Nonlinear Dynamics and Systems Theory, **3**(1) (2003) 23–41

Existence and Uniqueness for the Weak Solutions of the Ginzburg-Landau Equations

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Received: February 14, 2002; Revised: December 13, 2002

Abstract: In this paper we study a gauge-invariant Ginzburg-Landau model which describes the phenomenon of the superconductivity, characterizing the state of the material by means of observable variables. We give a definition of weak solutions for the steady and the time-dependent Ginzburg-Landau equations and prove theorems of existence and uniqueness.

Keywords: *Superconductivity, Ginzburg-Landau equations.*

Mathematics Subject Classification (2000): 82D55, 35A05.

1 Introduction

The Ginzburg-Landau theory gives a macroscopic model which explains the main experimental phenomena related to the superconductivity, i.e. the absence of electrical resistance and the Meissner effect ([5, 13]). In their model, Ginzburg and Landau describe the behaviour of a superconducting material in steady conditions, through the introduction of a free energy functional and assume that the state of the system minimizes such a functional. They identify the state of the superconductor with the pair $(\psi, \mathbf{A})$, where ψ is a complex order parameter, whose squared modulus coincides with the number density of the superconducting electrons and $\mathbf{A}$ is the vector magnetic potential.

Later, the model was extended to the non stationary case by Gor'kov and Èliashberg [8], who deduce the time-dependent Ginzburg-Landau equations from the microscopic theory BCS. Such equations constitute a non linear differential system for which theorems of existence and uniqueness are proved ([4, 12, 15]).

Recently, Fabrizio [6, 7] has proposed a macroscopic model which characterizes the state of the material by means of real and observable variables. Therefore, while in the

†Research performed under the auspices of G.N.F.M. - I.N.D.A.M. and partially supported by Italian M.I.U.R. through the project "*Mathematical Methods in Continuum Mechanics*".

classical formulation the unknown quantities are defined up to a gauge transformation, the variables involved in this model have a well determined physical meaning, so that they are gauge-invariant.

In this paper, the real Ginzburg-Landau equations are studied both in the steady and in the time-dependent case. The two models are presented in Section 2. In Section 3, we introduce a new definition of weak solutions which allows to prove existence and uniqueness theorems. In the steady case, the uniqueness of the weak solution is shown, provided that the coefficients of the equations and the domain occupied by the superconducting material are sufficiently small. In the time-dependent case, the uniqueness is proved in two-dimensional domains, with L^2 initial data.

Both in the stationary and in the time-dependent problem the results are obtained with the same method used in [1] and [2], namely by introducing a suitable decomposition of the unknown variables and reducing the original system to an equivalent one.

2 Ginzburg-Landau Model of Superconductivity

The electromagnetic behaviour of a superconducting material is described by Maxwell equations

$$\varepsilon\,\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \quad \nabla \cdot \mathbf{E} = \rho, \tag{2.1}$$

$$\mu\,\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{H} = 0, \tag{2.2}$$

where ε, μ, ρ are respectively the dielectric constant, the magnetic permeability and the charge density. For simplicity ε and μ are assumed constant.

According to London theory, the electrons in a superconductor behave like a fluid which may appear either in the normal or in the superconducting phase. Therefore, the current density $\mathbf{J}$ inside the material can be expressed through the sum

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s \tag{2.3}$$

of the normal and the superconducting current. The conduction current $\mathbf{J}_n$ is required to obey Ohm's law

$$\mathbf{J}_n = \sigma \mathbf{E}, \tag{2.4}$$

while the superconducting current satisfies London constitutive equation

$$\nabla \times (\Lambda \mathbf{J}_s) = -\mu \mathbf{H}, \qquad L = \frac{m}{e^2 f^2}, \tag{2.5}$$

where m, e, f^2 denote respectively the mass, the charge and the number density of the superconducting electrons.

By means of the equation (2.5), London theory describes the superconducting features of a material in the hypothesis that the parameter Λ is constant, so that the density of superconducting electrons is uniform. However, near the transition temperature there occurs a mixed state consisting of alternating domains of normal and superconducting phase. Therefore the material cannot be considered spatially homogeneous.

The Ginzburg-Landau model extends London theory since it allows spatial variations of the density of superconducting electrons. In the stationary case and without free charge, Maxwell equations $(2.1)_2$, $(2.2)_1$ together with the boundary condition

$$\mathbf{E}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{2.6}$$

imply $\mathbf{E}=\mathbf{0}$, so that the system (2.1)–(2.2) reduces to

$$\nabla\times\mathbf{H}=\mathbf{J}_s, \qquad \nabla\cdot\mathbf{H}=0. \tag{2.7}$$

Since the electric field can be neglected, the state of the material is identified with the pair $(f,\mathbf{J}_s)$.

According to the Ginzburg-Landau theory, the material is in a state which minimizes the free energy. If we denote by Ω the domain occupied by the superconductor and by $\partial\Omega$ its boundary, the free energy can be written as a functional of the variables $(f,\mathbf{J}_s)$ in the form ([6, 7])

$$\begin{aligned}\mathcal{E}(f,\mathbf{J}_s)=&\int\limits_\Omega\Big[-\alpha f^2+\frac{\beta}{2}f^4+\frac{1}{2\mu}|\nabla\times(\Lambda(f)\mathbf{J}_s)|^2+\frac{\hbar^2}{2m}|\nabla f|^2\Big]dx\\&+\int\limits_\Omega\frac{1}{2}\Lambda(f)\mathbf{J}_s^2\,dx-2\int\limits_{\partial\Omega}\Lambda(f)\mathbf{J}_s\times\mathbf{H}_{ex}^\tau\cdot\mathbf{n}\,d\sigma,\end{aligned} \tag{2.8}$$

where α, β are positive constants depending on the temperature, $\hbar$ is the Planck constant and $\mathbf{H}_{ex}^\tau$ is the tangential component of the external magnetic field.

Henceforth, we consider external magnetic fields $\mathbf{H}_{ex}$ which satisfy the relation

$$\int\limits_{\partial\Omega}\nabla\varphi\times\mathbf{H}_{ex}\cdot\mathbf{n}\,d\sigma=0, \tag{2.9}$$

where φ is the trace on $\partial\Omega$ of an arbitrary function φ.

By introducing the quantity

$$\mathbf{p}_s=\Lambda(f)\mathbf{J}_s=\frac{m\mathbf{v}_s}{e}, \tag{2.10}$$

identified with the linear momentum of the superconducting electrons per unit charge, the free energy (2.8) can be expressed in terms of the variables $(f,\mathbf{p}_s)$ in the form

$$\begin{aligned}\mathcal{E}(f,\mathbf{p}_s)=&\int\limits_\Omega\Big[-\alpha f^2+\frac{\beta}{2}f^4+\frac{1}{2\mu}|\nabla\times\mathbf{p}_s|^2+\frac{\hbar^2}{2m}|\nabla f|^2+\frac{e^2}{2m}f^2\mathbf{p}_s^2\Big]dx\\&-2\int\limits_{\partial\Omega}\mathbf{p}_s\cdot\mathbf{H}_{ex}^\tau\times\mathbf{n}\,d\sigma.\end{aligned} \tag{2.11}$$

The stationariety of the functional (2.11) with respect to $(f,\mathbf{p}_s)$ leads to the system

$$\frac{\hbar^2}{2m}\nabla^2 f-\frac{e^2}{2m}f\mathbf{p}_s^2+\alpha f-\beta f^3=0, \tag{2.12}$$

$$\nabla\times\nabla\times\mathbf{p}_s+\frac{\mu e^2}{m}f^2\mathbf{p}_s=\mathbf{0}, \tag{2.13}$$

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega}=0, \quad (\nabla\times\mathbf{p}_s)\times\mathbf{n}|_{\partial\Omega}=\mu\mathbf{H}_{ex}\times\mathbf{n}|_{\partial\Omega}. \tag{2.14}$$

In order to reduce our notations, we introduce the following non-dimensional quantities

$$f = \left(\frac{\alpha}{\beta}\right)^{1/2} f', \qquad \mathbf{p}_s = \left(\frac{2m\alpha}{e^2}\right)^{1/2} \mathbf{p}_s', \tag{2.15}$$

$$x = \left(\frac{m\beta}{e^2\alpha\mu}\right)^{1/2} x', \qquad \mathbf{H}_{ex} = \left(\frac{2\alpha^2}{\beta\mu}\right)^{1/2} \mathbf{H}_{ex}', \tag{2.16}$$

$$\mathcal{E} = \left(\frac{m\alpha^3}{\beta^2\mu}\right)^{1/2} \mathcal{E}', \qquad k = \left(\frac{2m^2\beta}{\hbar^2 e^2\mu}\right)^{1/2}. \tag{2.17}$$

With such positions, dropping the primes, the free energy (2.11) assumes the form

$$\begin{aligned} \mathcal{E}(f, \mathbf{p}_s) = \int_\Omega \left[\frac{1}{2}(f^2-1)^2 + |\nabla \times \mathbf{p}_s|^2 + \frac{1}{k^2}|\nabla f|^2 + f^2\mathbf{p}_s^2\right] dx \\ - 2\int_{\partial\Omega} \mathbf{p}_s \cdot \mathbf{H}_{ex}^\tau \times \mathbf{n}\, d\sigma \end{aligned} \tag{2.18}$$

and the ensuing Ginzburg-Landau system is

$$\frac{1}{k^2}\nabla^2 f - f\mathbf{p}_s^2 + f - f^3 = 0, \tag{2.19}$$

$$\nabla \times \nabla \times \mathbf{p}_s + f^2\mathbf{p}_s = \mathbf{0}, \tag{2.20}$$

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{p}_s) \times \mathbf{n}|_{\partial\Omega} = \mathbf{H}_{ex}^\tau \times \mathbf{n}|_{\partial\Omega}. \tag{2.21}$$

Moreover, we assume the boundary condition

$$\mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

The generalization of the Ginzburg-Landau model to the time-dependent case is obtained by introducing a further variable ϕ_s which is related to the charge density ρ. In non-dimensional variables the time-dependent Ginzburg-Landau equations are ([6, 7])

$$\frac{\partial f}{\partial t} - \nabla^2 f + k^2(f^2-1)f + f\mathbf{p}_s^2 = 0, \tag{2.22}$$

$$\eta\,\frac{\partial \mathbf{p}_s}{\partial t} + \nabla \times \nabla \times \mathbf{p}_s + \eta\nabla\phi_s + f^2\mathbf{p}_s = \mathbf{0}, \tag{2.23}$$

with boundary conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla \times \mathbf{p}_s - \mathbf{H}_{ex}^\tau) \times \mathbf{n}|_{\partial\Omega} = 0 \tag{2.24}$$

and initial data

$$f(x,0) = f_0(x), \quad \mathbf{p}_s(x,0) = \mathbf{p}_{s0}(x). \tag{2.25}$$

As in the steady model, the equation (2.23) can be obtained from Maxwell equation $(2.1)_1$. However, (2.23) coincides with $(2.1)_1$ only if the time derivative of the electric field is negligible. In such a case, the equation $(2.1)_1$ assumes the form

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \mathbf{J}_s \tag{2.26}$$

and, by using London equation (2.5), we get

$$\nabla \times \nabla \times \mathbf{p}_s + \mu\sigma \mathbf{E} + \frac{\mu e^2}{m} f^2 \mathbf{p}_s = \mathbf{0}. \tag{2.27}$$

On the other hand, the equation $(2.2)_1$ yields

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{p}_s) - \nabla \times \mathbf{E} = \mathbf{0},$$

hence there exists a function ϕ_s such that

$$\mathbf{E} = \frac{\partial}{\partial t} \mathbf{p}_s + \nabla \phi_s. \tag{2.28}$$

Substitution in (2.23) leads to the equation

$$\mu\sigma \frac{\partial}{\partial t} \mathbf{p}_s + \nabla \times \nabla \times \mathbf{p}_s + \mu\sigma \nabla \phi_s + \frac{\mu e^2}{m} f^2 \mathbf{p}_s = \mathbf{0},$$

which coincides (in non-dimensional form) with (2.23).

We assume the following constitutive equation for ϕ_s

$$f^2 \phi_s = -\frac{\hbar^2}{2m\tau} \nabla \cdot (f^2 \mathbf{p}_s), \tag{2.29}$$

or in non-dimensional form

$$f^2 \phi_s = -\nabla \cdot (f^2 \mathbf{p}_s). \tag{2.30}$$

The choice of the equation (2.29) corresponds to a particular choice of the charge density ρ. Indeed, the relation (2.26) implies

$$\nabla \cdot \mathbf{J}_s = -\nabla \cdot \mathbf{J}_n = -\sigma \nabla \cdot \mathbf{E} = -\frac{\sigma}{\varepsilon} \rho,$$

so that, by substituting in (2.29), we obtain

$$f^2 \phi_s = -\frac{\hbar^2}{2e^2\tau} \nabla \cdot \mathbf{J}_s = \frac{\sigma \hbar^2}{2e^2 \varepsilon \tau} \rho.$$

3 Existence and Uniqueness of Solutions

3.1 The stationary case

In this section we prove that the functional (2.18) admits at least a minimizer and, under suitable hypotheses on the coefficients of the equations, such a minimizer is unique.

Let $\mathcal{D}(\Omega)$ be the domain of the functional (2.18), constituted by the pairs $(f, \mathbf{p}_s)$ such that the free energy is finite, namely

$$\mathcal{D}(\Omega) = \big\{(f, \mathbf{p}_s)\colon f \in \mathrm{H}^1(\Omega),\ \nabla \times \mathbf{p}_s \in \mathbf{L}^2(\Omega),\ f\mathbf{p}_s \in \mathbf{L}^2(\Omega)\big\}.$$

We introduce a new variable $\mathbf{p}_1$ which satisfies

$$\nabla \times \mathbf{p}_1 = \nabla \times \mathbf{p}_s, \qquad \nabla \cdot \mathbf{p}_1 = 0, \qquad \mathbf{p}_1 \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{3.1}$$

Therefore $\mathbf{p}_s$ can be decomposed as

$$\mathbf{p}_s = \mathbf{p}_1 + \frac{1}{k}\nabla\theta, \tag{3.2}$$

with $\mathbf{p}_1 \in \mathcal{R}_0(\Omega) = \{\mathbf{v} \colon \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega),\ \nabla \cdot \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n}|_\Omega = 0\}$.

Observe that for each $\mathbf{p}_s$ such that $\nabla \times \mathbf{p}_s \in \mathbf{L}^2(\Omega)$, there exist unique $\mathbf{p}_1$ and $\nabla\theta$ which satisfy (3.1), (3.2) and $\nabla\theta \cdot \mathbf{n}|_\Omega = 0$. Moreover, the condition $\mathbf{p}_1 \in \mathcal{R}_0(\Omega)$ implies $\mathbf{p}_1 \in \mathbf{H}^1(\Omega)$.

With such positions, the functional (2.11) can be expressed in terms of the variables $(f, \theta, \mathbf{p}_1)$ as

$$\begin{aligned}\mathcal{E}(f, \theta, \mathbf{p}_1) = \int_\Omega \left[\frac{1}{2} f^4 - f^2 + |\nabla \times \mathbf{p}_1|^2 + \frac{1}{k^2}|\nabla f|^2 + f^2\left|\mathbf{p}_1 + \frac{1}{k}\nabla\theta\right|^2\right] dx \\ - 2\int_{\partial\Omega} \mathbf{p}_1 \cdot \mathbf{H}_{ex}^\tau \times \mathbf{n}\, d\sigma\end{aligned} \tag{3.3}$$

and the Ginzburg-Landau system (2.19) – (2.21) assumes the form

$$\frac{1}{k^2}\nabla^2 f - f\left|\mathbf{p}_1 + \frac{1}{k}\nabla\theta\right|^2 + f - f^3 = 0, \tag{3.4}$$

$$\nabla \times \nabla \times \mathbf{p}_1 + f^2\left(\mathbf{p}_1 + \frac{1}{k}\nabla\theta\right) = \mathbf{0}, \tag{3.5}$$

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla \times \mathbf{p}_1) \times \mathbf{n}|_{\partial\Omega} = \mathbf{H}_{ex}^\tau \times \mathbf{n}|_{\partial\Omega}. \tag{3.6}$$

Lemma 3.1 *A pair $(f, \mathbf{p}_s) \in \mathcal{D}(\Omega)$ is a minimizer of the functional (2.18) if and only if the triplet $(f, \theta, \mathbf{p}_1) \in \mathcal{D}_1(\Omega)$, with*

$$\mathcal{D}_1(\Omega) = \{(f, \theta, \mathbf{p}_1) \colon f \in \mathrm{H}^1(\Omega),\ \mathbf{p}_1 \in \mathcal{R}_0(\Omega),\ f\nabla\theta \in \mathbf{L}^2(\Omega)\}$$

and θ, $\mathbf{p}_1$ satisfying (3.2), is a minimizer of the functional (3.3).

Proof It suffices to prove that $(f, \mathbf{p}_s) \in \mathcal{D}(\Omega)$ if and only if $(f, \theta, \mathbf{p}_1) \in \mathcal{D}_1(\Omega)$. Let $(f, \mathbf{p}_s) \in \mathcal{D}(\Omega)$. In view of the embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$, we have $f \in \mathrm{L}^4(\Omega)$ and $\mathbf{p}_1 \in \mathbf{L}^4(\Omega)$, thus $f\mathbf{p}_1 \in \mathbf{L}^2(\Omega)$. In this way, $f\nabla\theta = k(f\mathbf{p}_s - f\mathbf{p}_1) \in \mathbf{L}^2(\Omega)$, so that $(f, \theta, \mathbf{p}_1) \in \mathcal{D}_1(\Omega)$.

Conversely, if $(f, \theta, \mathbf{p}_1) \in \mathcal{D}_1(\Omega)$, it results $f\mathbf{p}_1 \in \mathbf{L}^2(\Omega)$. Therefore, by (3.2), we obtain $f\mathbf{p}_s \in \mathbf{L}^2(\Omega)$.

As shown in [1], the functional (3.3) admits at least a minimizer, so that, in view of Lemma 3.1, the existence theorem can be proved.

Theorem 3.1 *For each $\mathbf{H}_{ex}^\tau \in \mathbf{H}^{-1/2}(\partial\Omega)$, satisfying the relation (2.9), there exists at least a pair $(f, \mathbf{p}_s) \in \mathcal{D}(\Omega)$ which minimizes the free energy (2.18).*

Definition 3.1 A *triplet* $(f,\theta,\mathbf{p}_1)\in\mathcal{D}_1(\Omega)$ is a *weak solution of the Ginzburg-Landau problem* if it satisfies

$$\int_\Omega \cos(\theta-\varphi)\Big[\frac{1}{k^2}\nabla f\cdot\nabla h + fh\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\varphi\Big)+h(f^3-f)\Big]\,dx$$
$$+\frac{1}{k}\int_\Omega \sin(\theta-\varphi)\Big[h\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\varphi\Big)-f\nabla h\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]dx=0, \tag{3.7}$$

$$\int_\Omega\Big[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\mathbf{q}_1\Big]dx+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}^\tau_{ex}\cdot\mathbf{n}\,d\sigma=0, \tag{3.8}$$

for each $(h,\varphi,\mathbf{q}_1)\in\mathcal{D}_1(\Omega)$.

Remark It is possible to give a different definition of weak solution for the problem (3.4)–(3.6), by replacing the equations (3.7)–(3.8) with the following

$$\int_\Omega\Big[\frac{1}{k^2}\nabla f\cdot\nabla g+fg\Big|\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big|^2+(f^3-f)g\Big]dx=0, \tag{3.9}$$

$$\int_\Omega\Big[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\cdot\mathbf{q}_1\Big]dx+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}^\tau_{ex}\cdot\mathbf{n}\,d\sigma=0. \tag{3.10}$$

Though the equations (3.9)–(3.10) can be obtained by (3.7)–(3.8), choosing suitably the functions $(g,\mathbf{q}_1)$, the definitions are not equivalent, since the spaces of test functions are different.

Proposition 3.1 *If $(f,\theta,\mathbf{p}_1)$ is a regular solution of the Ginzburg-Landau problem (3.4)–(3.6), then it is a weak solution in the sense of Definition 3.1.*

Proof By taking the divergence of (3.5), we obtain

$$\nabla\cdot\Big[f^2\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]=0.$$

Hence

$$f\Big[2\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)+f\nabla\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]=0,$$

which leads to

$$\int_\Omega\Big[2\nabla f\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)+f\nabla\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)\Big]h\sin(\theta-\varphi)dx=0$$

for each (h,φ) such that $h\in H^1(\Omega)$, $h\nabla\varphi\in\mathbf{L}^2(\Omega)$. An integration by parts yields

$$\frac{1}{k}\int_\Omega[(h\nabla f-f\nabla h)\sin(\theta-\varphi)-fh(\nabla\theta-\nabla\varphi)\cos(\theta-\varphi)]\cdot\Big(\mathbf{p}_1+\frac{1}{k}\nabla\theta\Big)dx=0. \tag{3.11}$$

Moreover, multiplying (3.4) by $h\cos(\theta-\varphi)$ and integrating on Ω, it results

$$\begin{aligned}&\int_\Omega \frac{1}{k^2}\,\nabla f\cdot(\nabla\varphi-\nabla\theta)\sin(\theta-\varphi)\,dx\\&+\int_\Omega\left[\frac{1}{k^2}\,\nabla f\cdot\nabla h+fh\left|\mathbf{p}_1+\frac{1}{k}\,\nabla\theta\right|^2-fh(1-f^2)\right]\cos(\theta-\varphi)\,dx=0.\end{aligned}\tag{3.12}$$

By adding (3.11) and (3.12), we obtain (3.7). Finally, the relation (3.8) can be proved by multiplying (3.5) by an arbitrary function $\mathbf{q}_1\in\mathcal{R}_0(\Omega)$ and integrating by parts.

Denoting by $f_c=f\cos\theta$, $f_s=f\sin\theta$, $h_c=h\cos\varphi$, $h_s=h\sin\varphi$, the equations (3.7) and (3.8) can be written in the form

$$\begin{aligned}&\int_\Omega\left[\frac{1}{k^2}(\nabla f_c\cdot\nabla h_c+\nabla f_s\cdot\nabla h_s)+\frac{1}{k}(h_c\nabla f_s-f_s\nabla h_c+f_c\nabla h_s)\cdot\mathbf{p}_1\right]dx\\&\qquad-\int_\Omega\left[\frac{1}{k}\,h_s\nabla f_c\cdot\mathbf{p}_1-(\mathbf{p}_1^2+f_c^2+f_s^2-1)(f_ch_c+f_sh_s)\right]dx=0,\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\int_\Omega\left[\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1+(f_c^2+f_s^2)\mathbf{p}_1\cdot\mathbf{q}_1+\frac{1}{k}(f_c\nabla f_s-f_s\nabla f_c)\cdot\mathbf{q}_1\right]dx\\&\qquad\qquad+\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}_{ex}^\tau\cdot\mathbf{n}\,d\sigma=0.\end{aligned}\tag{3.14}$$

It is easy to verify that $(f,\theta,\mathbf{p}_1)\in\mathcal{D}(\Omega)$ if and only if $(f_c,f_s,\mathbf{p}_1)\in H^1(\Omega)\times H^1(\Omega)\times\mathcal{R}_0(\Omega)$. Moreover the equations (3.13)–(3.14) can be obtained by writing the free energy (3.3) as a functional of the variables $(f_c,f_s,\mathbf{p}_1)$

$$\begin{aligned}\mathcal{E}(f_c,f_s,\mathbf{p}_1)=&\int_\Omega\left[\frac{1}{2}(f_c^2+f_s^2)^2-f_c^2-f_s^2+\frac{1}{2}(|\nabla f_c|^2+|\nabla f_s|^2)+|\nabla\times\mathbf{p}_1|^2\right]dx\\&+\int_\Omega\left[\mathbf{p}_1^2(f_c^2+f_s^2)+\frac{2}{k}\,\mathbf{p}_1\cdot(f_c\nabla f_s-f_s\nabla f_c)\right]dx+2\int_{\partial\Omega}\mathbf{p}_1\cdot\mathbf{H}_{ex}^\tau\times\mathbf{n}\,d\sigma\end{aligned}$$

and then by computing the first variation with respect to such variables.

To reduce our notations, we put $\Pi=(f_c,f_s,\mathbf{p}_1)$, $\Sigma=(g_c,g_s,\mathbf{r}_1)$, $\Theta=(h_c,h_s,\mathbf{q}_1)$ and define

$$a(\Pi,\Theta)=\int_\Omega\left[\frac{1}{k^2}(\nabla f_c\cdot\nabla h_c+\nabla f_s\cdot\nabla h_s)+\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1\right]dx,\tag{3.15}$$

$$\begin{aligned}l(\Sigma,\Theta)=&-\int_\Omega\frac{1}{k}(h_c\nabla g_s-g_s\nabla h_c+g_c\nabla h_s-h_s\nabla g_c)\cdot\mathbf{r}_1\,dx\\&-\int_\Omega[(\mathbf{r}_1^2+g_c^2+g_s^2-1)(g_ch_c+g_sh_s)+(g_c^2+g_s^2)\mathbf{r}_1\cdot\mathbf{q}_1]dx\\&-\int_\Omega\frac{1}{k}(g_c\nabla g_s-g_s\nabla g_c)\cdot\mathbf{q}_1\,dx-\int_{\partial\Omega}\mathbf{q}_1\times\mathbf{H}_{ex}^\tau\cdot\mathbf{n}\,d\sigma=0.\end{aligned}\tag{3.16}$$

Hence (3.13) and (3.14) can be written as

$$a(\Pi,\Theta)=l(\Pi,\Theta). \tag{3.17}$$

Consider now the equation

$$a(\Pi,\Theta)=l(\Sigma,\Theta), \tag{3.18}$$

for each $\Theta\in\mathcal{V}(\Omega)$, where $\mathcal{V}(\Omega)\subset \mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\times\mathcal{R}_0(\Omega)$ is a closed subspace which does not contain triplets $(h_c,h_s,\mathbf{q}_1)$ with constant h_c and h_s. It can be proved ([10]) that $\mathcal{V}(\Omega)$ is a Hilbert space with respect to the norm

$$\|(h_c,h_s,\mathbf{q}_1)\|^2_{\mathcal{V}}=\frac{1}{k^2}(\|\nabla h_c\|_2^2+\|\nabla h_s\|_2^2)+\|\mathbf{q}_1\|^2_{\mathcal{R}_0}, \tag{3.19}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ and $\|\mathbf{q}_1\|_{\mathcal{R}_0}=\|\nabla\times\mathbf{q}_1\|_2$.

We will prove that the equation (3.18) admits a unique solution $\Pi=T(\Sigma)\in\mathcal{V}(\Omega)$. Moreover, with suitable hypotheses, T is a contraction, whose fixed point satisfies the relation (3.17).

Lemma 3.2 *For each $\Sigma\in\mathcal{V}(\Omega)$ there exists a unique $T(\Sigma)\in\mathcal{V}(\Omega)$ such that*

$$a(T(\Sigma),\Theta)=l(\Sigma,\Theta) \tag{3.20}$$

for all $\Theta\in\mathcal{V}(\Omega)$.

Proof In view of the definition (3.16), for each $\Sigma\in\mathcal{V}(\Omega)$, the map

$$l(\Sigma,\cdot):\mathcal{V}(\Omega)\to\mathbb{R}$$

is linear. Moreover, the following inequalities can be easily proved

$$\begin{aligned}
&\Big|\int_\Omega \mathbf{r}_1\cdot(h_c\nabla g_s-g_s\nabla h_c+g_c\nabla h_s-h_s\nabla g_c)\,dx\Big|\\
&\le\|\mathbf{r}_1\|_4(\|h_c\|_4\|\nabla g_s\|_2+\|g_s\|_4\|\nabla h_c\|_2+\|g_c\|_4\|\nabla h_s\|_2+\|h_s\|_4\|\nabla g_c\|_2)\\
&\le(c_1+1)\|\mathbf{r}_1\|_4[(\|\nabla g_s\|_2+\|g_s\|_4)\|h_c\|_{\mathrm{H}^1(\Omega)}+(\|\nabla g_c\|_2+\|g_c\|_4)\|h_s\|_{\mathrm{H}^1(\Omega)}],
\end{aligned}$$

$$\Big|\int_\Omega(g_ch_c+g_sh_s)\mathbf{r}_1^2\,dx\Big|\le c_1\|\mathbf{r}_1\|_4^2(\|g_c\|_4\|h_c\|_{\mathrm{H}^1(\Omega)}+\|g_s\|_4\|h_s\|_{\mathrm{H}^1(\Omega)}),$$

$$\begin{aligned}
&\Big|\int_\Omega(g_ch_c+g_sh_s)(1-g_c^2-g_s^2)\,dx\Big|\le(\|g_c\|_2+\|g_c\|_6^3\\
&+c_1\|g_c\|_4\|g_s\|_4^2)\|h_c\|_{\mathrm{H}^1(\Omega)}+(\|g_s\|_2+c_1\|g_c\|_4^2\|g_s\|_4+\|g_s\|_6^3)\|h_s\|_{\mathrm{H}^1(\Omega)},
\end{aligned}$$

$$\Big|\int_\Omega(g_c^2+g_s^2)\mathbf{r}_1\cdot\mathbf{q}_1\,dx\Big|\le c_2(\|g_c\|_4^2+\|g_s\|_4^2)\|\mathbf{r}_1\|_4\|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

$$\Big|\int_\Omega(g_c\nabla g_s-g_s\nabla g_c)\cdot\mathbf{q}_1\,dx\Big|\le c_2(\|g_c\|_4\|\nabla g_s\|_2+\|g_s\|_4\|\nabla g_c\|_2)\|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

$$\left| \int_{\partial\Omega} \mathbf{q}_1 \times \mathbf{H}_{ex}^{\tau} \cdot \mathbf{n}\, d\sigma \right| \le \|\mathbf{H}_{ex}^{\tau}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \|\mathbf{q}_1 \times \mathbf{n}\|_{\mathbf{H}^{1/2}(\partial\Omega)}$$
$$\le c_3 \|\mathbf{H}_{ex}^{\tau}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \|\mathbf{q}_1\|_{\mathcal{R}_0(\Omega)},$$

where c_1, c_2, c_3 satisfy respectively the inequalities

$$\|f\|_4 \le c_1 \|f\|_{\mathrm{H}^1(\Omega)}, \qquad \|\mathbf{p}_1\|_{\mathbf{H}^1(\Omega)} \le c_2 \|\mathbf{p}_1\|_{\mathcal{R}_0}, \qquad \|\mathbf{p}_1 \times \mathbf{n}\|_{\mathbf{H}^{1/2}(\Omega)} \le c_3 \|\mathbf{p}_1\|_{\mathcal{R}_0}. \quad (3.21)$$

Therefore, keeping the definition (3.19) into account, we get

$$l(\Sigma, \Theta) \le C(\Sigma) \|\Theta\|_{\mathcal{V}},$$

where $C(\Sigma)$ denotes a positive constant depending on Σ.

Since $l(\Sigma, \cdot)$ is continuous, the Riesz theorem ensures the existence of a unique $\Theta(\Sigma) \in \mathcal{V}(\Omega)$ such that

$$a(T(\Sigma), \Theta) = l(\Sigma, \Theta), \qquad \forall\, \Theta \in \mathcal{V}(\Omega). \quad (3.22)$$

Lemma 3.3 *For each $M > 0$ and Σ_1, $\Sigma_2 \in \mathcal{V}(\Omega)$ satisfying $\|\Sigma_i\|_{\mathcal{V}} \le M$, $i = 1, 2$, there exists a constant $\delta_M > 0$ such that*

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le \delta_M \|\Sigma_1 - \Sigma_2\|_{\mathcal{V}}.$$

Proof From the equation (3.22) we obtain the identity

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} = \sup_{\|\Theta\|_{\mathcal{V}}=1} [a(T(\Sigma_1), \Theta) - a(T(\Sigma_2), \Theta)] = \sup_{\|\Theta\|_{\mathcal{V}}=1} [l(\Sigma_1, \Theta) - l(\Sigma_2, \Theta)]$$
$$= \sup_{\|\Theta\|_{\mathcal{V}}=1} [I_1(\Theta) + I_2(\Theta) + I_3(\Theta) + I_4(\Theta) + I_5(\Theta) + I_6(\Theta)],$$

where

$$I_1(\Theta) = \frac{1}{k} \int_{\Omega} [(\nabla g_{c1} \cdot \mathbf{r}_{11} - \nabla g_{c2} \cdot \mathbf{r}_{12}) h_s - (\nabla g_{s1} \cdot \mathbf{r}_{11} - \nabla g_{s2} \cdot \mathbf{r}_{12}) h_c] dx,$$

$$I_2(\Theta) = \frac{1}{k} \int_{\Omega} [(g_{s1}\mathbf{r}_{11} - g_{s2}\mathbf{r}_{12}) \cdot \nabla h_c - (g_{c1}\mathbf{r}_{11} - g_{c2}\mathbf{r}_{12}) \cdot \nabla h_s] dx,$$

$$I_3(\Theta) = -\int_{\Omega} (\mathbf{r}_{11}^2 g_{c1} + g_{c1}^3 + g_{c1} g_{s1}^2 - g_{c1} - \mathbf{r}_{12}^2 g_{c2} - g_{c2}^3 - g_{c2} g_{s2}^2 + g_{c2}) h_c dx$$

$$I_4(\Theta) = -\int_{\Omega} (\mathbf{r}_{11}^2 g_{s1} + g_{s1}^3 + g_{s1} g_{c1}^2 - g_{s1} - \mathbf{r}_{12}^2 g_{s2} - g_{s2}^3 - g_{s2} g_{c2}^2 + g_{s2}) h_s dx$$

$$I_5(\Theta) = -\int_{\Omega} (g_{c1}^2 \mathbf{r}_{11} + g_{s1}^2 \mathbf{r}_{11} - g_{c2}^2 \mathbf{r}_{12} - g_{s2}^2 \mathbf{r}_{12}) \cdot \mathbf{q}_1 dx$$

$$I_6(\Theta) = \frac{1}{k} \int_{\Omega} (g_{c1} \nabla g_{s1} - g_{s1} \nabla g_{c1} - g_{c2} \nabla g_{s2} + g_{s2} \nabla g_{c2}) \cdot \mathbf{q}_1\, dx.$$

Since $\|\Sigma_i\|_{\mathcal{V}} \le M,\ i = 1, 2$, we have the inequalities

$$\|\nabla g_{ci}\|_2 \le kM, \quad \|\nabla g_{si}\|_2 \le kM, \quad \|\mathbf{r}_{1i}\|_{\mathcal{R}_0} \le M$$

Therefore, by the definition of $\mathcal{V}(\Omega)$, g_{ci} and g_{si} satisfy the estimates

$$\|g_{ci}\|_{\mathrm{H}^1(\Omega)} \le c_4 kM, \quad \|g_{si}\|_{\mathrm{H}^1(\Omega)} \le c_4 kM.$$

Let $\delta g_c = g_{c1} - g_{c2},\ \delta g_s = g_{s1} - g_{s2},\ \delta\mathbf{r}_1 = \mathbf{r}_{11} - \mathbf{r}_{12}$. We deduce the estimates

$$\begin{aligned}
|I_1(\Theta)| &\le Mc_1c_2[(\|h_c\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)})\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} \\
&\quad + \frac{1}{k}(\|h_c\|_{\mathrm{H}^1(\Omega)}\|\nabla(\delta g_s)\|_2 + \|h_s\|_{\mathrm{H}^1(\Omega)}\|\nabla(\delta g_c)\|_2)], \\
|I_2(\Theta)| &\le Mc_1c_2[c_4(\|h_c\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)})\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} \\
&\quad + \frac{1}{k}(\|h_c\|_{\mathrm{H}^1(\Omega)}\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + \|h_s\|_{\mathrm{H}^1(\Omega)}\|\delta g_c\|_2)] \\
|I_3(\Theta)| &\le M^2c_1^2c_2[c_2\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + 2c_2c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + 2c_1c_4^2k^2\|\delta g_s\|_{\mathrm{H}^1(\Omega)} \\
&\quad + 4c_1c_4^2k^2\|\delta g_c\|_{\mathrm{H}^1(\Omega)}]\|h_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_c\|_2\|h_c\|_2 \\
|I_4(\Theta)| &\le M^2c_1^2c_2[c_2\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + 2c_2c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + 2c_1c_4^2k^2\|\delta g_c\|_{\mathrm{H}^1(\Omega)} \\
&\quad + 4c_1c_4^2k^2\|\delta g_s\|_{\mathrm{H}^1(\Omega)}]\|h_s\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_2\|h_s\|_2 \\
|I_5(\Theta)| &\le 2M^2c_1^2c_2^2c_4k(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)} + c_4k\|\delta\mathbf{r}_1\|_{\mathcal{R}_0})\|\mathbf{q}_1\|_{\mathcal{R}_0} \\
|I_6(\Theta)| &\le Mc_1c_2k(1 + c_4)(\|\delta g_s\|_{\mathrm{H}^1(\Omega)} + \|\delta g_c\|_{\mathrm{H}^1(\Omega)})\|\mathbf{q}_1\|_{\mathcal{R}_0}.
\end{aligned}$$

Thus

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le A\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + B(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)}),$$

where

$$\begin{aligned}
A &= 2Mc_1c_2c_4k(1 + c_4 + 3Mc_1c_2c_4k) \\
B &= Mc_1c_2(Mc_2c_4k + 6Mc_1^2c_4^3k^3 + 2Mc_1c_2c_4k + 1 + 3c_4) + c_4k.
\end{aligned}$$

By using the inequality $2xy \le \varepsilon x^2 + \frac{1}{\varepsilon}y^2$, we obtain

$$\begin{aligned}
\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}}^2 &\le [A\|\delta\mathbf{r}_1\|_{\mathcal{R}_0} + B(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)})]^2 \\
&\le A^2(1+\varepsilon)\|\delta\mathbf{r}_1\|_{\mathcal{R}_0}^2 + B^2\,\frac{1+\varepsilon}{\varepsilon}\,(\|\delta g_c\|_{\mathrm{H}^1(\Omega)} + \|\delta g_s\|_{\mathrm{H}^1(\Omega)})^2 \\
&\le A^2(1+\varepsilon)\|\delta\mathbf{r}_1\|_{\mathcal{R}_0}^2 + 2B^2c^2\,\frac{1+\varepsilon}{\varepsilon}\,(\|\nabla(\delta g_c)\|_2^2 + \|\nabla(\delta g_s)\|_2^2).
\end{aligned}$$

Hence, the choice

$$\delta_M^2 = \max\left\{A^2(1+\varepsilon), 2B^2c^2k^2\,\frac{1+\varepsilon}{\varepsilon}\right\}$$

yields

$$\|T(\Sigma_1) - T(\Sigma_2)\|_{\mathcal{V}} \le \delta_M\|\Sigma_1 - \Sigma_2\|_{\mathcal{V}}.$$

Lemma 3.4 *If the following inequalities*

$$2Mc_4k[2Mc_1c_2 + M^2c_1^2c_4k(c_2^2 + 2c_1^2c_4^2k^2) + c_4k] \leq M, \tag{3.23}$$

$$4M^2c_1c_2c_4k(Mc_1c_2k + 1) + 2c_3\|\mathbf{H}_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)} \leq M, \tag{3.24}$$

hold, T defined through (3.20), maps $\mathcal{B}_M = \{\Sigma \in \mathcal{V}(\Omega)\colon \|\Sigma\|_{\mathcal{V}} \leq M\}$ in itself.

Proof By the definition (3.22) we have

$$\|T(\Sigma)\|_{\mathcal{V}}^2 = l(\Sigma, T(\Sigma)).$$

Therefore, proceeding as in the proof of Lemma 3.4, it results

$$\|T(\Sigma)\|_{\mathcal{V}}^2 \leq \frac{D}{k}(\|\nabla f_c\|_2 + \|\nabla f_s\|_2) + E\|\mathbf{p}_1\|_{\mathcal{R}_0},$$

where

$$D = Mc_4k[2Mc_1c_2 + M^2c_1^2c_4k(c_2^2 + 2c_1^2c_4^2k^2) + c_4k],$$
$$E = 2M^2c_1c_2c_4k(Mc_1c_2k + 1) + c_3\|\mathbf{H}_{ex}\|_{\mathbf{H}^{-1/2}(\partial\Omega)}.$$

The hypotheses (3.23) and (3.24) imply

$$\|T(\Sigma)\|_{\mathcal{V}}^2 \leq M\|T(\Sigma)\|_{\mathcal{V}},$$

so that $T(\Sigma) \in \mathcal{B}_M$.

By applying the previous lemmas, we get the uniqueness result.

Theorem 3.2 *Let $(f_c, f_s, \mathbf{p}_1) \in \mathcal{B}_M$ and $(g_c, g_s, \mathbf{q}_1) \in \mathcal{B}_M$, satisfying the Definition 3.1. If the inequalities (3.23) and (3.24) hold and there exists $\varepsilon > 0$ such that $\delta_M < 1$, then $(f_c, f_s, \mathbf{p}_1) = (g_c, g_s, \mathbf{q}_1)$.*

The Theorem 3.2 ensures the uniqueness of weak solutions, provided that the parameter k, the external field and the domain Ω are sufficiently small. A non-uniqueness result can be obtained with the same method used in [9, 11], if the domain Ω is sufficiently large.

3.2 The time-dependent case

The weak formulation of the evolution problem (2.22) – (2.25) is obtained by introducing the functional space

$$\begin{aligned}\mathcal{H}(Q) = \Big\{&(f, \mathbf{p}_s, \phi_s)\colon f \in \mathrm{L}^2(0,T; H^1(\Omega)) \cap H^1(0,T; (H^1(\Omega))'),\\ &\nabla\times\mathbf{p}_s \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega)),\ \dot{\mathbf{p}}_s + \nabla\phi_s \in \mathrm{L}^2(0,T;(\mathbf{H}^1_{\mathbf{n}}(\Omega))'),\\ &f\mathbf{p}_s \in \mathrm{L}^2(0,T;\mathbf{L}^2(\Omega)),\ f\phi_s \in \mathrm{L}^2(0,T;(H^1(\Omega))'),\ \mathbf{p}_s\cdot\mathbf{n}|_{\partial\Omega} = 0\Big\},\end{aligned}$$

where $Q = \Omega \times (0,T)$, $\mathbf{H}^1_{\mathbf{n}}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\colon \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}$ and X' denotes the dual space of X.

Proceeding as in the stationary case, in order to prove existence and uniqueness theorems, we decompose $\mathbf{p}_s$ and ϕ_s in the following form

$$\mathbf{p}_s = \mathbf{p}_1 - \nabla\theta, \tag{3.25}$$

$$\phi_s = \phi + \dot{\theta}, \tag{3.26}$$

with $\mathbf{p}_1 \in \mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))$ and

$$\phi = -\frac{1}{\eta}\,\nabla\cdot\mathbf{p}_1. \tag{3.27}$$

By means of the positions (3.25)–(3.26), the system (2.22)–(2.25) can be written in the form

$$\dot{f} - \nabla^2 f + k^2(f^2-1)f + f(\mathbf{p}_1 - \nabla\theta)^2 = 0, \tag{3.28}$$

$$\eta\dot{\mathbf{p}}_1 + \nabla\times\nabla\times\mathbf{p}_1 + \eta\nabla\phi + f^2(\mathbf{p}_1 - \nabla\theta) = \mathbf{0}, \tag{3.29}$$

$$\nabla f\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{p}_1 + \mathbf{H}^\tau_{ex})\times\mathbf{n}|_{\partial\Omega} = 0, \tag{3.30}$$

$$\nabla\theta\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \mathbf{p}_1\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.31}$$

$$f(x,0) = f_0(x), \qquad \mathbf{p}_1(x,0) - \nabla\theta(x,0) = \mathbf{p}_{s0}(x). \tag{3.32}$$

Let

$$\begin{aligned}\mathcal{K}(Q) = \Big\{(f,\theta,\mathbf{p}_1)\colon\ & f\in \mathrm{L}^2(0,T;H^1(\Omega))\cap H^1(0,T;(H^1(\Omega))'),\\ & f\nabla\theta\in\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega)),\ \ f\dot{\theta}\in\mathrm{L}^2(0,T;(\mathrm{H}^1(\Omega))'),\\ & \mathbf{p}_1\in\mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))\cap H^1(0,T;(\mathbf{H}^1_{\mathbf{n}}(\Omega))')\Big\}.\end{aligned}$$

From the definition of $\mathcal{H}(Q)$, it follows that if $\mathbf{p}_1$ satisfies (3.25)–(3.27), then the following equation holds

$$\dot{\mathbf{p}}_1 - \frac{1}{\eta}\,\nabla^2\mathbf{p}_1 = \mathbf{F}, \tag{3.33}$$

where $\mathbf{F} = \dot{\mathbf{p}}_s + \nabla\phi_s + \frac{1}{\eta}\,\nabla\times\nabla\times\mathbf{p}_s \in \mathrm{L}^2(0,T;(\mathrm{H}^1_{\mathbf{n}}(\Omega))')$. The equation (3.33), with the boundary and initial conditions $(3.30)_2$, $(3.31)_2$, $(3.32)_2$ admits a unique solution $\mathbf{p}_1\in\mathrm{L}^2(0,T;\mathbf{H}^1_{\mathbf{n}}(\Omega))\cap H^1(0,T;(\mathbf{H}^1_{\mathbf{n}}(\Omega))')$ (see [14]). Hence, by putting $\phi = -\frac{1}{\eta}\,\nabla\cdot\mathbf{p}_1$, we have that for each $(f,\mathbf{p}_s,\phi_s)\in\mathcal{H}(Q)$, there exists a unique triplet $(f,\theta,\mathbf{p}_1)\in\mathcal{K}(Q)$ which satisfies (3.25)–(3.27).

Definition 3.2 A *triplet* $(f,\theta,\mathbf{p}_1)\in\mathcal{K}(Q)$ is a *weak solution* of the problem (3.28)–(3.32), with $\mathbf{H}^\tau_{ex}\in\mathbf{H}^{-1/2}(\partial\Omega)$ if

$$\begin{aligned}&\int_\Omega [\dot{f}g + k^2(f^2-1)fg + \nabla f\cdot\nabla g + fg(\mathbf{p}_1-\nabla\theta)\cdot(\mathbf{p}_1-\nabla\varphi)]\cos(\theta-\varphi)\,dx \\ &-\int_\Omega [fg\dot{\theta} - \frac{1}{\eta}\,fg\nabla\cdot\mathbf{p}_1 + g\nabla f\cdot(\mathbf{p}_1-\nabla\varphi) - f\nabla g\cdot(\mathbf{p}_1-\nabla\theta)]\sin(\theta-\varphi)\,dx = 0,\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\int_\Omega \big\{\nabla\times\mathbf{p}_1\cdot\nabla\times\mathbf{q}_1 + \nabla\cdot\mathbf{p}_1\nabla\cdot\mathbf{q}_1 + [\eta\dot{\mathbf{p}}_1 + f^2(\mathbf{p}_1-\nabla\theta)]\cdot\mathbf{q}_1\big\}\,dx\\ &\qquad -\int_{\partial\Omega}\mathbf{H}^\tau_{ex}\times\mathbf{q}_1\cdot\mathbf{n}\,d\sigma = 0,\end{aligned} \tag{3.35}$$

for each $(g, \varphi, \mathbf{q}_1) \in \mathcal{K}(Q)$, a.e. $t \in (0,T)$ and

$$f(x,0) = f_0(x), \qquad \mathbf{p}_1(x,0) - \nabla\theta(x,0) = \mathbf{p}_{s0}(x).$$

By means of the decomposition (3.25) – (3.26), the constitutive equation (2.30) yields

$$f^2\dot{\theta} + \left(1 - \frac{1}{\eta}\right) f^2 \nabla \cdot \mathbf{p}_1 + 2f\nabla f \cdot (\mathbf{p}_1 - \nabla\theta) - f^2\nabla^2\theta = 0. \tag{3.36}$$

Proposition 3.2 *Every regular solution $(f, \theta, \mathbf{p}_1)$ of the Ginzburg-Landau equations (3.28) – (3.32) is a weak solution in the sense of Defintion 3.2.*

Proof By multiplying (3.28) by $g\cos(\theta - \varphi)$ and integrating on Ω, it results

$$\begin{aligned} &\int_\Omega [\dot{f}g + \nabla f \cdot \nabla g + k^2 fg(f^2 - 1) + fg|\mathbf{p}_1 - \nabla\theta|^2] \cos(\theta - \varphi)\, dx \\ &\qquad - \int_\Omega g\nabla f \cdot (\nabla\theta - \nabla\varphi) \sin(\theta - \varphi)\, dx = 0. \end{aligned} \tag{3.37}$$

On the other hand, from (3.36), we obtain

$$\begin{aligned} &\int_\Omega [f\dot{\theta} + \left(1 - \frac{1}{\eta}\right) fg \nabla \cdot \mathbf{p}_1 + 2g\nabla f \cdot (\mathbf{p}_1 - \nabla\theta)] \sin(\theta - \varphi)\, dx \\ &+ \int_\Omega [(f\nabla g + g\nabla f) \cdot \nabla\theta \sin(\theta - \varphi) + fg(\nabla\theta - \nabla\varphi) \cdot \nabla\theta \cos(\theta - \varphi)]\, dx = 0. \end{aligned} \tag{3.38}$$

By subtracting (3.37) and (3.38), we get (3.34).

Finally, inner multiplication of (3.29) by $\mathbf{q}_1$ and an integration by parts, lead to (3.35).

Let $(f, \theta, \mathbf{p}_1)$ be a weak solution of the problem (3.28) – (3.32). The choice $g = f$, $\varphi = \theta$ in the Definition 3.2 yields

$$\int_\Omega \left[\frac{1}{2}\frac{\partial f^2}{\partial t} + |\nabla f|^2 + f^2|\mathbf{p}_1 - \nabla\theta|^2 + k^2 f^4\right] dx = \int_\Omega k^2 f^2\, dx. \tag{3.39}$$

Therefore, by applying Gronwall's inequality, we obtain

$$\|f(t)\|_2 \le \|f_0\|_2 \exp(2k^2 t) \quad 0 \le t \le T. \tag{3.40}$$

By integrating (3.39) in the interval $(0,t)$, with $0 \le t \le T$, it follows[1]

$$\int_0^t \int_\Omega [|\nabla f|^2 + f^2|\mathbf{p}_1 - \nabla\theta|^2 + k^2 f^4]\, dx\, d\tau \le k_1 \|f_0\|_2^2, \tag{3.41}$$

[1] Henceforth k_i denotes a function of the variable t, belonging to $L^1(0,T)$.

so that $\|f\|^2_{\mathrm{H}^1(\Omega)}$ and $\|f(\mathbf{p}_1 - \nabla\theta)\|_2^2$ are $\mathrm{L}^1(0,T)$-functions.

Analogously, by choosing $\mathbf{q}_1 = \mathbf{p}_1$ in (3.35) and integrating on $(0,t)$, we obtain the identity

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + \int_0^t \int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$= \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 - \int_0^t\int_\Omega f^2\mathbf{p}_1\cdot(\mathbf{p}_1 - \nabla\theta)\,dx\,d\tau - \int_0^t\int_{\partial\Omega} \mathbf{H}^\tau_{ex}\times\mathbf{p}_1\cdot\mathbf{n}\,d\sigma\,d\tau,$$

so that

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + \int_0^t \int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$\le \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 + \int_0^t \|\mathbf{H}^\tau_{ex}\times\mathbf{n}\|_{\mathbf{H}^{-1/2}(\Omega)}\|\mathbf{p}_1\times\mathbf{n}\|_{\mathbf{H}^{1/2}(\Omega)}\,d\tau$$

$$+ \int_0^t \left[\frac{k_2}{2}\|f\|_4^2 + \frac{1}{2k_2}\|\mathbf{p}_1\|_4^2 + \|f(\mathbf{p}_1 - \nabla\theta)\|_2^2\right]d\tau.$$

The inequality (3.41), implies

$$\frac{\eta}{2}\|\mathbf{p}_1(t)\|_2^2 + k_3\int_0^t \int_\Omega [|\nabla\times\mathbf{p}_1|^2 + |\nabla\cdot\mathbf{p}_1|^2]\,dx\,d\tau$$

$$\le \frac{\eta}{2}\|\mathbf{p}_{10}\|_2^2 + \int_0^t \left[\frac{k_2}{2}\|f\|_4^2 + k_4\|\mathbf{p}_1\|_2^2 + \|f(\mathbf{p}_1 - \nabla\theta)\|_2^2\right]d\tau \tag{3.42}$$

$$+ \int_0^t k_5\|\mathbf{H}^\tau_{ex}\times\mathbf{n}\|^2_{\mathbf{H}^{-1/2}(\Omega)}\,d\tau \le k_6 + k_4\int_0^t \|\mathbf{p}_1\|_2^2\,d\tau.$$

Thus Gronwall's inequality yields

$$\|\mathbf{p}_1(t)\|_2 \le k_6(1 + k_4 t\exp(k_4 t)), \quad 0 \le t \le T. \tag{3.43}$$

Therefore $\|\mathbf{p}_1\|^2_{\mathrm{H}^1(\Omega)} \in \mathrm{L}^1(0,T)$.

Theorem 3.3 *If* $\Omega \subset \mathbb{R}^2$, $\mathbf{H}^\tau_{ex} \in \mathbf{H}^{-1/2}(\partial\Omega)$, $f_0 \in \mathrm{L}^2(\Omega)$ *and* $\mathbf{p}_{s0} = \mathbf{p}_{10} - \nabla\theta_0$ *with* $\mathbf{p}_{10} \in \mathbf{L}^2(\Omega)$, *there exists a unique triplet* $(f,\theta,\mathbf{p}_1)$ *satisfying the Definition 3.2.*

Proof Let $(f_1,\theta_1,\mathbf{p}_{11})$ and $(f_2,\theta_2,\mathbf{p}_{12})$ be weak solutions of the problem (3.28)–(3.32).

Denote by $\delta f_c = f_{c1} - f_{c2} = f_1 \cos\theta_1 - f_2 \cos\theta_2$, $\delta f_s = f_{s1} - f_{s2} = f_1 \sin\theta_1 - f_2 \sin\theta_2$ and $\delta \mathbf{p}_1 = \mathbf{p}_{11} - \mathbf{p}_{12}$. Write the equation (3.34), first with $(f, \theta, \mathbf{p}_1) = (f_1, \theta_1, \mathbf{p}_{11})$, then with $(f, \theta, \mathbf{p}_1) = (f_2, \theta_2, \mathbf{p}_{12})$ and subtract the ensuing relations. By choosing (g, φ) such that $g \cos\varphi = \delta f_c$, $g \sin\varphi = \delta f_s$, we obtain

$$\begin{aligned}\int_\Omega \left[\frac{1}{2}\frac{d}{dt}(\delta f_c)^2 + \frac{1}{2}\frac{d}{dt}(\delta f_s)^2 + |\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2\right] dx \\ = \int_\Omega k^2[(\delta f_c)^2 + (\delta f_s)^2]\,dx + J_1 + J_2 + J_3 + J_4,\end{aligned} \tag{3.44}$$

where

$$\begin{aligned}
J_1 &= -\frac{1}{\eta}\int_\Omega (\nabla\cdot\mathbf{p}_{11} f_1 \sin\theta_1 - \nabla\cdot\mathbf{p}_{12} f_2 \sin\theta_2)\delta f_c\,dx \\
&\quad + \frac{1}{\eta}\int_\Omega (\nabla\cdot\mathbf{p}_{11} f_1 \cos\theta_1 - \nabla\cdot\mathbf{p}_{12} f_2 \cos\theta_2)\delta f_s\,dx \\
&= -\frac{1}{\eta}\int_\Omega [f_1 \sin\theta_1 \nabla\cdot(\delta\mathbf{p}_1)\delta R - f_1\cos\theta_1\nabla\cdot(\delta\mathbf{p}_1)\delta f_s]\,dx, \\
J_2 &= -k^2\int_\Omega [(f_1^3\cos\theta_1 - f_2^3\cos\theta_2)\delta f_c + (f_1^3\sin\theta_1 - f_2^3\sin\theta_2)\delta f_s]\,dx \\
&= -k^2\int_\Omega \{f_1^2[(\delta f_c)^2 + (\delta f_s)^2] + (f_1^2 - f_2^2)(f_2\cos\theta_2\delta f_c + f_2\sin\theta_2\delta f_s)\}\,dx \\
J_3 &= -\int_\Omega [(\mathbf{p}_{11}^2 f_1\cos\theta_1 - \mathbf{p}_{12}^2 f_2\cos\theta_2)\delta f_c + (\mathbf{p}_{11}^2 f_1\sin\theta_1 - \mathbf{p}_{12}^2 f_2\sin\theta_2)\delta f_s]\,dx \\
&\quad + \int_\Omega [\nabla(f_1\sin\theta_1)\cdot\mathbf{p}_{11} - \nabla(f_2\sin\theta_2)\cdot\mathbf{p}_{12}]\delta f_c\,dx \\
&\quad - \int_\Omega [\nabla(f_1\cos\theta_1)\cdot\mathbf{p}_{11} - \nabla(f_2\cos\theta_2)\cdot\mathbf{p}_{12}]\delta f_s\,dx \\
&= -\int_\Omega (f_1\cos\theta_1\delta f_c + f_1\sin\theta_1\delta f_s)(\mathbf{p}_{11} + \mathbf{p}_{12})\cdot\delta\mathbf{p}_1\,dx \\
&\quad + \int_\Omega \{\mathbf{p}_{12}^2[(\delta R)^2 + (\delta I)^2] + [\nabla(f_1\sin\theta_1)\delta f_c - \nabla(f_1\cos\theta_1)]\cdot\delta\mathbf{p}_1\}\,dx \\
&\quad + \int_\Omega \mathbf{p}_{12}\cdot[\nabla(\delta f_s)\delta f_c - \nabla(\delta f_c)\delta f_s]\,dx, \\
J_4 &= -\int_\Omega (\mathbf{p}_{11} f_1\sin\theta_1 - \mathbf{p}_{12} f_2\sin\theta_2)\cdot\nabla(\delta R)\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int\limits_{\Omega} (\mathbf{p}_{11} f_1 \cos\theta_1 - \mathbf{p}_{12} f_2 \cos\theta_2) \cdot \nabla(\delta I)\, dx \\
&= - \int\limits_{\Omega} (\delta f_s \mathbf{p}_{11} + f_2 \sin\theta_2 \delta \mathbf{p}_1) \cdot \nabla(\delta f_c) \\
&- \int\limits_{\Omega} (\delta f_c \mathbf{p}_{11} + f_2 \cos\theta_2 \delta \mathbf{p}_1) \cdot \nabla(\delta f_s)\, dx.
\end{aligned}
$$

Keeping the estimates (3.40) and (3.43) into account, we deduce the following inequality

$$
\begin{aligned}
|J_1| &\le \frac{1}{\eta} \|f_1\|_4 \|\nabla \cdot \delta \mathbf{p}_1\|_2 (\|\delta f_c\|_4 + \|\delta f_s\|_4) \\
&\le \frac{\varepsilon}{2} \|\nabla \cdot \delta \mathbf{p}_1\|_2^2 + \frac{C}{2\varepsilon} \|f_1\|_4^2 (\|\delta f_c\|_2 \|\delta f_c\|_{\mathrm{H}^1(\Omega)} + \|\delta f_s\|_2 \|\delta f_s\|_{\mathrm{H}^1(\Omega)}) \\
&\le C(\varepsilon)[\|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 + \|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2] + C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2],
\end{aligned}
$$

where $\varepsilon > 0$, $C(\varepsilon)$ is a positive constant and $C(t)$ is a L^1-function.

Analogously, we can prove the estimates

$$
\begin{aligned}
|J_2| &\le C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2\big) + C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2\big), \\
|J_3| &\le C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2\big) \\
&\quad + C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 + \|\nabla \times (\delta \mathbf{p}_1)\|_2^2\big), \\
|J_4| &\le C(t)\big(\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2\big) \\
&\quad + C(\varepsilon)\big(\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 + \|\nabla \times (\delta \mathbf{p}_1)\|_2^2\big).
\end{aligned}
$$

Therefore, from (3.44) we get

$$
\begin{aligned}
&\int\limits_{\Omega} \left[\frac{1}{2} \frac{d}{dt} (\delta f_c)^2 + \frac{1}{2} \frac{d}{dt} (\delta f_s)^2 + |\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2\right] dx \\
&\le C(\varepsilon)\big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 + \|\nabla \times (\delta \mathbf{p}_1)\|_2^2\big] \\
&\qquad + C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2].
\end{aligned} \tag{3.45}
$$

With the same technique, from (3.35), we have

$$
\int_{\Omega} \left[\frac{\eta}{2} \frac{d}{dt} (\delta \mathbf{p}_1)^2 + |\nabla \times (\delta \mathbf{p}_1)|^2 + |\nabla \cdot (\delta \mathbf{p}_1)|^2\right] dx = -J_5 + J_6, \tag{3.46}
$$

with

$$J_5 = \int_\Omega (f_1^2 \mathbf{p}_{11} - f_2^2 \mathbf{p}_{12}) \cdot \delta \mathbf{p}_1 \, dx$$
$$= \int_\Omega [f_1^2 \delta \mathbf{p}_1 + \mathbf{p}_{12}(f_1 - f_2)(f_1 + f_2)] \cdot \delta \mathbf{p}_1 \, dx,$$
$$J_6 = \int_\Omega (f_1^2 \nabla\theta_1 - f_2^2 \nabla\theta_2) \cdot \delta \mathbf{p}_1 \, dx$$
$$= \int_\Omega [f_1(f_1 \nabla\theta_1 - f_2 \nabla\theta_2) + f_2 \nabla\theta_2 (f_1 - f_2)] \cdot \delta \mathbf{p}_1 \, dx.$$

Therefore

$$\begin{aligned} &\int_\Omega \left[\frac{\eta}{2} \frac{d}{dt} (\delta \mathbf{p}_1)^2 + |\nabla \times (\delta \mathbf{p}_1)|^2 + |\nabla \cdot (\delta \mathbf{p}_1)|^2\right] dx \\ &\leq C(\varepsilon) \big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 + \|\nabla \times (\delta \mathbf{p}_1)\|_2^2\big] \\ &\qquad + C(t) \big[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2\big]. \end{aligned} \tag{3.47}$$

From the relations (3.45) and (3.47) we have

$$\begin{aligned} &\int_\Omega \frac{1}{2} \frac{d}{dt} (\delta f_c)^2 + (\delta I)^2 + \eta(\delta \mathbf{p}_1)^2] \, dx \\ &+ \int_\Omega [|\nabla(\delta f_c)|^2 + |\nabla(\delta f_s)|^2 + |\nabla \times (\delta \mathbf{p}_1)|^2 + |\nabla \cdot (\delta \mathbf{p}_1)|^2] \, dx \\ &\leq C(\varepsilon) \big[\|\nabla(\delta f_c)\|_2^2 + \|\nabla(\delta f_s)\|_2^2 + \|\nabla \cdot (\delta \mathbf{p}_1)\|_2^2 \\ &+ \|\nabla \times (\delta \mathbf{p}_1)\|_2^2\big] + C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2] \end{aligned}$$

so that, by a suitable choice of the constant $C(\varepsilon)$, we conclude

$$\int_\Omega \frac{1}{2} \frac{d}{dt} [(\delta f_c)^2 + (\delta I)^2 + \eta(\delta \mathbf{p}_1)^2] \, dx \leq C(t)[\|\delta f_c\|_2^2 + \|\delta f_s\|_2^2 + \|\delta \mathbf{p}_1\|_2^2].$$

Gronwall's inequality yields $\delta f_c = 0$, $\delta f_s = 0$, $\delta \mathbf{p}_1 = 0$. Hence $f_1 = f_2$, $\nabla\theta_1 = \nabla\theta_2$, $\mathbf{p}_{11} = \mathbf{p}_{12}$.

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