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Robustness Analysis of a Class of Discrete-Time Systems with Applications to Neural Networks

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Abstract: In this paper we study the robust stability properties of a large class of nonlinear discrete-time systems by addressing the following question: given a nonlinear discrete-time system with specified exponentially stable equilibria, under what conditions will a perturbed model of the discrete-time system possess exponentially stable equilibria that are close (in distance) to the exponentially stable equilibria of the unperturbed discrete-time system? In arriving at our results, we establish robust stability results for the perturbed discrete-time systems considered herein. We apply the above results in the robustness analysis of a large class of discrete-time recurrent neural networks.

Keywords: *Discrete-time systems; robust stability; neural networks.*

Mathematics Subject Classification (2000): 34C35, 34D05, 34D20, 34D45, 34H05, 54H20, 93C10, 93C15, 93C50, 93C60, 93D05, 93D20, 93D30.

1 Introduction

We consider discrete-time systems described by first-order ordinary difference equations of the form

$$x(k+1) = f(x(k)) + h(x(k)), \tag{1}$$

where $x(k)$ is a real n-vector, $k \in Z_+$ (the set of nonnegative integers) and f and h are continuously differentiable n-vector valued functions. We view (1) as a perturbation model of systems described by

$$x(k+1) = f(x(k)). \tag{2}$$

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Thus, $h(x(k))$ in (1) represents uncertainties or perturbation terms.

In the present paper we study robustness properties of system (2) with respect to perturbations. Of particular interest to us will be the robust stability of equilibria and estimates of the perturbations of the equilibrium locations. To demonstrate applicability, we apply these results in the qualitative analysis of a large class of discrete-time recurrent neural networks.

Qualitative robustness results for linear and nonlinear dynamical systems abound (refer to the references cited in pp. II-144 – II-147 of [1] concerning robustness for linear systems and pp. II-147 – II-148 of [1] concerning robustness for nonlinear systems). Although several of these works are tangentially related to the present work, to the best of our knowledge the present results are new. In particular, results involving perturbations of equilibrium locations for discrete-time systems do not seem to have received much attention. Rather, the present results are more in the spirit of those established in [24] for the case of continuous-time systems. We emphasize, however, that the present results are not straightforward translations of the results given in [24] to the case of discrete-time systems.

In Section 2 we provide the necessary notation and definitions used throughout the paper. Given an exponentially stable equilibrium x_e for (2), we establish in Section 3 sufficient conditions for the exponential stability of an equilibrium $\bar{x}_e$ for (1) with the property the $\bar{x}_e$ is near x_e, i.e., $|x_e - \bar{x}_e|_\infty < \epsilon$, where ϵ is sufficiently small. To establish these results, we require several preliminary results which are established in the appendix.

In Section 4, we apply the above results in a perturbation analysis of a large class of discrete-time recurrent neural networks described by systems of first-order ordinary difference equations

$$x_i(k+1) = b_i x_i(k) + c_i s_i\left(\sum_{j=1}^{n} T_{ij} x_j(k) + I_i\right), \qquad i = 1, \cdot, n, \tag{3}$$

where x_i represents the state of the i-th neuron, $T = (T_{ij})_{n\times n}$ is the real-valued matrix of the synaptic connection weights, I_i is a constant external input to the i-th neuron, $s_i(\cdot)$ is the i-th nonlinear activation function, and the self-feedback constant and the neural gain are assumed to satisfy $-1 \le b_i \le 1$ and $c_i \ne 0$, $k \in Z_+$, respectively.

The paper is concluded with some pertinent remarks in Section 5.

2 Notation and Definitions

Let R denote the set of real numbers, let $R_+ = [0, \infty)$, and let R^n denote real n-dimensional vector space. If $x \in R^n$, then $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. Let Z and Z_+ denote the set of integers and the set of nonnegative integers, respectively.

If X and Y are subsets of R^n and R^m, respectively, we let $C[X, Y]$ denote the set of all continuous functions from X to Y. When X is an open subset of R^n, we let $C^N[X, Y]$ denote the set of all functions from X to Y whose partial derivatives up to order N are continuous, $N \ge 1$.

In R^n, we let $|\cdot|$ denote any equivalent norm if we do not specify a particular norm. The norms $|\cdot|_p$, $p \ge 1$, are defined by $|x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, and, in particular, when

$p = 1,\ p = 2$, and $p = \infty$, then $|x|_1 = \sum\limits_{i=1}^{n} |x_i|,\ |x|_2 = \Big(\sum\limits_{i=1}^{n} x_i^2\Big)^{1/2}$, and $|x|_\infty = \max\limits_{1\le i\le n} |x_i|$, respectively.

Let $A = [a_{ij}]$ denote an $n \times n$ matrix and let A^T denote the transpose of A. The matrix norms $|\cdot|_p,\ 1 \le p \le \infty$, induced by the norms $|\cdot|_p$ on $R^n,\ 1 \le p \le \infty$, are defined as $|A|_p = \sup\limits_{0\ne x\in R^n} [|Ax|_p/|x|_p],\ 1 \le p \le \infty$. In particular, we have $|A|_1 = \max\limits_{1\le j\le n} \sum\limits_{i=1}^{n} |a_{ij}|$, $|A|_2 = \Big(\sum\limits_{i,j=1}^{n} a_{ij}^2\Big)^{1/2}$, and $|A|_\infty = \max\limits_{1\le i\le n} \sum\limits_{j=1}^{n} |a_{ij}|$.

Let $x_e \in R^n$ and $\epsilon_0 > 0$ be an appropriate positive number. We define $B(x_e, \epsilon_0)$ by $B(x_e, \epsilon_0) = \{x\colon |x - x_e| < \epsilon_0\}$. In this paper, we assume that $f,\, h \in C^2[B(x_e, \epsilon_0), R^n]$.

Definition 2.1 A *square matrix* A is said to be *Schur stable*, if all eigenvalues of A are located within the unit circle.

Definition 2.2 For $f\colon R^n \to R^n$ and $x_e \in R^n$, $\frac{\partial f}{\partial x_i}(x_e)$ is defined by $\frac{\partial f}{\partial x_i}(x_e) = \Big(\frac{\partial f_1}{\partial x_i}(x_e), \dots, \frac{\partial f_n}{\partial x_i}(x_e)\Big)^T$ and $Df(x_e)$ is defined by the Jacobian matrix $\frac{\partial f}{\partial x}(x)\big|_{x=x_e}$.

In the present paper we use E to denote the $n \times n$ identity matrix.

3 Robustness Analysis of Perturbed Discrete-Time Systems

This section consists of three parts.

3.1 Robust stability: Perturbed discrete-time systems with fixed equilibria

In this subsection we first consider the special case where an equilibrium x_e of the original system (2) is unchanged in the resulting perturbed system (1).

In order to establish our first result, we consider the discrete-time systems with uncertainties and perturbations of the form

$$x(k+1) = (A + \Delta A)x(k) + m(x(k)), \tag{4}$$

where $x(k) \in R^n$, A and ΔA are constant and uncertain $n\times n$ matrices, respectively, $k \in Z_+$, $x(k) \equiv 0$ is an equilibrium of (4), $m \in C[U, R^n]$ satisfies the condition $\lim\limits_{x\to 0} \frac{|m(x)|}{|x|} = 0$, $U \subset R^n$ is an open subset containing x_e.

Lemma 3.1 *In addition to the assumptions $x_e = 0$ and $\lim\limits_{x\to 0} \frac{|m(x)|}{|x|} = 0$, we assume for system (4) that*

(i) *A is Schur stable;*

(ii) *$|\Delta A|_\infty < \sigma$, where $\sigma \in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big)$, and P is a symmetric and positive definite matrix determined by $A^TPA - P = -E$.*

Then the equilibrium $x(k) \equiv 0$ of (4) is exponentially stable.

Proof In applying the second method of Lyapunov, we choose the Lyapunov function given by $v(x(k)) = x^T(k)Px(k)$. Let $\Delta v(x(k))_{(4)} = v(x(k+1)) - v(x(k))$, where $x(k+1)$

satisfies the difference equation (4). For all $x(k) \in U$, we have, using condition (i) and (ii) as well as the relation $|\Delta A|_\infty \le |\Delta A|_2$,

$$\begin{aligned}
\Delta v(x(k))_{(4)} &= [Ax(k) + (\Delta A)x(k) + m(x(k))]^T P[Ax(k) \\
&\quad + (\Delta A)x(k) + m(x(k))] - x^T(k)Px(k) \\
&= x^T(k)[A^T PA - P]x(k) + x^T(k)[(\Delta A)^T PA \\
&\quad + A^T P(\Delta A) + (\Delta A)^T P(\Delta A)]x(k) \\
&\quad + 2[Ax(k) + (\Delta A)x(k)]^T Pm(x(k)) \\
&\quad + m(x(k))^T Pm(x(k)) \\
&= x^T(k)[-E + (\Delta A)^T PA + A^T P(\Delta A) \\
&\quad + (\Delta A)^T P(\Delta A)]x(k) + 2[Ax(k) \\
&\quad + (\Delta A)x(k)]^T Pm(x(k)) + m(x(k))^T Pm(x(k)) \\
&\le [-1 + 2\sigma|P|_2|A|_2 + \sigma^2|P|_2]x^T(k)x(k) \\
&\quad + 2x^T(k)[A + \Delta A]^T Pm(x(k)) + m(x(k))^T Pm(x(k)) \\
&< -4\epsilon x^T(k)x(k) + 2x^T(k)[A + \Delta A]^T Pm(x(k)) \\
&\quad + m(x(k))^T Pm(x(k)),
\end{aligned} \tag{5}$$

where $-4\epsilon = -1 + 2\sigma|P|_2|A|_2 + \sigma^2|P|_2 < 0$ by condition (ii). Since $\lim_{x\to 0}(|m(x)|/|x|) = 0$, it is clear that there exists an open subset of the origin, $V \subset U$, such that for all $x \in V$, $2x^T[A + (\Delta A)]^T Pm(x) < 2\epsilon x^T x$ and $m(x)^T Pm(x) < \epsilon x^T x$. Therefore, from (5) we obtain for $x(k) \in V$, $\Delta v(x(k))_{(4)} < -\epsilon x^T(k)x(k)$. By the basic stability theorem of Lyapunov, the equilibrium $x(k) \equiv 0$ of (4) is exponentially stable.

Remark 3.1 The existence and uniqueness of solutions of the Lyapunov equation $A^T PA - P = -E$ are guaranteed by the assumption that A is Schur stable (see, e.g., [3]).

We will require the following assumption.

Assumption 3.1 *For systems (1) and (2), it is true that*

(i) *x_e is an equilibrium of both (1) and (2);*
(ii) *$A = Df(x_e)$ is Schur stable;*
(iii) *$|\Delta A|_\infty < \sigma$, where $\Delta A = Dh(x_e)$, $\sigma \in \left(0, -|A|_2 + \left(|A|_2^2 + \frac{1}{|P|_2}\right)^{1/2}\right)$, and P is a symmetric and positive definite matrix determined by $A^T PA - P = -E$.*

Theorem 3.1 *Under Assumption 3.1, the equilibrium $x(k) \equiv x_e$ of system (1) is exponentially stable.*

Proof By the assumption that $f, h \in C^2[B(x_e, \epsilon_0), R^n]$ and $x(k) \equiv x_e$ is an equilibrium of (1), we can express (1) in the following equivalent form

$$x(k+1) - x_e = f(x(k)) - f(x_e) + h(x(k)) - h(x_e). \tag{6}$$

The right-hand side of (6) can be rewritten in the form

$$\begin{aligned}
&f(x(k)) - f(x_e) + h(x(k)) - h(x_e) \\
&= Df(x_e)(x(k) - x_e) + Dh(x_e)(x(k) - x_e) + m(x(k) - x_e) \\
&= (A + \Delta A)(x(k) - x_e) + m(x(k) - x_e),
\end{aligned} \tag{7}$$

where $m(\cdot)$ denotes the remaining higher-order terms with respect to $(x(k)-x_e)$.

Let $y(k)=x(k)-x_e$. Then system (1) can be rewritten in the following equivalent form

$$y(k+1)=(A+\Delta A)y(k)+m(y(k)). \tag{8}$$

It is clear that $y(k)\equiv 0$ is an equilibrium of (8) and all conditions of Lemma 3.1 are satisfied. Therefore, the equilibrium $y(k)\equiv 0$ of (8) is exponentially stable and thus the equilibrium $x(k)\equiv x_e$ of (1) is exponentially stable.

3.2 Robust stability: Perturbed discrete-time systems with perturbed equilibria

In this subsection, we will consider the case where the equilibrium $\bar{x}_e$ of the perturbed discrete-time system (1) differs from the equilibrium x_e of the unperturbed discrete-time system (2).

Assumption 3.2 *Let $\bar{x}_e$ and x_e denote the equilibrium of (1) and (2), respectively. Assume that*

(i) *$A=Df(x_e)$ is Schur stable;*
(ii) $|Dh(x_e)|_\infty < a_1$,

where $a_1=\frac{\sigma}{2}$, $\sigma\in\Big(0,-|A|_2+\Big(|A|_2^2+\frac{1}{|P|_2}\Big)^{\frac{1}{2}}\Big)$, A is given in (i) and P is a positive definite and symmetric matrix which is determined by $A^TPA-A=-E$; and

(iii) $|\bar{x}_e-x_e|_\infty<\epsilon$,

where $0<\epsilon<\epsilon_1$, $\epsilon_1=\min\Big\{\frac{\sigma}{2M_2},\epsilon_0\Big\}$, $M_2=\sup\limits_{x,y\in B(x_e,\epsilon_0)}|Q_2(x,y)|_\infty$, σ is given in part (ii), and $Q_2(x,y)$ satisfies the properties of Lemma A.1 with respect to $q=f+h$ (see the Appendix).

Theorem 3.2 *If Assumption 3.2 is satisfied, then the equilibrium $x(k)\equiv\bar{x}_e$ of the perturbed system (1) is exponentially stable.*

Proof Since $x(k)\equiv\bar{x}_e$ is an equilibrium of (1), we can rewrite (1) as

$$x(k+1)-\bar{x}_e=f(x(k))+h(x(k))-(f(\bar{x}_e)+h(\bar{x}_e)) \tag{9}$$

or its equivalent form

$$x(k+1)-\bar{x}_e=(Df(\bar{x}_e)+Dh(\bar{x}_e))(x(k)-\bar{x}_e)+m(x(k)-\bar{x}_e). \tag{10}$$

Let $A=Df(x_e)$ and $\Delta A=Df(\bar{x}_e)+Dh(\bar{x}_e)-Df(x_e)$. Then we can rewrite (10) as

$$x(k+1)-\bar{x}_e=(A+\Delta A)(x(k)-\bar{x}_e)+m(x(k)-\bar{x}_e). \tag{11}$$

Letting $y(k)=x(k)-\bar{x}_e$, (11) can be rewritten as

$$y(k+1)=(A+\Delta A)y(k)+m(y(k)). \tag{12}$$

Using Lemma A.1 in the Appendix and Remark A.1, we have

$$\begin{aligned}\Delta A&=Df(\bar{x}_e)+Dh(\bar{x}_e)-Df(x_e)\\&=Df(\bar{x}_e)+Dh(\bar{x}_e)-(Df(x_e)+Dh(x_e))+Dh(x_e)\\&=Q_2(\bar{x}_e,x_e)\Lambda(\bar{x}_e-x_e)+Dh(x_e),\end{aligned} \tag{13}$$

where Q_2 and Λ satisfy the properties of Lemma A.1 with respect to $q = f + h$ (see the Appendix).

Using parts (ii) and (iii) of Assumption 3.2, we have

$$\begin{aligned} |\Delta A|_\infty &\le |Q_2(\bar{x}_e, x_e)|_\infty \cdot |\bar{x}_e - x_e|_\infty + |Dh(x_e)|_\infty \\ &\le M_2|\bar{x}_e - x_e|_\infty + |Dh(x_e)|_\infty \le M_2\epsilon + a_1 < \frac{1}{2}\sigma + \frac{1}{2}\sigma = \sigma. \end{aligned} \tag{14}$$

It is clear that all conditions of Lemma 3.1 are satisfied for (12). We conclude that the equilibrium $y(k) \equiv 0$ of (12) is exponentially stable and thus the equilibrium $x(k) \equiv \bar{x}_e$ of (1) is exponentially stable.

3.3 Example

In the following, we utilize a specific example to demonstrate the applicability of Theorem 3.1. In the next section, we consider a general class of problems.

In (1) and (2), let $x = [x_1, x_2]^T$, $f(x) = [f_1(x), f_2(x)]^T$, $h(x) = [h_1(x), h_2(x)]^T$, $f_1(x) = x_1 - \frac{1}{2}\arctan x_1$, $f_2(x) = x_2 - \frac{1}{2}\arctan(x_1 + x_2)$, $h_1(x) = \delta_1 \arctan x_1$, and $h_2(x) = \delta_2 \arctan(x_1 + x_2)$, where δ_1 and δ_2 are perturbation parameters. Presently, systems (1) and (2) assume the form

$$\begin{aligned} x_1(k+1) &= x_1(k) - \frac{1}{2}\arctan x_1(k) + \delta_1 \arctan x_1(k), \\ x_2(k+1) &= x_2(k) - \frac{1}{2}\arctan(x_1(k) + x_2(k)) + \delta_2 \arctan(x_1(k) + x_2(k)) \end{aligned} \tag{15}$$

and

$$\begin{aligned} x_1(k+1) &= x_1(k) - \frac{1}{2}\arctan x_1(k), \\ x_2(k+1) &= x_2(k) - \frac{1}{2}\arctan(x_1(k) + x_2(k)), \end{aligned} \tag{16}$$

respectively.

$x_e = 0$ is an equilibrium for both (15) and (16). We have

$$A = Df(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

which is Schur stable. Also, $A^T P A - P = -E$ with $P = P^T$ yields

$$P = \begin{bmatrix} \frac{56}{27} & -\frac{4}{9} \\ -\frac{4}{9} & \frac{3}{4} \end{bmatrix}.$$

In our result we have $\sigma_M = -|A|_2 + \left(|A|_2^2 + \frac{1}{|P|_2}\right)^{1/2} = 0.2432$ and $\Delta A = Dh(0) = \begin{bmatrix} \delta_1 & 0 \\ \delta_2 & \delta_2 \end{bmatrix}$, $|\Delta A|_\infty| = \min\{|\delta_1|, 2|\delta_2|\}$. If $\min\{|\delta_1|, 2|\delta_2|\} < \sigma < 0.2030$. Theorem 3.1 implies that the state $x_e = 0$ is an exponentially stable equilibrium of (15).

4 Applications to Neural Networks

This section consists of three parts.

4.1 Model of discrete-time recurrent neural networks

In the present section we consider discrete-time recurrent neural networks described by systems of nonlinear difference equations of the form

$$x_i(k+1) = b_i x_i(k) + c_i s_i\left(\sum_{j=1}^{n} T_{ij} x_j(k) + I_i\right), \qquad i = 1, \cdots, n, \tag{17}$$

where x_i represents the state of the i-th neuron, $T = (T_{ij})_{n\times n}$ is the real-valued matrix of the synaptic connection weights, I_i is a constant external input to the i-th neuron, $s_i(\cdot)$ is the i-th nonlinear activation function, and the self-feedback constant and the neural gain are assumed to be $-1 \le b_i \le 1$ and $c_i \ne 0$, $k \in Z_+$, respectively.

In (17), the neural activation function $s_i(\cdot)$ is chosen to be a continuously differentiable nonlinear sigmoidal function (i.e., $s_i(\cdot)$ maps the real axis R into the real interval $(-1, 1)$, it is smooth and monotonically increasing, and its graph in the plane is symmetric with respect to the origin). Typical examples of activation functions include: $s_i(y_i) = \frac{2}{\pi} \arctan\left(\frac{\pi}{2}\, y_i\right)$, $s_i(y_i) = \frac{1-e^{-y_i}}{1+e^{-y_i}}$, and $s_i(y_i) = \tanh(y_i) = \frac{e^{y_i}-e^{-y_i}}{e^{y_i}+e^{-y_i}}$.

We can represent the neural network (17) in vector form as

$$x(k+1) = Bx(k) + Cs(Tx(k) + I), \tag{18}$$

where $x = (x_1, \cdots, x_n)^T$ is the state vector and $s(y) = (s_1(y_1), \cdots, s_n(y_n))^T$ for $y = (y_1, \cdots, y_n)^T \in R^n$. Also, $B = \operatorname{diag}[b_1, \cdots, b_n]$, $C = \operatorname{diag}[c_1, \cdots, c_n]$, $T = (T_{ij})_{n\times n}$, and $I = (I_1, \cdots, I_n)^T$.

Stability properties of recurrent discrete-time neural networks have been widely studied (see, e.g., [4, 10, 16, 18, 19, 21, 22]). Some of the most important applications of such networks concern associative memories (see, e.g., [4, 16, 18]).

For system (18) we consider the perturbation model

$$x(k+1) = (B + \Delta B)x(k) + (C + \Delta C)s[(T + \Delta T)x(k) + (I + \Delta I)], \tag{19}$$

where ΔB, ΔC, ΔT, and ΔI are the uncertain or perturbation matrices with the same dimension as B, C, T, and I, respectively.

In Feng and Michel [5], a robustness analysis for the neural network (18) is given. In the present section, we will consider the neural network (18) as a special case of (2) and apply the robustness results in Section 3 to the discrete-time system (2) to establish robustness results for the neural network (18).

4.2 Stability of perturbed neural networks with unperturbed equilibria

In this subsection we first consider the special case where an equilibrium x_e of the original system (18) is unchanged in the resulting perturbed system (19).

Let x_e be an equilibrium of system (18), let ϵ_0 be an appropriate fixed positive number, and let R_0, L_1, and L_2 denote positive real numbers satisfying $R_0 \ge |x_e|_\infty$, $L_1 \ge$

$\sup_{x\in B(x_e,\epsilon_0)} |s'(x)|$, and $L_2 \geq \sup_{x\in B(x_e,\epsilon_0)} |s''(x)|$, where $s'(x) = \operatorname{diag}[s'_1(x_1), \cdots, s'_n(x_n)]$, and $s''(x) = \operatorname{diag}[s''_1(x_1), \cdots, s''_n(x_n)]$, $s'_i(\cdot)$ and $s''_i(\cdot)$ denote the first-order and the second-order derivatives of $s_i(\cdot)$, respectively. In practice, L_1 and L_2 can frequently be chosen independently of x_e and ϵ_0. For example, if $s_j(x_j) = \arctan(\lambda_j x_j)$ with $\lambda_j > 0$, $1 \leq j \leq n$, then for all $x \in R^n$ we have $|s'(x)|_\infty \leq \max_{1\leq j\leq n}\{\lambda_j\}$ and $|s''(x)|_\infty \leq \max_{1\leq j\leq n}\{\lambda_j^2\}$. Therefore, in the present example, we may choose $L_1 = \max_{1\leq j\leq n}\{\lambda_j\}$ and $L_2 = \max_{1\leq j\leq n}\{\lambda_j^2\}$.

We will require the following assumption.

Assumption 4.1 *For systems (18) and (19), it is true that*

(i) x_e *is an equilibrium of both (18) and (19);*
(ii) $A = B + Cs'(Tx_e + I)T$ *is Schur stable;*
(iii) $\max\{|\Delta B|_\infty, |\Delta C|_\infty, |\Delta T|_\infty, |\Delta I|_\infty\} < K_0$, *where* K_0 *is given by*

$$K_0 = \frac{1}{2L_1}\big[-\beta + (\beta^2 + L_1\sigma)^{1/2}\big],$$

where

$$\beta = 1 + L_1|T|_\infty + L_1|C|_\infty + L_2|C|_\infty|T|_\infty(R_0 + 1),$$
$$\sigma \in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big),$$

and where $P = P^T$ *is a positive definite matrix that is determined by* $A^T PA - P = -E$, *and* A *is defined in (ii) above.*

We note that in Assumption 4.1, K_0 is a positive number determined by system (18) and is independent of the system perturbations. The following result shows that K_0 is an admissible bound for robust stability.

Proposition 4.1 *Under Assumption 4.1, the equilibrium* $x = x_e$ *of system (18) is exponentially stable.*

Proof Let

$$f(x) = Bx + Cs(Tx + I) \tag{20}$$

and

$$h(x) = (B + \Delta B)x + (C + \Delta C)s[(T + \Delta T)x + (I + \Delta I)] - [Bx + Cs(Tx + I)]. \tag{21}$$

Then (19) can be expressed in the form of $x(k + 1) = f(x(k)) + h(x(k))$, or in the form of (1). We have that

$$Df(x_e) = B + Cs'(Tx_e + I)T \tag{22}$$

and

$$Dh(x_e) = (\Delta B) + (C + \Delta C)s'[(T + \Delta T)x_e + (I + \Delta I)](T + \Delta T) - Cs'(Tx_e + I)T. \tag{23}$$

To show that the equilibrium $x = x_e$ of (19) is exponentially stable, we only need to verify that all conditions of Theorem 3.1 are satisfied, or to verify that all statements in

Assumption 3.1 are true. By part (ii) of Assumption 4.1, $B + Cs'(Tx_e + I)T$ is Schur stable and thus part (ii) of Assumption 3.1 is satisfied.

To show that part (iii) of Assumption 3.1 is also satisfied, it suffices to show that $|Dh(x_e)|_\infty < \sigma$, where $\sigma \in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big)$, and where P is given by $A^T PA - P = -E$. Using part (iii) of Assumption 4.1, we have

$$\begin{aligned} Dh(x_e) &= \Delta B + (C + \Delta C)s'[(T + \Delta T)x_e \\ &\qquad + (I + \Delta I)](T + \Delta T) - Cs'(Tx_e + I)T \\ &= \Delta B + Cs'[(T + \Delta T)x_e + (I + \Delta I)](\Delta T) \\ &\qquad + (\Delta C)s'[(T + \Delta T)x_e + (I + \Delta I)]T \\ &\qquad + (\Delta C)s'[(T + \Delta T)x_e + (I + \Delta I)](\Delta T) \\ &\qquad + CQ_2((T + \Delta T)x_e + (I + \Delta I), Tx_e + I)\Lambda((\Delta T)x_e + \Delta I)T, \end{aligned} \tag{24}$$

where Q_2 and Λ satisfy the properties of Lemma A.1 in the Appendix with respect to $q = s$. Using part (iii) of Assumption 4.1 and noticing that

$$\sup_{x,y \in B(x_e,\epsilon_0)} |Q_2(x, y)|_\infty \le L_2 = \sup_{x \in B(x_e,\epsilon_0)} |s''(x)|_\infty,$$

we obtain

$$\begin{aligned} |Dh(x_e)|_\infty &\le |\Delta B|_\infty + L_1|C|_\infty|\Delta T|_\infty + L_1|\Delta C|_\infty|T|_\infty + L_1|\Delta C|_\infty|\Delta T|_\infty \\ &\qquad + L_2R_0|C|_\infty|\Delta T|_\infty|T|_\infty + L_2|C|_\infty|\Delta I|_\infty|T|_\infty \le L_1K_0^2 + \beta K_0 < \sigma. \end{aligned} \tag{25}$$

This shows that part (iii) of Assumption 3.1 is satisfied. Therefore, the results follow from Theorem 3.1.

4.3 Stability of perturbed neural networks with perturbed equilibria

In this subsection, we will consider the case where the equilibrium $\bar{x}_e$ of the perturbed neural network (19) differs from the equilibrium x_e of the original neural network (18).

Assumption 4.2 *Let x_e and $\bar{x}_e$ denote equilibria of systems (18) and (19), respectively. Assume that*

(i) $A = B + Cs'(Tx_e + I)T$ *is Schur stable and therefore there exists a positive definite matrix* $P = P^T$ *determined by the matrix equation* $A^T PA - P = -E$;

(ii) $\max\{|\Delta B|_\infty, |\Delta C|_\infty, |\Delta T|_\infty, |\Delta I|_\infty\} < K_1$, *where K_1 is given by*

$$K_1 = \frac{1}{2L_1}\Big[-\beta + \Big(\beta^2 + \frac{L_1\sigma}{2}\Big)^{1/2}\Big],$$

where

$$\begin{aligned} \beta &= 1 + L_1(|T|_\infty + |C|_\infty) + L_2|C|_\infty|T|_\infty(R_0 + 1), \\ \sigma &\in \Big(0, -|A|_2 + \Big(|A|_2^2 + \frac{1}{|P|_2}\Big)^{1/2}\Big); \end{aligned}$$

and

(iii) $|\bar{x}_e - x_e| \le \epsilon$, *where* $0 < \epsilon < \bar{\epsilon}_1$, $\bar{\epsilon}_1 = \min\Big\{\frac{\sigma}{2\alpha_1 L_2}, \epsilon_0\Big\}$, *where* $\alpha_1 = (|C|_\infty + K_1)L_2(|T|^2 + 2|T|K_1 + K_1^2)$ *and ϵ_0 is given in the previous section.*

Proposition 4.2 *If Assumption 4.2 is true, then the equilibrium $\bar{x}_e$ of the perturbed system (19) is exponentially stable.*

Proof Let $f(x)$ and $h(x)$ be given by (20) and (21), respectively. To prove the result, it suffices to verify all conditions in Assumption 3.2.

From part (i) of Assumption 4.2, it follows that $A = Df(x_e)$ is Schur stable and thus part (i) of Assumption 3.2 is true.

Using similar statements as in the proof of Proposition 4.1 (see (24) and (25)), we can prove that part (ii) of Assumption 4.1 implies part (ii) of Assumption 3.1.

To show part (iii) of Assumption 3.1 is also satisfied, it suffices to verify that $\bar{\epsilon}_1 \leq \epsilon_1$, where $\epsilon_1 = \min\left\{\frac{\sigma}{2M_2}, \epsilon_0\right\}$ or $M_2 \leq \alpha_1 L_2$, where $M_2 = \sup\limits_{x,y \in B(x_e,\epsilon_0)} |Q_2(x,y)|$, where Q_2 is a function satisfying the properties of Lemma A.1 with respect to $f + h$ with $f(x) + h(x) = (B + \Delta B)x + (C + \Delta C)s[(T + \Delta T)x + (I + \Delta I)]$. Using part (iii) of Assumption 4.2 and the definition of L_2, we have

$$
\begin{aligned}
M_2 = \sup_{x,y\in B(x_e,\epsilon_0)} |Q_2(x,y)|_\infty &= \sup_{x,y\in B(x_e,\epsilon_0)} \Bigg| \int_0^1 (C+\Delta C) \\
&\times s''[(T+\Delta T)(x+t(y-x)) + (I+\Delta I)](T+\Delta T)^2\,dt \Bigg|_\infty \\
&\leq |C+\Delta C|_\infty \sup_{x\in B(x_e,\epsilon_0)} |s''(x)|_\infty |T+\Delta T|^2 \\
&\leq (|C|_\infty + |\Delta C|_\infty) L_2 (|T|_\infty^2 + 2|T|_\infty |\Delta T|_\infty + |\Delta T|_\infty^2) \leq \alpha_1 L_2
\end{aligned}
\tag{26}
$$

which implies $\bar{\epsilon}_1 \leq \epsilon_1$.

This shows that all conditions of Assumption 3.2 are satisfied. Therefore, the result of Proposition 4.2 follows from Theorem 3.2.

Remark 4.1 It should be noted that in Assumption 4.2, the existence of an equilibrium of the perturbed system (19) is hypothesized to be not far away from the corresponding equilibrium of the unperturbed system (18). It is reasonable to expect that when the perturbations of the system in question are sufficiently small, this assumption will be satisfied.

5 Concluding Remarks

A robustness analysis was conducted for a large class of nonlinear discrete-time systems described by ordinary difference equations under perturbations. The results presented aimed to give an answer to the following question: given a nonlinear discrete-time system with specified exponentially stable equilibria, under what conditions will a perturbed model of the discrete-time system possess exponentially stable equilibria that are close (in distance) to the exponentially stable equilibria of the unperturbed model? Robustness stability results for perturbed nonlinear discrete-time systems were established. Using these results, a set of sufficient conditions was established for robust stability of a large class of discrete-time recurrent neural networks for associative memories under perturbations of system parameters.

Appendix

We require the following result in the proofs of Theorem 3.2 and Proposition 4.2.

Lemma A.1 *Let $q \in C^2[\bar{U}, R^n]$, where $U \subset R^n$ is a convex open set and $\bar{U}$ denotes the closure of U. Then there exists a $Q_1 \in C^1[U \times U, R^{n\times n}]$ and $Q_2 \in C^1[U \times U, R^{n\times n^2}]$ satisfying the following properties for all $x, y \in U$:*

(i) *$q(x) - q(y) = Q_1(x,y)(x-y)$, where $Q_1(x,y)$ is given by*

$$Q_1(x,y) = \int_0^1 Dq^T(x + t(y-x))\,dt; \tag{A.1}$$

(ii) *$Dq(x) - Dq(y) = Q_2(x,y)\Lambda(x-y)$, where $Q_2(x,y)$ and $\Lambda(x-y)$ are given by*

$$Q_2(x,y) = [Q_{21}(x,y), \cdots, Q_{2n}(x,y)] \tag{A.2}$$

with

$$Q_{2i}(x,y) = \int_0^1 D\left(\frac{\partial q}{\partial x_i}\right)(x + t(y-x))\,dt, \tag{A.3}$$

and

$$\Lambda(x-y) = \begin{bmatrix} x-y & 0 & \dots & 0 \\ 0 & x-y & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x-y \end{bmatrix}, \tag{A.4}$$

respectively.

Proof Part (i) can be proved by using the following formula from the calculus (refer to pp. 48–49 in Chapter 2 of [3]):

$$q(x) - q(y) = \left(\int_0^1 Dq(x + t(y-x))\,dt\right)(x-y). \tag{A.5}$$

Part (ii) can be obtained by using part (i) for every column of Dq:

$$Dq(x) - Dq(y) = \left[\int_0^1 D\left(\frac{\partial q}{\partial x_1}(x+t(y-x)\right)dt, \cdots, \int_0^1 D\left(\frac{\partial q}{\partial x_n}(x+t(y-x)\right)dt\right]. \tag{A.6}$$

Remark A.1 In the following we assume that $U = B(x_e, \epsilon_0)$, where $x_e \in R^n$, $\epsilon_0 > 0$. As a consequence of Lemma A.1, for any $x, y \in U$, if $U \in R^n$ is bounded, then we have

$$|q(x) - q(y)|_\infty \le |Q_1(x,y)|_\infty \cdot |x-y|_\infty \le M_1|x-y|_\infty \tag{A.7}$$

and

$$|Dq(x) - Dq(y)|_\infty \le |Q_2(x,y)|_\infty \cdot |x-y|_\infty \le M_2|x-y|_\infty, \tag{A.8}$$

where $M_1 = \sup_{x\in U} |Dq(x)|_\infty$ and $M_2 = \sup_{x,y\in U} |Q_2(x,y)|_\infty$.

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