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Subharmonic Solutions of a Class of Hamiltonian Systems

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Abstract: In this paper, we prove the existence of subharmonic solutions for the non autonomous Hamiltonian system: $\dot{u}(t) = J\nabla H(t, u(t))$ when H is convex and non coercive.

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1 Introduction and Statement of Results

Let $G \in C^1(\mathbb{R}^n, \mathbb{R})$ be a convex function, A, $B \in C(\mathbb{R}, \mathcal{M}_n(\mathbb{R}))$ be periodic with minimal period T $(T > 0)$, $B(t)$ be invertible for all $t \in \mathbb{R}$ and $h = (f, g) \in C(\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^n)$ be T-periodic with mean value zero.

Let $H(t,(r,p)) = G(A(t)r + B(t)p) + \prec h(t),(r,p) \succ$, $\forall\,(r,p) \in \mathbb{R}^n \times \mathbb{R}^n$, $\forall\, t \in \mathbb{R}$.

In this paper we consider the Hamiltonian system of ordinary differential equations

$$\dot{u}(t) = J\nabla H(t, u(t)), \tag{$\mathcal{H}_h$}$$

where ∇H is the first derivative of the Hamiltonian H with respect to (r, p) and J is the standard symplectic $(2n \times 2n)$-matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The motion of a relativist particle submitted to an electromagnetic field is governed by a noncoercive Hamiltonian system. However, most of results proving the existence of solutions to systems like $(\mathcal{H}_h)$ have been made use of a coercivity assumption on H, i.e., $\lim\limits_{|x|\to+\infty} H(t, x) = \infty$, see for example [5, 8, 9, 12] and references therein.

Timoumi investigates the case of non coercivity when H is convex (see [10, 11]). The purpose of this paper is to improve and complete the results obtained in [10, 11] dealing with this problem.

In the first theorem we establish the existence of subharmonic solutions, i.e., periodic solutions with minimal period in the set $\{kT,\, k \in \mathbb{N},\, k \geq 2\}$ for the Hamiltonian system of ordinary differential equations $(\mathcal{H}_0)$.

The problem of search for subharmonics is classical, it has been dealt with using various methods, especially index theories in different settings, see [3, 5, 6, 12].

In [10], Timoumi studied the question when the Hamiltonian has the form

$$H(t,(r,p)) = f(|p - A(t)r|),$$

where $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\exists\, \lambda,\, \mu > 0 / f(t) \leq \lambda t + \mu \quad \forall\, t \geq 0$$

and the matrix $A(t)$ satisfies

1. $A^*(t) = -A(t)\ \forall\, t \in \mathbb{R}$
2. $\int\limits_0^T A(t)\, dt \neq 0.$

Here, we try to conserve the same results when the Hamiltonian is subquadratic and $A(t)$ belongs to a larger set of matrices.

Precisely, we assume

(H_1) $\lim\limits_{|x|\to+\infty} G(x) = +\infty;$

(H_2) $\lim\limits_{|x|\to\infty} \frac{G(x)}{|x|^2} = 0;$

(H_3) G' is one to one;

(H_4) $C_0 = \int\limits_0^T B^{-1}(t)A(t)\, dt$ is non symmetric.

Theorem 1.1 *Under the above assumptions, for all $k \in \mathbb{N}^*$, $(\mathcal{H}_0)$ possesses a kT periodic solution $u_k = (r_k, p_k)$ satisfying*

(i) $\lim\limits_{k\to+\infty} \|Ar_k + Bp_k\|_\infty = +\infty.$

(ii) *The minimal period of u_k is kT for any sufficiently large and prime integer k.*

Corollary 1.1 *Under the assumptions (H_2), (H_4) and*

(H_5) *G is strictly convex;*

(H_6) $\lim\limits_{|x|\to\infty} \frac{G(x)}{|x|} = +\infty$

the conclusion of Theorem 1.1 holds.

The second result concerns the forced case ($h \neq 0$), where h is interpreted as exterior forcing term. Here we prove the existence of a non constant T-periodic solution for $(\mathcal{H}_h)$ without the following assumption, needed in [11]

$$\forall\, r \in \mathbb{R}^n \backslash \{0\} \quad t \longmapsto A(t)r \quad \text{is non constant.}$$

Assume that

(H_7) $G(x) > G(0),\ \forall\, x \in \mathbb{R}^n \backslash \{0\};$

(H_8) $(B^{-1}A)^* g \neq f.$

Theorem 1.2 *Under assumptions* (H_1), (H_2), (H_7), (H_8), *the problem* $(\mathcal{H}_h)$ *possesses a non constant* T*-periodic solution.*

Remark 1.1 The assumption (H_8) is technical, it will be used only to guarantee the non constancy of solution for $(\mathcal{H}_h)$.

2 Proof of Theorem 1.1

Proof of the first part:

We use the dual action of Clarke-Ekeland.

Denote $H_0(t,r,p) = G(A(t)r + B(t)p)$. H_0 is convex with respect to (r,p) and its Fenchel's conjugate H_0^* is given by

$$\forall\,(s,q) \in \mathbb{R}^n \times \mathbb{R}^n, \quad H_0^*(t,s,q) = \begin{cases} G^*(B^{-1*}q) & \text{if } s = (B^{-1}A)^*q, \\ +\infty & \text{otherwise.} \end{cases}$$

For all $k \in \mathbb{N}^*$ we consider the functional

$$\Phi_k(w) = \frac{1}{2}\int_0^{kT} \prec Jw, \pi w \succ\, dt + \int_0^{kT} H_0^*(t,w)\,dt$$

defined on the space

$$L_0^2(0,kT,\mathbb{R}^{2n}) = \left\{ w \in L^2(0,kT,\mathbb{R}^{2n}) \Big/ \int_0^{kT} w(t)\,dt = 0 \right\},$$

where πw is the primitive of w with mean value zero.

Also, for all $v \in L_0^2(0,kT,\mathbb{R}^n)$ we define

$$\Psi_k(v) = \int_0^{kT} \prec B^{-1}A\pi v, v \succ\, dt + \int_0^{kT} G^*(B^{-1*}v)\,dt.$$

Obviously, we have $\Phi_k(w) = \Psi_k(v)$ for all $w = ((B^{-1}A)^*v, v) \in L_0^2(0,kT,\mathbb{R}^{2n})$.

Hence, we use the functional Ψ_k on the space $E_k = L_0^2(0,kT,\mathbb{R}^n)$.

For $v \in E_k$ we set

$$g(v) = \int_0^{kT} G^*(B^{-1*}v)\,dt$$

and

$$Q(v) = \int_0^{kT} \prec B^{-1}A\pi v, v \succ\, dt.$$

Lemma 2.1 *Ψ_k has a global minimum on E_k attained in $\bar{v}_k$.*

Proof Using Wirtinger's inequality, there exists a constant $\alpha_0 > 0$ such that

$$Q(v) \geq -\alpha_0 \|v\|^2_{L^2}, \quad \forall\, v \in E_k. \tag{1}$$

By (H_2), for all $\alpha > 0$ there exists $\beta > 0$ such that

$$G(x) \leq \alpha |x|^2 + \beta, \quad \forall\, x \in \mathbb{R}^n \tag{2}$$

and by going to the conjugate, we get

$$G^*(y) \geq \frac{1}{4\alpha}\, |y|^2 - \beta, \quad \forall\, y \in \mathbb{R}^n$$

so

$$g(v) \geq \frac{1}{4\alpha} \|B^{-1*} v\|^2_{L^2} - \beta kT, \quad \forall\, v \in E_k. \tag{3}$$

From (1) and (3) there exists a constant $\gamma > 0$ such that

$$\Psi_k(v) \geq \gamma \|v\|^2_{L^2} - \beta kT, \quad \forall\, v \in E_k. \tag{4}$$

Let $(v_n) \in E_k$ be a minimizing sequence of Ψ_k. From (4), (v_n) is bounded and since E_k is reflexive, there exists a subsequence (v_{n_j}) weakly convergent to $\bar{v}_k$.

Moreover, g is weakly lower semi-continuous, so

$$\underline{\lim} \int_0^{kT} G^*(B^{-1*} v_{n_j})\, dt \geq \int_0^{kT} G^*(B^{-1*} \bar{v}_k)\, dt.$$

Since the operator π is compact then

$$\pi v_{n_j} \longrightarrow \pi \bar{v}_k$$

and so

$$\lim_{j \to +\infty} \int_0^{kT} \prec B^{-1} A \pi v_{n_j}, v_{n_j} \succ\, dt = \int_0^{kT} \prec B^{-1} A \pi \bar{v}_k, \bar{v}_k \succ\, dt.$$

Consequently

$$\min_{E_k} \Psi_k = \Psi_k(\bar{v}_k).$$

Lemma 2.2 *For all $v \in E_k$ on which g is finite we have*

$$\bar{\partial} g(v) = \Big\{ u \in L^2(0, kT, \mathbb{R}^n) / \exists\, \xi \in \mathbb{R}^n \colon B(t)(u(t) + \xi) \in \partial G^*(B^{-1*} v) \ a.e. \Big\},$$

where $\bar{\partial} g$ denotes the restriction of g on E_k.

Proof Let $u \in L^2(0, kT, \mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$ such that

$$B(t)(u(t) + \xi) \in \partial G^*(B^{-1*} v) \quad a.e.$$

so it's easy to show that $u \in \bar{\partial} g(v)$.

Conversely, it's clear that for $v \in E_k$

$$\bar{\partial} g(v) = \partial(g + \delta_{E_k})(v),$$

where

$$\delta_{E_k}(v) = \begin{cases} 0 & \text{if } v \in E_k, \\ +\infty & \text{otherwise.} \end{cases}$$

Since

$$\partial g(v) = \{u \in L^2(0, kT, \mathbb{R}^n) / B(t)u(t) \in \partial G^*(B^{-1*}v) \ a.e.\}$$

and

$$\partial \delta_{E_k} = \mathbb{R}^n$$

the result will be proved if

$$\partial(g + \delta_{E_k}) = \partial g + \partial \delta_{E_k}.$$

The functionals g and δ_{E_k} are proper convex and l.s.c., it suffices to prove that the inf-convolute $g^* \nabla \delta^*_{E_k}$ is exact (i.e., the infimum is attained).

Indeed, we have

$$(g^* \nabla \delta^*_{E_k})(v) = \inf_{x \in \mathbb{R}^n} \int_0^{kT} G(B(t)v + B(t)x)\, dt.$$

The function

$$F(x) = \int_0^{kT} G(B(t)v + B(t)x)\, dt, \quad \forall\, x \in \mathbb{R}^n$$

is continuous on $\mathbb{R}^n$, so by (H_1) and the fact that $B(t)$ is invertible it's clear that $\lim\limits_{|x| \to +\infty} F(x) = +\infty$ and consequently F attains its minimum on $\mathbb{R}^n$.

Conclusion of the first part:

Let $\bar{v}_k \in E_k$, where Ψ_k attains its minimum, we have

$$0 \in Q'(\bar{v}_k) + \bar{\partial} g(\bar{v}_k)$$

which implies that

$$-Q'(\bar{v}_k) \in \bar{\partial} g(\bar{v}_k).$$

By Lemma 2.2, there exists $\xi_k \in \mathbb{R}^n$ such that

$$B(-B^{-1}A\pi\bar{v}_k + \pi(B^{-1}A)^*\bar{v}_k + \xi_k) \in \partial G^*(B^{-1*}\bar{v}_k) \quad a.e.$$

Setting

$$r_k = -\pi\bar{v}_k, \quad p_k = \pi(B^{-1}A)^*\bar{v}_k + \xi_k, \quad u_k = (r_k, p_k). \tag{5}$$

We get, by Fenchel's reciprocity

$$B^{-1^*}\bar{v}_k = \nabla G(Ar_k + Bp_k) \tag{6}$$

and

$$\begin{cases} \dot{r}_k = -\bar{v}_k = -B^*\nabla G(Ar_k + Bp_k) = -\dfrac{\partial H_0}{\partial p}(t, u_k(t)) \\ \dot{p}_k = (B^{-1}A)^*\bar{v}_k = A^*\nabla G(Ar_k + Bp_k) = \dfrac{\partial H_0}{\partial r}(t, u_k(t)). \end{cases}$$

Therefore u_k is a solution of $(\mathcal{H}_0)$, moreover since $\bar{v}_k \in E_k$, r_k is kT periodic.

In the other hand r_k is C^1 so $\dot{r}_k$ is kT periodic. By (H_3) and (6), we have

$$p_k = B^{-1}[\nabla G^{-1}(-B^{-1*}\dot{r}_k) - Ar_k]$$

so p_k is kT periodic and then u_k is kT periodic.

Proof of the second part:

By (H_1) and the convexity assumption of G there exist two constants $m, M > 0$ such that

$$G(x) \geq m|x| - M, \quad \forall\, x \in \mathbb{R}^n \tag{7}$$

so for all $y \in \mathbb{R}^n$ such that $|y| \leq m$ we have

$$-G(0) \leq G^*(y) \leq M. \tag{8}$$

Let

$$q(t) = a\cos\left(\frac{2\pi}{kT}\,t\right) + b\sin\left(\frac{2\pi}{kT}\,t\right)$$

with any $(a, b) \in \mathbb{R}^{2n}$.

It's clear that $q \in E_k$ and a simple computation gives for all $k \geq 3$

$$Q(q) = \frac{k^2T^2}{4\pi} \prec (C_0 - C_0^*)a, b \succ .$$

By the assumption (H_4), we can choose (a, b) such that

$$\begin{cases} \prec (C_0 - C_0^*)a, b \succ < 0 \\ \|B^{-1^*}q\|_\infty \leq m. \end{cases} \tag{9}$$

Setting $\delta = -\frac{T}{4\pi} \prec (C_0 - C_0^*)a, b \succ$, we have

$$Q(q) = -\delta T k^2, \text{ with } \delta > 0 \text{ independent of } k.$$

Now, by (8) and (9) we have

$$\Psi_k(\bar{v}_k) \leq \Psi_k(q) \leq -\delta T k^2 + MkT, \quad \forall\, k \geq 3 \tag{10}$$

and

$$Q(\bar{v}_k) \leq -\delta T k^2 + MkT + G(0)kT \leq 0 \tag{11}$$

for all $k \geq k_0$ sufficiently large.

In the other hand, by duality we have

$$G(Ar_k+Bp_k)+G^*(B^{-1*}\bar v_k)=\prec Ar_k+Bp_k, B^{-1*}\bar v_k\succ$$

and by integration, we obtain

$$\int_0^{kT} G(Ar_k+Bp_k)\,dt+\int_0^{kT} G^*(B^{-1^*}\bar v_k)\,dt=-2\int_0^{kT}\prec B^{-1}A\pi\bar v_k,\bar v_k\succ\ dt.$$

Then it follows from (10) and (11) that

$$\int_0^{kT} G(Ar_k+Bp_k)\,dt=-Q(\bar v_k)-\Psi_k(\bar v_k)\ge \delta Tk^2-MkT,\quad \forall\, k\ge k_0$$

which gives

$$\frac{1}{kT}\int_0^{kT} G(Ar_k+Bp_k)\,dt\ge \delta k-M,\quad \forall\, k\ge k_0.$$

Hence by (2) we obtain

$$\delta k-M\le \frac{\alpha}{kT}\int_0^{kT}|Ar_k+Bp_k|^2dt+\beta\le \alpha\|Ar_k+Bp_k\|_\infty^2+\beta,\quad \forall\, k\ge k_0$$

and consequently

$$\lim_{k\to+\infty}\|Ar_k+Bp_k\|_\infty=+\infty.$$

To prove (ii) of Theorem 1.1, we need the following lemma:

Lemma 2.3 *For all T-periodic solution $u=(r,p)$ of $(\mathcal{H}_0)$ we have*

1. $\int_0^T|\dot u|^2dt\le \frac{2\alpha(\beta+M)\pi T}{\pi-\alpha T}$,
2. $\frac{1}{T}\int_0^T|Ar+Bp|\,dt\le \frac{(\beta+M)\pi}{m(\pi-\alpha T)}$.

Proof By (H_2) and (7), for all $\alpha\in\left]0,\frac{\pi}{T}\right[$ there exists $\beta>0$ only dependent on α such that

$$-M\le H_0(t,x)\le\frac{\alpha}{2}|x|^2+\beta,\quad \forall\, x\in\mathbb{R}^{2n},\quad \forall\, t\in[0,T].$$

A result of convex analysis gives

$$\frac{1}{2\alpha}|\nabla H_0(t,x)|^2\le \prec\nabla H_0(t,x),x\succ+\beta+M,\quad \forall\, x\in\mathbb{R}^{2n}.$$

It follows from $(\mathcal{H}_0)$ that

$$\frac{1}{2\alpha}\int_0^T|\dot u|^2dt+\int_0^T\prec J\dot u,u\succ\ dt\le(\beta+M)T$$

so

$$\left(\frac{1}{2\alpha} - \frac{T}{2\pi}\right) \int_0^T |\dot{u}|^2 dt \leq (\beta + M)T$$

and therefore

$$\int_0^T |\dot{u}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T}{\pi - \alpha T}. \tag{12}$$

By convexity and (7), for all T-periodic solution $u = (r, p)$ of $(\mathcal{H}_0)$ we have

$$m \int_0^T |Ar + Bp|\, dt - MT \leq TG(0) + \frac{T}{2\pi} \int_0^T |\dot{u}|^2 dt. \tag{13}$$

By (12) and (13), we deduce the desired result.

Now, we shall prove that the minimal period of u_k tends to $+\infty$ as k tends to $+\infty$. If not, there exists $\tau > 0$ and a subsequence (k_n) such that the minimal period T_{k_n} of u_{k_n} satisfies $T_{k_n} \leq \tau$, $\forall\, n \in \mathbb{N}$. By Lemma 2.3, with T replaced by T_{k_n}, we get

$$\int_0^{T_{k_n}} |\dot{u}_{k_n}|^2 dt \leq \frac{2\alpha(\beta + M)\pi T_{k_n}}{\pi - \alpha T_{k_n}} \leq \frac{2\alpha(\beta + M)\pi\tau}{\pi - \alpha\tau} \tag{14}$$

and

$$\frac{1}{T_{k_n}} \int_0^{T_{k_n}} |Ar_{k_n} + Bp_{k_n}|\, dt \leq \frac{\pi(\beta + M)}{m(\pi - \alpha\tau)}. \tag{15}$$

Writing $u_k = \bar{u}_k + \tilde{u}_k$ with $\bar{u}_k = \frac{1}{T_k} \int_0^{T_k} u_k(t)\, dt$.

By Sobolev's inequality and (14), we obtain

$$\|\tilde{u}_{k_n}\|_\infty^2 \leq \frac{\tau}{12}\left(\frac{2\alpha(\beta + M)\pi\tau}{\pi - \alpha\tau}\right)$$

thus $\|\tilde{u}_{k_n}\|_\infty$ is bounded. By (5) we have

$$\bar{u}_{k_n} = (\bar{r}_{k_n}, \bar{p}_{k_n}) = (0, \xi_{k_n}).$$

Since $\|u_{k_n}\|_\infty \longrightarrow +\infty$ and $\|\widetilde{u}_{k_n}\|_\infty$ is bounded so $|\xi_{k_n}| \longrightarrow +\infty$.

In the other hand, by (15) we deduce that

$$\frac{1}{T} \int_0^T |B(t)\xi_{k_n}|\, dt = \frac{1}{T_{k_n}} \int_0^{T_{k_n}} |A(t)\bar{r}_{k_n} + B(t)\bar{p}_{k_n}|\, dt$$

is bounded, but this is in contradiction with the fact that

$$|B(t)\xi_{k_n}| \longrightarrow +\infty, \quad \forall\, t \in [0, T].$$

Then, the minimal period T_k of u_k tends to $+\infty$ as k tends to $+\infty$ and so for sufficiently large prime integer k, the minimal period of u_k is kT.

3 Proof of Theorem 1.2

We consider the functional Φ defined on the space $L_0^2 = L_0^2(0,T,\mathbb{R}^{2n})$ by

$$\Phi(w) = \frac{1}{2}\int_0^T \prec Jw, \pi w \succ\, dt + \int_0^T H_0^*(t, w-h)\, dt.$$

Let for $w \in L_0^2$

$$Q(w) = \frac{1}{2}\int_0^T \prec Jw, \pi w \succ\, dt \quad \text{and} \quad \psi(w) = \int_0^T H_0^*(t, w-h)\, dt.$$

We follow the same ideas of the proof of Theorem 1.1.

Lemma 3.1 *Φ achieves its minimum over L_0^2 in $\bar{v}$.*

Proof By (H_2), for all $\alpha \in]0, \frac{2\pi}{T}[$ there exists $\beta > 0$ such that

$$H_0(t,x) \le \frac{\alpha}{2}|x|^2 + \beta, \quad \forall\, x \in \mathbb{R}^{2n}, \quad \forall\, t \in [0,T],$$

and by going to the conjugate, we get

$$H_0^*(t,y) \ge \frac{1}{2\alpha}|y|^2 - \beta, \quad \forall\, y \in \mathbb{R}^{2n}, \quad \forall\, t \in [0,T]$$

so

$$\int_0^T H_0^*(t,w)\, dt \ge \frac{1}{2\alpha}\|w\|_{L^2}^2 - \beta T, \quad \forall\, w \in L_0^2.$$

Moreover, by Wirtinger's inequality, we get for all $w \in L_0^2$

$$\Phi(w) \ge \frac{1}{2}\left(\frac{1}{\alpha} - \frac{T}{2\pi}\right)\|w\|_{L^2}^2 + \frac{1}{2\alpha}\|h\|_{L^2}^2 - \frac{1}{\alpha}\|w\|_{L^2}\|h\|_{L^2} - \beta T. \tag{16}$$

Let $(v_n) \in L_0^2$ be a minimizing sequence of Φ. From (16), (v_n) is bounded and since L_0^2 is reflexive, there exists a subsequence (v_{n_k}) weakly convergent to $\bar{v}$.

Moreover, ψ is weakly l.s.c., so

$$\underline{\lim}\int_0^T H_0^*(t, v_{n_k} - h)\, dt \ge \int_0^T H_0^*(t, \bar{v} - h)\, dt$$

and

$$\lim_{k\to+\infty}\int_0^T \prec Jv_{n_k}, \pi v_{n_k} \succ\, dt = \int_0^T \prec J\bar{v}, \pi\bar{v} \succ\, dt.$$

Consequently

$$\min_{L_0^2} \Phi = \Phi(\bar{v}).$$

Lemma 3.2 *For every $v \in L_0^2$ on which ψ is finite, we have*

$$\bar{\partial}\psi(v) = \{u \in L^2 / \exists \xi \in \mathbb{R}^{2n} : u(t) + \xi \in \partial H_0^*(t, v(t) - h(t)) \ a.e.\}.$$

Proof Let $I(v) = \int_0^T H_0^*(t,v)\,dt,\ \forall\, v \in L^2$, then $\psi(v) = I(v-h)$.

For $u, v \in L_0^2$ and $\xi \in \mathbb{R}^{2n}$ such that

$$u(t) + \xi \in \partial H_0^*(t, v(t)) \ a.e.,$$

we can prove easily that $u \in \bar{\partial} I(v)$.

Conversely, it's clear that for $v \in L_0^2$ we have

$$\bar{\partial} I(v) = \partial(I + \delta_{L_0^2})(v),$$

where

$$\delta_{L_0^2}(v) = \begin{cases} 0 & \text{if } v \in L_0^2, \\ +\infty & \text{otherwise.} \end{cases}$$

Arguing as in proof of Lemma 2.2, it suffices to prove that the inf-convolution $I^* \nabla \delta^*_{L_0^2}$ is exact.

In fact, for $u = (r,p) \in L^2$ we have

$$(I^* \nabla \delta^*_{L_0^2})(u) = \inf_{x \in \mathbb{R}^{2n}} \int_0^T H_0(t, u(t) + x)\,dt$$

$$= \inf_{(a,b) \in \mathbb{R}^{2n}} \int_0^T G[A(t)r + B(t)p + A(t)a + B(t)b]\,dt.$$

We need the following lemma:

Lemma 3.3 *The function*

$$F(a,b) = \int_0^T G(A(t)r + B(t)p + A(t)a + B(t)b)\,dt, \quad \forall\,(a,b) \in \mathbb{R}^{2n}$$

attains its minimum on $\mathbb{R}^{2n}$.

Proof Let

$$E = \Big\{a \in \mathbb{R}^n / B^{-1}(t)A(t)a = B^{-1}(0)A(0)a,\ \forall\, 0 \le t \le T\Big\},$$

E is a linear subspace of $\mathbb{R}^n$, so for all $a \in \mathbb{R}^n$ there exists $a_0 \in \mathbb{R}^n$ such that $a - a_0 \in E^\perp$. Notice that

$$F(a,b) = F(a - a_0, b + B^{-1}A(0)a_0) \in F(E^\perp \times \mathbb{R}^n)$$

so

$$\inf_{\mathbb{R}^{2n}} F = \inf_{E^{\perp}\times\mathbb{R}^n} F.$$

Arguing by contradiction, we suppose that $\inf\limits_{E^{\perp}\times\mathbb{R}^n} F$ is not attained so there exists a sequence $(a_n, b_n) \in E^{\perp} \times \mathbb{R}^n$ such that

$$\lim_{n\to+\infty}(a_n^2 + b_n^2) = +\infty \quad \text{and} \quad \lim_{n\to+\infty} F(a_n, b_n) = \inf F.$$

It follows that

$$\lim_{n\to+\infty} \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} = 0.$$

In the other hand, by convexity of G, we have for n large enough

$$\int_0^T G\left(\frac{A(t)r + B(t)p + A(t)a_n + B(t)b_n}{\sqrt{a_n^2 + b_n^2}}\right) dt \le \frac{F(a_n, b_n)}{\sqrt{a_n^2 + b_n^2}} + \left(1 - \frac{1}{\sqrt{a_n^2 + b_n^2}}\right) G(0)T.$$

The sequence

$$\left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \frac{b_n}{\sqrt{a_n^2 + b_n^2}}\right) \in E^{\perp} \times \mathbb{R}^n$$

is bounded, then by going to the limit in the above inequality through a subsequence, we obtain

$$\int_0^T G(A(t)a + B(t)b)\, dt \le G(0)T$$

for some $(a, b) \in E^{\perp} \times \mathbb{R}^n$ such that $a^2 + b^2 = 1$. Then

$$\int_0^T [G(A(t)a + B(t)b) - G(0)]\, dt \le 0$$

and by (H_7) we obtain

$$A(t)a + B(t)b = 0, \quad \forall\, t \in [0, T]$$

which is equivalent to

$$B^{-1}(t)A(t)a + b = 0, \quad \forall\, t \in [0, T],$$

but this is in contradiction with $a \in E^{\perp}$ and $a^2 + b^2 = 1$.

Conclusion of the proof

Let $\bar{v} \in L_0^2$ where Φ attains its minimum so

$$0 \in J\pi\bar{v} + \bar{\partial}\psi(\bar{v}).$$

By Lemma 3.2, there exists $\xi \in \mathbb{R}^{2n}$ such that

$$J\pi\bar{v} + \xi \in \partial H_0^*(t, \bar{v}(t) - h(t)) \;\; a.e.$$

Let $u = J\pi\bar{v} + \xi$, by Fenchel's reciprocity, we get

$$\dot{u} = J\bar{v} = J\nabla H(t, u(t))$$

and it's clear that $u(0) = u(T)$.

It remains to prove that u is not constant.

Setting $u = (r, p), (\mathcal{H}_h)$ is equivalent to

$$\dot{u}(t) = \begin{pmatrix} \dot{r} \\ \\ \dot{p} \end{pmatrix} = J\left[\begin{pmatrix} A^* \\ \\ B^* \end{pmatrix} \nabla G(Ar + Bp) + \begin{pmatrix} f \\ \\ g \end{pmatrix}\right]$$

but $\dot{u} = 0$ gives

$$-\begin{pmatrix} f \\ \\ g \end{pmatrix} = \begin{pmatrix} A^* \\ \\ B^* \end{pmatrix} \nabla G(Ar + Bp)$$

and then $(B^{-1}A)^* g = f$, which is in contradiction with the assumption (H_8).

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