]; for discussions on variants and ramifications of Brockett's condition, see, for example, [3−11]. In either case of an uncontrolled (1) or controlled (2) system, if $f\colon D \to \mathbb{R}^N$ is such that $f(D)$ contains an open neighbourhood of 0, we say that f has the BKZ (Brockett-Krasnosel'skiĭ-Zabreĭko) property.

In this paper, the necessity of the BKZ property is investigated in a wider context of differential inclusions under hypotheses weaker than asymptotic stability/stabilizability of equilibria. For example, amongst other consequences for (1), the results of the paper imply that, if any of the following hold, then f has the BKZ property:

(a) some compact set C is *globally attractive* for solutions of (1);
(b) some closed ball is a *locally asymptotically stable* (Lyapunov stable and locally attractive) set for (1);
(c) (1) is L^p*-stable* for some $1 \le p < \infty$ (in the sense that every maximal solution has interval of existence $\mathbb{R}_+$ and is of class L^p).

Within the control framework of (2), these observations have natural counterparts: f has the BKZ property if there exists a (possibly discontinuous) feedback k such that the feedback-controlled system (a) has a globally attractive compact set, or (b) has a locally asymptotically stable closed ball, or (c) is L^p-stable (in the above sense).

2 Notation and Terminology

For a Banach space X and non-empty $C \subset X$, d_C denotes the distance function given by

$$d_C(x) := \inf_{c\in C} \|x - c\| \quad \forall\, x \in X.$$

For non-empty $B, C \subset X$,

$$d(B, C) := \sup_{b\in B} d_C(b).$$

The open ball of radius $r \ge 0$ centred at $z \in \mathbb{R}^N$ is denoted $\mathbb{B}_r(z)$ (with closure $\overline{\mathbb{B}}_r(z)$), to which the conventions $\mathbb{B}_0(z) := \emptyset$ and $\overline{\mathbb{B}}_0(z) := \{z\}$ apply; if $z = 0$, then we simply write $\mathbb{B}_r$ (respectively, $\overline{\mathbb{B}}_r$) in place of $\mathbb{B}_r(0)$ (respectively, $\overline{\mathbb{B}}_r(0)$). The boundary of a set Ω is denoted $\partial\Omega$. We write $\mathbb{R}_+ := [0, \infty)$.

Throughout, a sequence (x_n) is regarded as synonymous with a map $n \mapsto x_n$ with domain $\mathbb{N}$. We shall frequently extract subsequences of sequences. In order to avoid proliferation of subscripts, the notation $(x_{\sigma(n)})$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing map, is adopted to indicate a subsequence of (x_n). If $\big((x_{\sigma_k(n)})\big)_{k\in\mathbb{N}}$ is a sequence of subsequences of (x_n) nested in the following sense

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots ,$$

then σ_k is to be interpreted as a k-fold composition of strictly increasing maps $\mathbb{N} \to \mathbb{N}$, with $\sigma_k = \hat{\sigma}_k \circ \sigma_{k-1}$ for all $k \geq 2$: the sequence $(x_{\sigma_n(n)}) \subset (x_n)$ will be referred to as the diagonal sequence.

$AC(I;\mathbb{R}^N)$ denotes the space of functions $I \to \mathbb{R}^N$ defined on an interval I and absolutely continuous on compact subintervals thereof.

$\mathcal{U}(D)$ denotes the space of upper semicontinuous maps $x \mapsto F(x) \subset \mathbb{R}^N$, defined on $D \subset \mathbb{R}^N$, with non-empty convex compact values: if $D = \mathbb{R}^N$, then we simply write $\mathcal{U}$.

We record the following well-known facts (see, for example, [12]):

Proposition 2.1 *Let $F \in \mathcal{U}(D)$.*

(i) *If $K \subset D$ is compact, then $F(K)$ is compact.*
(ii) *For each $\epsilon > 0$, there exists locally Lipschitz $f_\epsilon \colon D \to \mathbb{R}^N$ such that*

$$d(\operatorname{graph}(f_\epsilon), \operatorname{graph}(F)) < \epsilon$$

(*any such f_ϵ is said to be an ϵ-approximate selection for F*).

3 Set-Valued Maps: Degree and the BKZ Property

If $F \in \mathcal{U}(D)$ is such that $F(D)$ contains an open neighbourhood of 0, then F is said to have the BKZ property.

Let $\mathcal{M} := \{(F,\Omega,p) \mid F \in \mathcal{U}(D),\ \Omega$ an open bounded subset of $D,\ p \in \mathbb{R}^N \backslash F(\partial\Omega)\}$. As discussed in [8] within the framework of [13] (see, also, [14−16]), there exists a map $\deg \colon \mathcal{M} \to \mathbb{Z}$ with the properties:

P1. $\deg(F,\Omega,p) = \deg_{\mathrm{B}}(f_\epsilon,\Omega,p)$ for all $\epsilon > 0$ sufficiently small, where $\deg_{\mathrm{B}}$ denotes Brouwer degree and $f_\epsilon \colon \overline{\Omega} \to \mathbb{R}^N$ is any ϵ-approximate selection for $F\big|_{\overline{\Omega}}$;
P2. if $q \colon [0,1] \to \mathbb{R}^N \backslash F(\partial\Omega)$ is continuous, then $\deg(F,\Omega,q(t))$ is independent of t;
P3. if $\deg(F,\Omega,p) \neq 0$, then $p \in F(x)$ for some $x \in \Omega$.

Lemma 3.1 *Let $(F,\Omega,0) \in \mathcal{M}$. If $\deg(F,\Omega,0) \neq 0$, then F has the BKZ property.*

Proof Since $0 \notin F(\partial\Omega)$, $d_{F(x)}(0) > 0$ for all $x \in \partial\Omega$. Let $(x_n) \subset \partial\Omega$ be a convergent sequence with limit $x \in \partial\Omega$. Let $(x_{\sigma(n)})$ be a subsequence with

$$\lim_{n\to\infty} d_{F(x_{\sigma(n)})}(0) = \liminf_{n\to\infty} d_{F(x_n)}(0).$$

For each n, let z_n be a minimizer of $\|\cdot\|$ over compact $F(x_{\sigma(n)})$ (and so $\|z_n\| = d_{F(x_{\sigma(n)})}(0)$). By upper semicontinuity of F, for each $\epsilon > 0$,

$$z_n \in F(x_{\sigma(n)}) \subset F(x) + \mathbb{B}_\epsilon.$$

By compactness of $F(x)$ and since $\epsilon > 0$ is arbitrary, we may conclude that (z_n) has a convergent subsequence (which we do not reliabel) with limit $z \in F(x)$. Therefore,

$$d_{F(x)}(0) \leq \|z\| = \lim_{n\to\infty} \|z_n\| = \liminf_{n\to\infty} d_{F(x_n)}(0)$$

and so $x \mapsto d_{F(x)}(0)$ is lower semicontinuous and positive-valued on compact $\partial\Omega$. It follows that there exists $\mu > 0$ such that $p \notin F(\partial\Omega)$ for all $p \in \mathbb{B}_\mu$. By properties P2 and P3,

$$p \in \mathbb{B}_\mu \quad \Longrightarrow \quad p \in F(x) \quad \text{for some} \quad x \in \Omega.$$

Therefore, F has the BKZ property.

4 Differential Inclusions

Let $F \in \mathcal{U}$ and consider the differential inclusion (subsuming (1))

$$\dot{x}(t) \in F(x(t)). \tag{3}$$

By an F-arc, we mean a function $x \in AC(I;\mathbb{R}^N)$ that satisfies (3) for almost all $t \in I$.

The following is a particular case of [17, Theorem 3.1.7].

Proposition 4.1 *Let $F \in \mathcal{U}$, let $K \subset \mathbb{R}^N$ be compact, let $I := [a,b]$, let $(\epsilon_n) \subset (0,\infty)$ be a decreasing sequence with $\epsilon_n \downarrow 0$ as $n \to \infty$ and, for each $n \in \mathbb{N}$, define $F_n\colon x \mapsto F(x) + \mathbb{B}_{\epsilon_n}$.*

Let sequence $(x_n) \subset AC(I;\mathbb{R}^N)$ be such that, for each $n \in \mathbb{N}$, x_n is an F_n-arc with $x_n(I) \subset K$. Then (x_n) has a subsequence that converges uniformly to an F-arc $x \in AC(I;\mathbb{R}^N)$.

Next, we prove (by arguments similar to those used in establishing [18, Lemma 5 (p.8)], see also remarks on page 78 therein) a variant of the above, tailored to our later purposes.

Proposition 4.2 *Let $F \in \mathcal{U}$ and let $(s_n) \subset [a,b]$ be a convergent sequence with limit $s \in (a,b]$. If $(x_n) \subset AC([a,b];\mathbb{R}^N)$ is a sequence of F-arcs and there exists $r > 0$ such that, for all $n \in \mathbb{N}$, $\|x_n(t)\| \le r$ for all $t \in [a,s_n]$, then (x_n) has a subsequence $(x_{\sigma(n)})$ such that $(x_{\sigma(n)}|_{[a,s]})$ converges to an F-arc $x \in AC([a,s];\mathbb{R}^N)$.*

Proof Let $(\delta_k) \subset (0, s-a)$ be a decreasing sequence with $\delta_k \downarrow 0$ as $k \to \infty$. Write $I_k := [a, s-\delta_k]$. By Proposition 4.1, the sequence (x_n) has a subsequence, which we label $(x_{\sigma_1(n)})$, such that $(x_{\sigma_1(n)}|_{I_1})$ converges uniformly to an F-arc $x^1 \in AC(I_1;\mathbb{R}^N)$. Again by Proposition 4.1, the sequence $(x_{\sigma_1(n)})$ has a subsequence, which we label $(x_{\sigma_2(n)})$, such that $(x_{\sigma_2(n)}|_{I_2})$ converges uniformly to an F-arc $x^2 \in AC(I_2;\mathbb{R}^N)$ (with $x^2|_{I_1} = x^1$). By induction, we generate a sequence of subsequences of (x_n),

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \cdots \supset (x_{\sigma_k(n)}) \supset \cdots$$

such that, for all k, $\big(x_{\sigma_k(n)}|_{I_k}\big)$ converges to an F-arc $x^k \in AC(I_k;\mathbb{R}^N)$ with $x^k|_{I_{k-1}} = x^{k-1}$ for all $k \ge 2$. Therefore, the diagonal sequence of restrictions to $[a,s)$, that is, the sequence $(x_{\sigma_n(n)}|_{[a,s)})$, converges to the F-arc $x\colon [a,s) \to \overline{\mathbb{B}}_r$ defined by the property:

$$\forall\, k \in \mathbb{N} \quad x(t) = x^k(t) \quad \forall\, t \in I_k = [a, s-\delta_k].$$

By compactness of $F(\overline{\mathbb{B}}_r)$, it follows that the bounded F-arc x is uniformly continuous and so extends to an F-arc on the closed interval $[a,s]$ by defining $x(s) := \lim_{t\uparrow s} x(t)$.

4.1 The initial-value problem

Let $F \in \mathcal{U}$. For each $x^0 \in \mathbb{R}^N$, the initial-value problem

$$\dot{x}(t) \in F(x(t)), \qquad x(0) = x^0 \tag{4}$$

has a solution and every solution can be extended to a maximal solution. By a solution, we mean an F-arc $x \in AC([0,\omega);\mathbb{R}^N)$, with $0 < \omega \le \infty$ and $x(0) = x^0$; by a maximal solution, we mean a solution having no proper right extension which is also a solution. Moreover, if $x\colon [0,\omega) \to \mathbb{R}^N$ is maximal and $\omega < \infty$, then $\limsup_{t\uparrow\omega} \|x(t)\| = +\infty$.

Proposition 4.3 *Let non-empty $K \subset \mathbb{R}^N$ be compact. Assume that, for each $x^0 \in K$, every maximal solution of* (4) *has interval of existence $\mathbb{R}_+$. For $T > 0$, define*

$$\Sigma_T(K) := \bigcup_{t\in[0,T]} \big\{x(t) \mid x \in AC([0,T];\mathbb{R}^N) \text{ is an } F\text{-arc with } x(0) \in K\big\} \subset \mathbb{R}^N$$

and write $\Sigma_\infty(K) := \bigcup_{T>0} \Sigma_T(K)$.

(a) *For all $T > 0$, the set $\Sigma_T(K)$ is compact.*

(b) *Let non-empty C_1, $C_2 \subset \mathbb{R}^N$ be compact, with $C_1 \subset C_2 \subset K$ and $C_1 \cap \partial C_2 = \emptyset = K \cap \partial C_2$. Assume that, for every maximal solution x of* (4) *with $x^0 \in K$, $d_{C_1}(x(t)) \to 0$ as $t \to \infty$. Then there exists $T > 0$ such that $\Sigma_T(K) = \Sigma_\infty(K)$ and, for all $x^0 \in \Sigma_\infty(K)$, every maximal solution x of* (4) *has interval of existence $\mathbb{R}_+$ and has the properties:*
   (i) *$x(\mathbb{R}_+) \subset \Sigma_\infty(K)$;*
   (ii) *$x(t) \in C_2$ for some $t \in [0,T]$.*

Proof (a) Let $T > 0$ be arbitrary. Seeking a contradiction, suppose that $\Sigma_T(K)$ is unbounded. Then there exist a constant $\delta > 0$, a sequence $(t_n) \subset [0,T]$ and a sequence (x_n) of maximal solutions of (4) such that

$$x_n(0) \in K \quad \text{and} \quad \|x_n(t_n)\| > (n+1)\delta \quad \forall\, n \in \mathbb{N}.$$

By continuity of the solutions, it follows that, for each $n \in \mathbb{N}$, there exist s_n^k, $k = 1,\dots,n$, such that

$$\|x_n(s_n^k)\| = (k+1)\delta \quad \text{and} \quad \|x_n(t)\| < (k+1)\delta \quad \forall\, t \in [0, s_n^k) \tag{5}$$

and $s_n^1 < s_n^2 < \dots < s_n^n$ for all $n \geq 2$.

From (s_n^1), extract a convergent subsequence $(s^1_{\sigma_1(n)})$ with limit $s^1 \in [0,T]$. By compactness of $F(\overline{\mathbb{B}}_{2\delta}(0))$, $s^1 > 0$. Write $I_1 := [0, s^1]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_1(n)}|_{I_1})$ converges uniformly to an F-arc $x^1 \in AC(I_1;\mathbb{R}^N)$; moreover, by (5), $\|x^1(s^1)\| = 2\delta$. From $(s^2_{\sigma_1(n)})$, extract a subsequence $(s^2_{\sigma_2(n)})$ with limit $s^2 \in [0,T]$. By compactness of $F(\overline{\mathbb{B}}_{3\delta}(0))$, $s^2 > s^1$. Write $I_2 := [0, s^2]$. By Proposition 4.2, and passing to a subsequence if necessary, we may assume that $(x_{\sigma_2(n)}|_{I_2})$ converges uniformly to an F-arc $x^2 \in AC(I_2;\mathbb{R}^N)$ with $x^2|_{I_1} = x^1$; moreover, by (5), $\|x^2(s^2)\| = 3\delta$. By induction, we generate a strictly increasing sequence $(s^k) \subset [0,T]$, with limit $s \in [0,T]$, and a sequence of subsequences of (x_n),

$$(x_n) \supset (x_{\sigma_1(n)}) \supset \dots \supset (x_{\sigma_k(n)}) \supset \cdots$$

such that the diagonal sequence of restricted functions $(x_{\sigma_n(n)}|_I)$, where $I := [0, s)$, converges to the F-arc $x \in AC(I;\mathbb{R}^N)$ defined by the property that, for each $k \in \mathbb{N}$,

$$x(t) = x^k(t) \quad \forall\, t \in I_k := [0, s^k].$$

Clearly, $x(0) \in K$. Furthermore, $\|x(s^k)\| = (k+1)\delta$ for all $k \in \mathbb{N}$ and so x has no proper right extension that is also an F-arc. This contradicts the hypothesis that all maximal solutions of (4), with $x^0 \in K$, have interval of existence $\mathbb{R}_+$. Therefore, $\Sigma_T(K)$ is bounded.

Let $(y_n) \subset \Sigma_T(K)$ be a convergent sequence with limit y. Then $y_n = x_n(t_n)$ for some sequence $(t_n) \subset [0,T]$ and some sequence of F-arcs $(x_n) \subset AC([0,T];\mathbb{R}^N)$ with $x_n(0) \in K$ for all n. Without loss of generality, we may assume that (t_n) is convergent, with limit $t \in [0,T]$. By boundedness of $\Sigma_T(K)$, there exists compact C such that $x_n([0,T]) \subset C$ for all n. By Proposition 4.1, passing to a subsequence if necessary, we may assume that (x_n) converges uniformly to an F-arc $x \in AC([0,T];\mathbb{R}^N)$, with $x(0) \in K$. Therefore,

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n(t_n) = x(t) \in \Sigma_T(K),$$

and so $\Sigma_T(K)$ is closed.

(b) It suffices to show that there exists $T > 0$ such that, for every maximal solution x of (4), with $x^0 \in K$, $x(t) \in C_2$ for some $t \in [0,T]$ (in which case, $\Sigma_T(K) = \Sigma_\infty(K)$). Seeking a contradiction, suppose that no such T exists. Then there is a sequence $(x_n) \subset AC(\mathbb{R}_+;\mathbb{R}^N)$ such that, for each $n \in \mathbb{N}$, $x_n(0) \in K$ and $d_{C_2}(x_n(t) > 0$ for all $t \in I_n := [0,n]$. By part (a) above, for each $k \in \mathbb{N}$, the sequence $(x_n|_{I_k})$ is bounded. Therefore, repeated application of Proposition 4.1 yields a sequence of subsequences $(x_n) \supset (x_{\sigma_1(n)}) \supset (x_{\sigma_2(n)}) \cdots$ such that, for each $k \in \mathbb{N}$, the sequence $(x_{\sigma_k(n)}|_{I_k})$ converges uniformly to an F-arc $x^k \in AC(I_k;\mathbb{R}^N)$ with $d_{C_2}(x^k(t)) \geq 0$ for all $t \in I_k$. It follows that the diagonal sequence $(x_{\sigma_n(n)})$ converges to the F-arc $x \in AC(\mathbb{R}_+;\mathbb{R}^N)$ defined by the property that, for each $k \in \mathbb{N}$, $x(t) = x^k(t)$ for all $t \in I_k$. Therefore, $d_{C_2}(x(t)) \geq 0$ for all $t \in \mathbb{R}_+$, which contradicts the hypothesis that every maximal solution approaches $C_1 \subset C_2$ (recall that $C_1 \cap \partial C_2 = \emptyset$).

Remark 4.1 Proposition 4.3(a) is closely akin to [18, Theorem 3 (p.79)]. Proposition 4.3(b-i) is essentially an assertion that $\Sigma_\infty(K)$ is compact and is an invariant set for (4) in the sense that, for each $x^0 \in \Sigma_\infty(K)$, every maximal solution of (4) has trajectory in $\Sigma_\infty(K)$. A similar observation occurs in the proof of [7, Theorem 11].

4.2 Persistence of the BKZ property

The following is essentially Theorem 1 of [8].

Theorem 4.1 *Let $F \in \mathcal{U}$. If there exist $0 < \tau < \delta < \rho$ and $T > 0$ such that*

$$\|x^0\| \leq \delta \quad \Longrightarrow \quad \begin{cases} \|x(t)\| \leq \rho & \forall\, t \in [0,T] \\ \|x(t)\| \leq \tau & \forall\, t \in [T,2T] \end{cases}$$

for every maximal solution x of (4)*, then F has the BKZ property.*

In view of Lemma 3.1, to prove this result it suffices to show that $\deg(F, \mathbb{B}_\delta, 0) \neq 0$. In the Appendix, we provide a proof which incorporates minor corrections to the proof in [8].

In what follows, several specific consequences of the above result are highlighted: simply stated, the first of these (Theorem 4.2) asserts that, if there exists a compact set that attracts all maximal solutions of (4), then F has the BKZ property.

A non-empty set $C \subset \mathbb{R}^N$ is said to be attractive for (4) if there exists an open neighbourhood $\mathcal{N}$ of C (that is, an open set containing the closure of C) with the property that, for each $x^0 \in \mathcal{N}$, every maximal solution $x\colon [0,\omega) \to \mathbb{R}^N$ of (4) is such that $d_C(x(t)) \to 0$ as $t \uparrow \omega$ (if C is compact, then $\omega = \infty$): C is globally attractive if the latter property holds with $\mathcal{N} = \mathbb{R}^N$. Non-empty C is said to be stable for (4) if, for each open neighbourhood $\mathcal{N}_1$ of C, there is an open neighbourhood $\mathcal{N}_2$ of C such that, for each $x^0 \in \mathcal{N}_2$, every maximal solution of (4) has trajectory in $\mathcal{N}_1$.

Theorem 4.2 *Let $F \in \mathcal{U}$. Let $C \subset \mathbb{R}^N$ be non-empty and compact. If C is globally attractive for* (4)*, then F has the BKZ property.*

Proof By global attractivity of compact C, every maximal solution of (4) has interval of existence $\mathbb{R}_+$. Fix $r > 0$ such that $\overline{\mathbb{B}}_r \supset C$. By Proposition 4.3, the set $\Sigma_\infty(\overline{\mathbb{B}}_{3r})$ is compact and positively invariant.

Let $\tau > 3r$ be sufficiently large so that $\Sigma_\infty(\overline{\mathbb{B}}_{3r}) \subset \overline{\mathbb{B}}_\tau$ and choose $\delta > \tau$. By Proposition 4.3(b), there exists $T > 0$ such that, for every F-arc $x \in AC(\mathbb{R}_+;\mathbb{R}^N)$ with $\|x(0)\| \le \delta$, $\|x(t)\| \le 3r$ for some $t \in [0,T]$. Since $\overline{\mathbb{B}}_{3r} \subset \Sigma_\infty(\overline{\mathbb{B}}_{3r})$, it follows that, for each x^0,

$$\|x^0\| \le \delta \quad \Longrightarrow \quad x(t) \in \Sigma_\infty(\overline{\mathbb{B}}_{3r}) \quad \text{for some} \quad t \in [0,T]$$

for every maximal solution of (4). Therefore, by (positive) invariance of $\Sigma_\infty(\overline{\mathbb{B}}_{3r}) \subset \overline{\mathbb{B}}_\tau$,

$$\|x^0\| \le \delta \quad \Longrightarrow \quad \|x(t)\| \le \tau \quad \forall\, t \in [T,\infty)$$

for every maximal solution of (4).

By Proposition 4.3(a), there exists $\rho > \delta$ such that

$$\|x^0\| \le \delta \quad \Longrightarrow \quad \|x(t)\| \le \rho \quad \forall\, t \in [0,T].$$

Therefore, the hypotheses of Theorem 4.1 hold and so the result follows.

Next, we highlight a further consequence of the above theorem which, for example, implies that, if (1) generates a global semiflow and is L^p stable in the sense that all solutions are of class L^p for some $1 \le p < \infty$, then f has the BKZ property.

Corollary 4.1 *Let $F \in \mathcal{U}$. Let $g\colon \mathbb{R}^N \to \mathbb{R}_+$ be lower semicontinuous with the properties:*

(a) $C := g^{-1}(0)$ *is compact;*
(b) $\inf\limits_{z\in K} g(z) > 0$ *for any closed set $K \subset \mathbb{R}^N$ with $K \cap C = \emptyset$.*

If, for each $x^0 \in \mathbb{R}^N$, every maximal solution of (4) *has interval of existence $\mathbb{R}_+$ and $\int_0^\infty g(x(t))\,dt < \infty$, then F has the BKZ property.*

Proof By [19, Theorem 10 (i)], the compact set $C = g^{-1}(0)$ is globally attractive for (4) and the result follows by Theorem 4.2.

In Theorem 4.2, in order to conclude that F has the BKZ property, hypotheses of a global nature were imposed (global in the sense that, for each $x^0 \in \mathbb{R}^N$, every maximal solution was posited to approach C). The following theorem imposes hypotheses of a local nature under which the BKZ property again persists: in particular, if there exists a closed ball that is locally asymptotically stable for (4), then F has the BKZ property.

Theorem 4.3 *If there exists a closed ball* $\overline{\mathbb{B}}_r(z) =: B$ *which is both stable and attractive for* (4)*, then* F *has the BKZ property.*

Proof Without loss of generality, we may assume $z = 0$ and so $B = \overline{\mathbb{B}}_r \equiv \overline{\mathbb{B}}_r(0)$. By stability and attractivity of compact B, there exist $\alpha, \beta \in \mathbb{R}_+$ such that, for all $x^0 \in \mathbb{R}^N$,

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad \begin{cases} d_B(x(t)) \le \beta & \forall\, t \in \mathbb{R}_+ \\ d_B(x(t)) \to 0 & \text{as} \quad t \to \infty \end{cases}$$

for every maximal solution of (4). Let $\gamma \in (0, \alpha)$ be arbitrary. By stability of B, there exists $\mu \in (0, \gamma)$ such that, for all x^0,

$$d_B(x^0) \le \mu \quad \Longrightarrow \quad d_B(x(t)) \le \gamma \quad \forall\, t \in \mathbb{R}_+ \tag{6}$$

for every maximal solution of (4). By Proposition 4.3(b), there exists $T > 0$ such that, for all x^0,

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad d_B(x(t)) \le \mu \quad \text{for some} \quad t \in [0, T]$$

which, together with (6), yields

$$d_B(x^0) \le \alpha \quad \Longrightarrow \quad d_B(x(t)) \le \gamma \quad \forall\, t \ge T$$

for every maximal solution x of (4). We may now conclude that the hypotheses of Theorem 4.1 hold (with $\tau = \gamma + r$, $\delta = \alpha + r$ and $\rho = \beta + r$) and the proof is complete.

5 Feedback Control

We now turn to the main concern of the paper, namely, the consequences of the above results in a context of feedback control systems.

Let $f \colon \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ be continuous and consider the controlled system

$$\dot{x} = f(x, u). \tag{7}$$

Henceforth, we assume that f has the property that, for every non-empty convex set $C \subset \mathbb{R}^M$, the set $f(x, C) \subset \mathbb{R}^N$ is convex for all $x \in \mathbb{R}^N$.

As *admissible feedback controls* for (7), we take the class $\mathcal{K}$ of upper semicontinuous maps $x \mapsto k(x) \subset \mathbb{R}^M$ on $\mathbb{R}^N$, with non-empty convex and compact values. Therefore, for every feedback $k \in \mathcal{K}$, the map $F_k \colon x \mapsto f(x, k(x))$ is of class $\mathcal{U}$.

5.1 Persistence of the BKZ property in feedback systems

For system (7), a feedback $k \in \mathcal{K}$ is said to render a compact set $C \subset \mathbb{R}^N$ stable (respectively, attractive) if C is stable (respectively, attractive) for (4) with $F = F_k$.

The following theorem and corollary are immediate consequences of Theorem 4.2 and Corollary 4.1.

Theorem 5.1 *Let* $k \in \mathcal{K}$ *and let* $C \subset \mathbb{R}^N$ *be non-empty and compact. If either of the following holds, then* f *has the BKZ property:*

(i) k *renders* C *globally attractive for* (4)*;*
(ii) k *renders some closed ball* B *stable and attractive for* (4)*.*

Corollary 5.1 *Let $k \in \mathcal{K}$ and let $g\colon \mathbb{R}^N \to \mathbb{R}_+$ be as in Corollary 4.1. If, for each $x^0 \in \mathbb{R}^N$, every maximal solution of* (4) *with $F = F_k$ has interval of existence $\mathbb{R}_+$ and $g \circ x \in L^1(\mathbb{R}_+)$, then f has the BKZ property.*

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Appendix: Proof of Theorem 4.1

Let $D := \overline{\mathbb{B}}_\rho$ and let $\hat{F} \in \mathcal{U}(D)$ denote the restriction of $F \in \mathcal{U}$ to D.

Observe that $0 \notin \hat{F}(\partial\mathbb{B}_\delta)$ (otherwise, there exists a constant solution $t \mapsto x^0$ of (4) with $\|x^0\| = \delta$, contradicting the hypotheses). Therefore $\deg(\hat{F}, \mathbb{B}_\delta, 0)$ is well-defined and, in view of Lemma 3.1, to complete the proof it suffices to show that $\deg(\hat{F}, \mathbb{B}_\delta, 0) \neq 0$.

By Proposition 2.1(ii) and property P1 of degree, there exists a sequence (f_n) of locally Lipschitz functions $D \to \mathbb{R}^N$ such that:

$$\begin{aligned} \deg(\hat{F}, \mathbb{B}_\delta, 0) &= \deg_{\mathrm{B}}(f_n, \mathbb{B}_\delta, 0) \quad \forall\, n\,; \\ d(\operatorname{graph}(f_n), \operatorname{graph}(\hat{F})) &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{8}$$

By compactness of $\hat{F}(D)$, the functions f_n are bounded and so, for each n, the equation $\dot{x} = f_n(x)$ generates a semiflow $\varphi_n \colon \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$.

Write $I := [0, 2T]$ and $X := C(I; \mathbb{R}^N)$ (with the uniform norm). On $\overline{\mathbb{B}}_\delta$ define

$$\mathcal{F} \colon x^0 \mapsto \{x \in X \mid x \text{ an } \hat{F}\text{-arc with } x(0) = x^0\}$$

with $\operatorname{graph}(\mathcal{F}) := \{(x^0, x) \mid x^0 \in \overline{\mathbb{B}}_\delta,\ x \in \mathcal{F}(x^0)\}$. For each n, define $\phi_n \colon \overline{\mathbb{B}}_\delta \to X$ by

$$(\phi_n(x^0))(t) := \varphi_n(t, x^0) \quad \forall\, t \in I.$$

Fix ϵ such that $0 < \epsilon < \delta - \tau$. We claim that

$$d(\operatorname{graph}(\phi_m), \operatorname{graph}(\mathcal{F})) < \epsilon \quad \text{for some} \quad m \in \mathbb{N}. \tag{9}$$

Suppose otherwise. Then there exists a sequence $(x_n^0) \subset \overline{\mathbb{B}}_\delta$ such that

$$d_{\operatorname{graph}(\mathcal{F})}((x_n^0, \phi_n(x_n^0))) \geq \epsilon \quad \forall\, n. \tag{10}$$

By Proposition 4.1, we may assume (without loss of generality) that $(\phi(x_n^0)) \subset X$ converges uniformly to an $\hat{F}$-arc $x \in AC(I; \mathbb{R}^N)$ with $x(0) \in \overline{\mathbb{B}}_\delta$ (and so $(x(0), x) \in \operatorname{graph}(\mathcal{F})$), which contradicts (10). Therefore, (9) is true.

Let $x^0 \in \overline{\mathbb{B}}_\delta$ be arbitrary. By (9), there exists $y^0 \in \overline{\mathbb{B}}_\delta$, with $\|x^0 - y^0\| < \epsilon$, and $y \in \mathcal{F}(y^0)$ such that $\|\varphi_m(t, x^0) - y(t)\| < \epsilon$ for all $t \in I$. Since the set $\{y(t) \mid y \in \mathcal{F}(\overline{\mathbb{B}}_\delta)\}$ lies in the ball $\mathbb{B}_\tau$ for all $t \in [T, 2T]$, we may conclude:

$$\text{for all} \quad x^0 \in \overline{\mathbb{B}}_\delta, \quad \varphi_m(t, x^0) \in \mathbb{B}_\delta \quad \text{for all} \quad t \in [T, 2T]. \tag{11}$$

Define continuous $h \colon [0,1] \times \overline{\mathbb{B}}_\delta \to \mathbb{R}^N$ by

$$h(s, x^0) := \begin{cases} f_m(x^0), & s = 0 \\ \frac{1}{sT}\left[(\phi_m(x^0))(sT) - x^0\right], & 0 < s \leq 1. \end{cases}$$

We conclude that $h(s, x^0) \neq 0$ for all $(s, x^0) \in [0,1] \times \partial\mathbb{B}_\delta$ by the following argument. Suppose $h(0, x^0) = f_m(x^0) = 0$ for some $x^0 \in \partial\mathbb{B}_\delta$. Then, $\varphi_m(t, x^0) = x^0 \in \partial\mathbb{B}_\delta$ for all $t \in I$, which contradicts (11). Now suppose $h(s, x^0) = 0$ for some $(s, x^0) \in (0,1] \times \partial\mathbb{B}_\delta$. Then $\varphi_m(nsT, x^0) = x^0 \in \partial\mathbb{B}_\delta$ for all $n \in \mathbb{N}$ with $ns \leq 2$. In particular, there exists $n \in \mathbb{N}$ such that $1 \leq ns \leq 2$ and $\varphi_m(nsT, x^0) = x^0 \in \partial\mathbb{B}_\delta$. This contradicts (11).

Therefore, by (8) and the homotopic invariance property of the Brouwer degree,

$$\begin{aligned} \deg(\hat{F}, \mathbb{B}_\delta, 0) &= \deg_{\mathrm{B}}(f_m, \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(h(0, \cdot), \mathbb{B}_\delta, 0) \\ &= \deg_{\mathrm{B}}(h(1, \cdot), \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(g_m, \mathbb{B}_\delta, 0), \end{aligned}$$

where, for notational convenience, g_m denotes the function

$$g_m : x^0 \mapsto [(\phi_m(x^0))(T) - x^0]/T.$$

Now consider the continuous map

$$h_0 : [0,1] \times \overline{\mathbb{B}}_\delta, \qquad (s, x^0) \mapsto (1-s) g_m(x^0) - s x^0.$$

Noting that h_0 is a homotopic connection of the function g_m and the odd map $o : x^0 \mapsto -x^0$ and $h_0(s, x^0) \neq 0$ for all $(s, x^0) \in [0,1] \times \partial\mathbb{B}_\delta$ by properties of the Brouwer degree, we may now conclude that

$$\deg(\hat{F}, \mathbb{B}_\delta, 0) = \deg_{\mathrm{B}}(o, \mathbb{B}_\delta, 0) \neq 0.$$

This completes the proof.