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A New Generalization of Direct Lyapunov Method for Uncertain Dynamical Systems

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Abstract: In this paper we study a class of uncertain dynamical systems and sufficient conditions, in terms of matrix-valued Liapunov functions are provided for the new concept of stability and uniform asymptotic stability.

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1 Introduction

In many cases the motion of uncertain systems is successfully analysed in terms of the development of ideas of direct Lyapunov method [1]. Recent surveys by Corless [2] and Leitmann [3] of papers in this direction provide a comprehensive idea of what has been done in the field of uncertain system investigations for the last decades. The aim of this paper is to give an account of results of qualitative investigation of solutions to uncertain systems with respect to the moving invariant set. To this end the method of matrix Lyapunov functions is applied.

It should be noted that the investigation of uncertain system dynamics in terms of matrix-valued functions allows the extension of the set of the direct Lyapunov method.

2 Statement of the Problem

2.1 Description of the system

We consider a mechanical system whose motion is modelled by the differential equations

$$\frac{dx}{dt} = f(t, x, \alpha), \qquad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in R^n$, $t \in \mathcal{T}_0 = [t_0, +\infty)$, $t_0 \in \mathcal{T}_i$, $\mathcal{T}_i \subseteq R$ and $f \in C(\mathcal{T}_0 \times R^n \times R^d,\, R^n)$. The parameter $\alpha \in R^d$, $d \geq 1$, represents the "uncertainties" of the system under consideration. Here and on it is assumed that the motion of system (2.1) described by the solution $x(t; t_0, x_0, \alpha) \triangleq x(t, \alpha)$ possesses the following properties.

(A_1) for an open neighborhood D of the state $x = 0$, $D \subseteq R^n$

- (a) system (2.1) has unique solution $x(t; t_0, x_0, \alpha)$ taking the value x_0 for $t = t_0$ for any $(t_0, x_0, \alpha) \in \mathcal{T}_i \times D \times \mathcal{S}$, $\mathcal{S} \subset R^d$, $\mathcal{S}$ is a compact set;
- (b) the motion $x(t; t_0, x_0, \alpha)$ of system (2.1) is defined and continuously (differentiable) in $(t, t_0, x_0) \in \mathcal{T}_0 \times \mathcal{T}_i \times D$ for any $\alpha \in \mathcal{S}$.

Note that the initial values x_0 and the uncertainty parameters α may be related by the correlations which ensure the presence of properties (A_1) for the motions of system (2.1).

Model (2.1) embraces many systems whose dynamics is modelled by ordinary differential equations with uncertain values of parameters.

According to Leitmann [4], Chen [5], etc., the parameter α:

- (a) can represent an uncertain value of some parameter of the system or the outer perturbation;
- (b) can be a function mapping R into R^d and representing some parameter value which is uncertainly time-varying or input effects;
- (c) can be a function mapping $\mathcal{T}_0 \times R^n$ into R^d and representing nonlinear elements of the mechanical system in question whose exact description is difficult;
- (d) can be just an index showing the existence of some uncertainties in the system;
- (e) can be a combination of (a)$-$(c).

Let a function $r = r(\alpha) > 0$ be given such that $r(\alpha) \to r_0$ ($r_0 = \text{const} > 0$) as $\|\alpha\| \to 0$ and $r(\alpha) \to +\infty$ as $\|\alpha\| \to +\infty$. In the Euclidean space $(R^n, \|\cdot\|)$ the moving set

$$A(r) = \{x \in R^n \colon \|x\| = r(\alpha)\} \tag{2.2}$$

is determined and the set $A(r)$ is assumed non-empty for any $(\alpha \neq 0) \in \mathcal{S} \subset R^d$.

Definition 2.0 The solution $x(t, \alpha)$ of system (2.1) is called *non-continuable*, if for any $x(t, \alpha)$ there is not a continuation, which would be different from $x(t, \alpha)$ on the interval of definition $J \subset \mathcal{T}_0$ for all $\alpha \in \mathcal{S} \subseteq R^d$.

Definition 2.1 The set $A(r)$ is called *moving invariant set of system (2.1)*, if for every $x_0 \in A(r)$ and all solutions $x(t, \alpha) = x(t; t_0, x_0, \alpha)$ of system (2.1) determined on some interval $J \subset \mathcal{T}_0$ and such that $x(t_0; t_0, x_0, \alpha) = x_0$ for all $(\alpha \neq 0) \in \mathcal{S} \subset R^d$ the inclusion $x(t, \alpha) \in A(r)$ is satisfied for every $t \in J$.

2.2 Definitions

Taking into account the results of paper [6] and monograph [7] we shall formulate definitions necessary for the subsequent presentation.

Definition 2.2 The solutions of system (2.1) are

- (a) *stable with respect to the sets $A(r)$ and $\mathcal{T}_i \subset R$*, iff given $r(\alpha) > 0$, $\varepsilon > 0$ and $t_0 \in \mathcal{T}_i$, given $\delta = \delta(t_0, \varepsilon) > 0$ such that under the initial conditions

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta,$$

the solution of system (2.1) satisfies the estimate

$$r(\alpha) - \varepsilon < \|x(t,\alpha)\| < r(\alpha) + \varepsilon,$$

for all $t \in \mathcal{T}_0$ and all $\alpha \in \mathcal{S} \subseteq R^d$;

(b) *uniformly stable with respect to the sets $A(r)$ and $\mathcal{T}_i$*, iff the conditions of Definition 2.2(a) are satisfied and for any $\varepsilon > 0$ the corresponding maximal value δ_M satisfying the conditions of Definition 2.2() is such that

$$\inf\,[\delta_M(t,\varepsilon)\colon t \in \mathcal{T}_i] > 0;$$

(c) *stable in the whole with respect to $\mathcal{T}_i$*, iff the condition of Remark 2.2 are satisfied as well as the conditions of Definition 2.2() with the function

$$\delta_M(t,\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to +\infty, \quad \forall\, t \in \mathcal{T}_i;$$

(d) *uniformly stable in the whole with respect to $\mathcal{T}_i$*, iff the conditions of Definitions 2.2(b) and 2.2(c) are satisfied.

Definition 2.3 For the solutions of system (2.1) the moving set $A(r)$ is called

(a) *attractive with respect to $\mathcal{T}_i$*, iff given function $r(\alpha) > 0$ and $t_0 \in \mathcal{T}_i$, there exists a $\delta(t_0) > 0$ and for any $\zeta > 0$ a $\tau(t_0, x_0, \zeta) \in [0,\infty)$ exists such that the condition

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta$$

implies the estimate

$$r(\alpha) - \zeta < \|x(t,\alpha)\| < r(\alpha) + \zeta$$

for all $t \in (t_0 + \tau(t_0, x_0, \zeta),\, +\infty)$ and all $\alpha \in \mathcal{S} \subseteq R^d$;

(b) *x_0-uniformly attractive with respect to $\mathcal{T}_i$*, iff the conditions of Definition 2.3(a) are satisfied and for any $t_0 \in \mathcal{T}_i$ there exists a $\delta(t_0) > 0$ and for any $\zeta \in (0,+\infty)$ a $\tau_u(t_0, \Delta(t_0), \zeta) \in [0,\infty)$ exists such that

$$\sup\{\tau_m(t_0,x_0,\zeta)\colon r(\alpha) - \Delta \le \|x_0\| < r(\alpha) + \Delta\} = \tau_u(t_0, \Delta(t_0), \zeta);$$

(c) *t_0-uniformly attractive with respect to $\mathcal{T}_i$*, iff the conditions of Definition 1.3(a) are satisfied, there exists a $\Delta^* > 0$ and for any

$$(x_0, \zeta) \in \{r(\alpha) - \Delta^* \le \|x_0\| < r(\alpha) + \Delta^*\} \times (0,+\infty)$$

there exists a $\tau_u(\mathcal{T}_i, x_0, \zeta) \in [0,+\infty)$ such that

$$\sup\{\tau_m(t_0,x_0,\zeta)\colon t_0 \in \mathcal{T}_i\} = \tau_u(\mathcal{T}_i, x_0, \zeta);$$

(d) *uniformly attractive with respect to $\mathcal{T}_i$*, if conditions of Definitions 2.3(b) and 2.3(c) are satisfied or, what is the same, the conditions of Definition 2.3() are satisfied and there exists a $\delta > 0$ and for any $\zeta \in (0,+\infty)$ a $\tau_u(\mathcal{T}_i, \Delta, \zeta) \in [0,\infty)$ exists such that

$$\begin{aligned}\sup\,[\tau_m(t_0,x_0,\zeta)\colon (t_0,x_0) \in \mathcal{T}_i \times \{r(\alpha) - \Delta < \|x_0\| < r(\alpha) + \Delta\}] = \\ = \tau_u(\mathcal{T}_i, \Delta, \zeta);\end{aligned}$$

(e) the attraction properties 2.3(a)−2.3(d) *take place in the whole*, if conditions of Definition 2.3(a) are satisfied for any $\Delta(t_0) \in (0,+\infty)$ and any $t_0 \in \mathcal{T}_i$, if $r(\alpha) \to +\infty$ as $\|\alpha\| \to +\infty$.

The expression "with respect to $\mathcal{T}_i$" in Definitions 2.3 is omitted, iff $\mathcal{T}_i = R$.

Definition 2.4 For system (2.1) the moving set $A(r)$ is called

(a) *asymptotically stable with respect to* $\mathcal{T}_i$, iff it is stable with respect to $\mathcal{T}_i$ and attractive with respect to $\mathcal{T}_i$;

(b) *equi-asymptotically stable with respect to* $\mathcal{T}_i$, if it is stable with respect to $\mathcal{T}_i$ and x_0-uniformly attractive with respect to $\mathcal{T}_i$;

(c) *quasi-uniformly asymptotically stable with respect to* $\mathcal{T}_i$, if it is uniformly stable with respect to $\mathcal{T}_i$ and t_0-uniformly attractive with respect to $\mathcal{T}_i$;

(d) *uniformly asymptotically stable with respect to the sets* $A(r)$ *and* $\mathcal{T}_i$, if it is uniformly stable with respect to the sets $A(r)$ and $\mathcal{T}_i$ and uniformly attractive with respect to the sets $A(r)$ and $\mathcal{T}_i$;

(e) *uniformly exponentially stable with respect to* $\mathcal{T}_i$, if given function $r(\alpha)$ and constants β_1, β_2 and λ, there exists a $\delta > 0$ such that the condition

$$r(\alpha) - \delta < \|x_0\| < r(\alpha) + \delta$$

implies the estimate

$$\begin{aligned} &r(\alpha) - \beta_1\|x_0\| \exp[-\lambda(t - t_0)] \le \|x(t, \alpha)\| \le \\ &\le r(\alpha) + \beta_2\|x_0\| \exp[-\lambda(t - t_0)] \qquad \forall\, t \in \mathcal{T}_0, \quad \forall\, t_0 \in \mathcal{T}_i; \end{aligned}$$

(f) *exponentially stable in the whole with respect to* $\mathcal{T}_i$, if the conditions of Definition 2.4(e) are satisfied for $r(\alpha) \to \infty$, $\|\alpha\| \to +\infty$ and $\delta \to +\infty$.

The expression "with respect to $\mathcal{T}_i$" in Definitions 1.4 is omitted, iff $\mathcal{T}_i = R$.

3 Properties of Matrix-Valued Functions on the Moving Set

Under some assumptions it is possible to construct for system (2.1) a two-index system of functions (see [8, 10])

$$U(t, x) = [u_{ij}(t, x)], \qquad i, j = 1, 2, \dots, s. \tag{3.1}$$

Here the elements $u_{ij} \in C(\mathcal{T}_0 \times R^n, R)$, for all $i, j = 1, 2, \dots, s$.

We construct by means of vector $y \in R^s$ $(y \ne 0)$ the scalar function

$$V(t, x, y) = y^T U(t, x,)y, \qquad (y \ne 0) \in R^s. \tag{3.2}$$

The total upper right Dini derivative of function (3.2) along solutions of system (2.1) is defined by the formula

$$D^+V(t, x, y) = y^T D^+U(t, x,)y, \qquad (y \ne 0) \in R^s. \tag{3.3}$$

Here the upper right Dini derivative of the matrix $U(t, x)$

$$D^+U(t, x) = \limsup \left\{[U(t + \theta,\, x + \theta f(t, x, \alpha)) - U(t, x)]\theta^{-1} \colon \theta \to 0^+\right\}$$

is computed element-wise.

Further for the set $A(r)$ moving in R^n we shall consider its moving σ-neighborhood and the internal and external parts $\operatorname{int} A(r)$ and $\operatorname{ext} A(r)$, i.e. the sets

$$N(A, \sigma) = \{x \in R^n \colon 0 < \rho(x, A) < \sigma\},$$

where $\rho(x, A) = \inf\limits_{q \in A(r)} \rho(x, q)$ and σ is some number,

$$\operatorname{int} A(r) = \{x \in R^n \colon \|x\| < r(\alpha)\} \quad \text{and} \quad \operatorname{ext} A(r) = \{x \in R^n \colon \|x\| > r(\alpha)\},$$

respectively.

In view of results of the monograph [10] we shall cite the following definitions.

Definition 3.1 The matrix-valued function $U\colon R\times R^n \to R^{s\times s}$ is called *positive semi-definite on $\mathcal{T}_\tau=[\tau,+\infty)$, $\tau\in R$ with respect to the moving set $A(r)$*, if

(i) U is continuous in $(t,x)\in\mathcal{T}_\tau\times N(A,\sigma)$,

$$U\in C\left(\mathcal{T}_\tau\times N(A,\sigma),\ R^{s\times s}\right);$$

(ii) U is nonnegative on $N(A,\sigma)$:

$$y^T U(t,x)y\geq 0 \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad \text{and} \quad \forall\, x\notin A(r);$$

(iii) U vanishes when $x\in A(r)$.

Definition 3.2 (see [11]) The continuous function $\varphi\colon[0,\beta]\to R_+$ belongs to the class K, i.e. $\varphi\in K$, if $\varphi(0)=0$ and $\varphi(u)$ is strictly increasing on $[0,\beta]$.

Definition 3.3 The matrix-valued function $U\colon R\times R^n\to R^{s\times s}$ is called *positive definite on $\mathcal{T}_\tau$, $\tau\in R$ with respect to the moving set $A(r)$*, if conditions (i)−(iii) of Definition 3.1 are satisfied and there exists a function a of class K satisfying the inequality

$$a(\|x\|)\leq y^T U(t,x)y, \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad \text{and} \quad \forall\, x\notin A(r).$$

The expression "on $\mathcal{T}_\tau$" in Definition 3.3(a) is omitted, iff all conditions of these definitions are satisfied for every $\tau\in R$.

Definition 3.4 The matrix-valued function $U\colon R\times R^n\to R^{s\times s}$ is called *decreasing on $\mathcal{T}_\tau$ with respect to the moving set $A(r)$*,

(i) U is continuous in $(t,x)\in\mathcal{T}_\tau\times N(A,\sigma)$,

$$U\in C\left(\mathcal{T}_\tau\times N(A,\sigma),\ R^{s\times s}\right);$$

(ii) there exists a function b of class K satisfying the inequality

$$y^T U(t,x)y\leq b(\|x\|), \quad \forall\,(t,y)\in\mathcal{T}_\tau\times R^s \quad \text{and} \quad \forall\, x\notin A(r);$$

(iii) U vanishes when $x\in A(r)$.

The expression "on $\mathcal{T}_\tau$" in Definition 3.4 is omitted, iff the conditions of Definition 3.4 are satisfied for every $\tau\in R$.

4 On Stability and Uniform Asymptotic Stability of Uncertain Systems

Theorem 4.1 *Assume that in system (2.1) $f(t,x,\alpha)$ is continuous on $\mathcal{T}_0\times R^n\times R^d$ and the following conditions are satisfied*

(1) *for every $\alpha\in\mathcal{S}\subseteq R^d$ there exists a function $r=r(\alpha)>0$ such that the set $A(r)$ is nonempty for all $\alpha\in\mathcal{S}\subseteq R^d$;*
(2) *there exists a matrix-valued function $U\in C(\mathcal{T}_0\times R^n, R^{s\times s})$, $U(t,x)$ is locally Lipschitzian in x, the vector $y\in R^s$, $s\times s$-matrices $\theta_1(r)$ and $\theta_2(r)$ are such that*
  (a) $\quad a(\|x\|)\leq V(t,x,y) \quad$ *for* $\quad \|x\|>r(\alpha)$,
  and

(b) $V(t,x,y) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K*;*

(c) $D^+V(t,x,y)|_{(2.1)} \le \varphi^T(\|x\|)\theta_1(r)\varphi(\|x\|)$ *if* $\|x\| > r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

and

(d) $D^+V(t,x,y)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $D^+V(t,x,y)|_{(2.1)} > \psi^T(\|x\|)\theta_2(r)\psi(\|x\|)$ *if* $\|x\| < r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

where $\varphi^T(\|x\|) = \Big(\varphi_1^{1/2}(\|x_1\|), \dots, \varphi_s^{1/2}(\|x_s\|)\Big)$, $\varphi_i \in K$,

$\psi^T(\|x\|) = \Big(\psi_1^{1/2}(\|x_1\|), \dots, \psi_s^{1/2}(\|x_s\|)\Big)$, $\psi_i \in K$*;*

(3) *there exist constant* $s \times s$ *-matrices* $\overline{\theta}_1$ *and* $\overline{\theta}_2$ *such that*

(a) $\dfrac{1}{2}\big(\theta_1(r) + \theta_1^T(r)\big) \le \overline{\theta}_1$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$

and

(b) $\dfrac{1}{2}\big(\theta_2(r) + \theta_2^T(r)\big) \ge \overline{\theta}_2$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

and moreover, $\overline{\theta}_1$ *is negative semi-definite and* $\overline{\theta}_2$ *is positive semi-definite;*

(4) *for any* $r(\alpha) > 0$ *and functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

Then the set $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are stable with respect to the set* $A(r)$*.*

For the *proof* see [12].

4.1 Corollary

In cases when it is possible to construct scalar Lyapunov function for system (2.1) the stability of solutions can be studied in terms of the following assertion.

Theorem 4.2 *The set* $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are stable with respect to the set* $A(r)$ *if*

(1) *for every* $\alpha \in \mathcal{S} \subseteq R^d$ *there exists a function* $r = r(\alpha) > 0$ *such that* $r(\alpha) \to r_0$ $(r_0 = \text{const} > 0)$ *as* $\|\alpha\| \to 0$ *and* $r(\alpha) \to +\infty$ *as* $\|\alpha\| \to +\infty$*;*

(2) *there exist scalar functions* $V \in C^1(\mathcal{T}_0 \times R^n, R_+)$, $W_1\colon R^n \times R^d \to R$ *and* $W_2\colon R^n \times R^d \to R$ *such that*

(a) $a(\|x\|) \le V(t,x)$ *for* $\|x\| > r(\alpha)$,

(b) $V(t,x) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K*;*

(c) $DV(t,x)|_{(2.1)} \le W_1(x,\alpha)$ *for* $\|x\| > r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

(d) $DV(t,x)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $DV(t,x)|_{(2.1)} \ge W_1(x,\alpha)$ *for* $\|x\| < r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$*;*

(3) *there exist functions* $\overline{W}_1(x)$ *and* $\underline{W}_2(x)$ *such that*

(a) $W_1(x,\alpha) \le \overline{W}_1(x) < 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(b) $W_2(x,\alpha) \ge \underline{W}_2(x) > 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$*;*

(4) *for any* $r(\alpha) > 0$ *and the functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

The assertion of Theorem 4.2 follows from Theorem 4.1.

4.2 Example

Considered is the uncertain equation

$$\frac{dx}{dt} = x - f^2(\alpha)x^3, \qquad x(0) \neq 0, \tag{4.1}$$

where $f(\alpha)$ is the function of the uncertainty parameter $\alpha \in \mathcal{S} \subseteq R^d$, $f(\alpha) \to f_0$ ($f_0 = \text{const} > 0$) as $\|\alpha\| \to 0$ and $f(\alpha) \to 0$ as $\|\alpha\| \to \infty$.

Zero solution $x = 0$ of this equation is unstable by Lyapunov, since its first approximation

$$\frac{dx}{dt} = x, \qquad x(0) \neq 0$$

has the eigenvalue $\lambda = 1 > 0$.

Let $r(\alpha) = (f(\alpha))^{-1} > 0$. It is clear that $r(\alpha) \to r_0$ as $\alpha \to 0$ and $r(\alpha) \to \infty$ as $\alpha \to \infty$.

The set $A(r)$ is

$$A(r) = \left\{ x \colon |x| = \frac{1}{f(\alpha)} \right\}. \tag{4.2}$$

We take $V = x^2$ and compute

$$\frac{dV}{dt} = 2x\,\frac{dx}{dt} = 2x^2(1 - f^2(\alpha)x^2).$$

Hence, it is clear

$$\begin{aligned} \frac{dV}{dt} &< 0 \quad \text{for} \quad |x| > \frac{1}{f(\alpha)}, \quad t \geq 0, \\ \frac{dV}{dt} &= 0 \quad \text{for} \quad |x| = \frac{1}{f(\alpha)}, \quad t \geq 0, \\ \frac{dV}{dt} &> 0 \quad \text{for} \quad |x| < \frac{1}{f(\alpha)}, \quad t \geq 0. \end{aligned}$$

Therefore, if $f(\alpha)$ satisfies the conditions $\lim\limits_{\alpha\to 0} f(\alpha) = f_0$ and $\lim\limits_{\alpha\to\infty} f(\alpha) = 0$, then by Theorem 4.2 the set $A(r)$ is invariant with respect to the equation (4.1) and all solutions of this equation are stable with respect to the set (4.2) in the sense of Definition 2.2(a).

Note that the equation (4.1) was considered in [13] for $f^2(\alpha) = \beta^2$, β is a control parameter.

Theorem 4.3 *Assume that in system (2.1) $f(t, x, \alpha)$ is continuous on $\mathcal{T}_0 \times R^n \times R^d$ and*

(1) *for any $\alpha \in \mathcal{S} \subseteq R^d$ there exists a function $r = r(\alpha) > 0$ such that the set $A(r)$ is nonempty for all $\alpha \in \mathcal{S} \subseteq R^d$;*

(2) *there exists a matrix-valued function* $U \in C(\mathcal{T}_0 \times R^n, R^{s\times s})$, $U(t,x)$ *is locally Lipschitzian in* x, *the vector* $y \in R^s$, $s \times s$*-matrices* $\Phi_1(r)$ *and* $\Phi_2(r)$ *are such that*

(a) $a(\|x\|) \le V(t,x,y)$ *for* $\|x\| > r(\alpha)$,

and

(b) $0 < V(t,x,y) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K*;*

(c) $D^+V(t,x,y)|_{(2.1)} < \varphi^T(\|x\|)\Phi_1(r)\varphi(\|x\|)$ *for* $\|x\| > r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

and

(d) $D^+V(t,x,y)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $D^+V(t,x,y)|_{(2.1)} > \psi^T(\|x\|)\Phi_2(r)\psi(\|x\|)$ *for* $\|x\| < r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

where

$$\varphi^T(\|x\|) = \left(\varphi_1^{1/2}(\|x_1\|), \dots, \varphi_s^{1/2}(\|x_s\|)\right), \qquad \varphi_i \in K,$$

$$\psi^T(\|x\|) = \left(\psi_1^{1/2}(\|x_1\|), \dots, \psi_s^{1/2}(\|x_s\|)\right), \qquad \psi_i \in K,$$

$$x_s \in R^{n_s}, \qquad n_1 + n_2 + \dots n_s = n;$$

(3) *there exist constant* $s \times s$*-matrices* $\overline{\Phi}_1$ *and* $\overline{\Phi}_2$ *such that*

(a) $\dfrac{1}{2}\left(\Phi_1(r) + \Phi_1^T(r)\right) \le \overline{\Phi}_1$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(b) $\dfrac{1}{2}\left(\Phi_2(r) + \Phi_2^T(r)\right) \ge \overline{\Phi}_2$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

and moreover $\overline{\Phi}_1$ *is negative definite and* $\overline{\Phi}_2$ *is positive definite;*

(4) *for any* $r(\alpha) > 0$ *and the functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

Then the set $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are uniformly asymptotically stable with respect to the set* $A(r)$.

For the *proof* see [14].

4.3 Corollary

Theorem 4.4 *The set* $A(r)$ *is invariant with respect to the solutions of system (2.1) and the solutions of system (2.1) are uniformly asymptotically stable with respect to the set* $A(r)$ *if*

(1) *for every* $\alpha \in \mathcal{S} \subseteq R^d$ *there exists a function* $r = r(\alpha)$ *such that* $r(\alpha) \to r_0$ *if* $\|\alpha\| \to 0$ *and* $r(\alpha) \to +\infty$ *if* $\|\alpha\| \to +\infty$*;*

(2) *there exist scalar functions* $V \in C^1(\mathcal{T}_0 \times R^n, R_+)$, $W_1\colon R^n \times R^d \to R$ *and* $W_2\colon R^n \times R^d \to R$ *such that*

(a) $a(\|x\|) \le V(t,x)$ *for* $\|x\| > r(\alpha)$,

(b) $V(t,x) \le b(\|x\|)$ *for* $\|x\| \le r(\alpha)$,

where a *and* b *are of class* K*;*

(c) $DV(t,x)|_{(2.1)} \le W_1(x,\alpha)$ *for* $\|x\| > r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$,

(d) $DV(t,x)|_{(2.1)} = 0$ *iff* $\|x\| = r(\alpha)$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(e) $DV(t,x)|_{(2.1)} \ge W_1(x,\alpha)$ *for* $\|x\| < r(\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$;

(3) *there exist functions* $\overline{W}_1(x)$ *and* $\underline{W}_2(x)$ *of definite sign in the sense of Lyapunov such that*

(a) $W_1(x,\alpha) \le \overline{W}_1(x) < 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$,

(b) $W_2(x,\alpha) \ge \underline{W}_2(x) > 0$ *for all* $\alpha \in \mathcal{S} \subseteq R^d$;

(4) *for any* $r(\alpha) > 0$ *and functions* $a(r)$ *and* $b(r)$

$$a(r) = b(r).$$

4.4 Examples

Example 4.4.1 Let the equations

$$\begin{aligned} \frac{dx}{dt} &= n(t)y + \left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right) x(x^2+y^2), \\ \frac{dy}{dt} &= -n(t)x + \left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right) y(x^2+y^2) \end{aligned} \tag{4.3}$$

be given, where $n(t) \in C(R,R)$, $m(\alpha)$ is the uncertainty function in system (4.3) with the same properties that the function $f(\alpha)$ in Example 4.2.

Let $r(\alpha) = \frac{a}{m(\alpha)}$, $\alpha \in \mathcal{S} \subseteq R$. The set $A(r)$ is determined as

$$A(r) = \left\{x,\, y \colon (x^2+y^2)^{1/2} = r(\alpha)\right\}. \tag{4.4}$$

We take the function V in the form

$$V = x^2 + y^2.$$

Its derivative by virtue of equations (4.3) is

$$\frac{dV}{dt} = 2\left(1 - \frac{1}{a^2}\, m^2(\alpha)(x^2+y^2)\right)\left(x^2+y^2\right)^2.$$

Hence, it follows

$$\begin{aligned} \frac{dV}{dt} &< 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} > r(\alpha), \quad t \ge t_0, \\ \frac{dV}{dt} &= 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} = r(\alpha), \quad t \ge t_0, \\ \frac{dV}{dt} &> 0 \quad \text{for} \quad \left(x^2+y^2\right)^{1/2} < r(\alpha), \quad t \ge t_0. \end{aligned}$$

It is easy to see that if the function $m(\alpha)$ satisfies the conditions $\lim\limits_{\|\alpha\|\to 0} m(\alpha) = m_0$ and $\lim\limits_{\|\alpha\|\to\infty} m(\alpha) = \infty$, all conditions of Theorem 4.4 are satisfied and the set $A(r)$ is invariant for system (4.3) and all solutions of the system are uniformly asymptotically stable with respect to the set $A(r)$.

Example 4.4.2 We consider the systems

$$\begin{aligned}\frac{dx}{dt} &= \mu x + y - g(x,y,\alpha)x(x^2+y^2),\\ \frac{dy}{dt} &= \mu y - x - g(x,y,\alpha)y(x^2+y^2), \qquad \alpha \in \mathcal{S} \subseteq R^d,\end{aligned} \tag{4.5}$$

where $\mu = \text{const} > 0$, $g(x,y,\alpha) > 0$ is a function characteristics of "uncertainties" of system (4.5) (cf. [15]).

In system (4.5) we substitute the variables

$$x = -r\,\cos\theta, \qquad y = r\,\sin\theta$$

and reduce the system to the form

$$\frac{dr}{dt} = \mu r - g(r,\theta,\alpha)r^3, \qquad \frac{d\theta}{dt} = 1, \tag{4.6}$$

where

$$g^m \le g(r,\theta,\alpha) \le g^M \tag{4.7}$$

for all $(r,\theta,\alpha) \in R_+ \times [0,2\pi] \times \mathcal{S}$, $g^m < g^M$ are given constants.

Note that the solution $r = 0$ of the first approximation equations (4.6) is unstable in the sense of Liapunov, since the linear approximation $\dfrac{dr}{dt} = \mu r$ has its eigen value $\lambda = \mu > 0$.

Together with system (4.6) consider function $V = r^2$.

For derivative dV/dt by virtue of system (4.6) we get

$$\left.\frac{dV}{dt}\right|_{(2.79)} = 2r\,\frac{dr}{dt} = 2r^2[\mu - g(r,\theta,\alpha)r^2], \quad \alpha \in \mathcal{S} \subseteq R^d. \tag{4.8}$$

Hence it follows that if for any function $g(r,\theta,\alpha)$ satisfying condition (4.7) the following inequalities hold

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(4.6)} &< 0 \qquad \text{for} \quad r^2 > \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\\ \left.\frac{dV}{dt}\right|_{(4.6)} &= 0 \qquad \text{for} \quad r^2 = \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\\ \left.\frac{dV}{dt}\right|_{(4.6)} &> 0 \qquad \text{for} \quad r^2 < \frac{\mu}{g(r,\theta,\alpha)}, \quad t \ge 0,\end{aligned}$$

then the moving set

$$\mathcal{S}^*(\varkappa) = \left\{ r \colon r^2 = \frac{\mu}{g(r,\theta,\alpha)} \right\}, \quad \alpha \in \mathcal{S} \subseteq R^d$$

is uniformly asymptotically stable.

Further consider the motion of system (4.6) with respect to the domains

$$S_1 = \{r\colon r^2 < H\}, \quad 0 < H < \infty,$$
$$S_2 = \{r\colon r^2 \le \delta\}, \quad \delta = \left(\frac{\mu}{g^M}\right)^{1/2},$$
$$S_3 = \{r\colon r^2 \ge \eta\}, \quad \eta = \left(\frac{\mu}{g^m}\right)^{1/2}$$

under restrictions (4.7).

Let the motion of system (4.6) begin outside the ring with radius $r_0 + \sigma$, where $r_0 = \left(\frac{\mu}{g^m}\right)^{1/2}$ and σ is an arbitrary small constant value. Since

$$\frac{dV}{dt}\bigg|_{(4.6)} = 2\mu V - 4g(r,\theta,\alpha)V^2,$$

by Theorem 1 from [16] the interval of time for which the solutions of system (4.6) will get to the moving surface

$$r^2 = \frac{\mu}{g(r,\theta,\alpha)}$$

is estimated by the inequality

$$\tau \le \int\limits_{\varkappa_1}^{\varkappa} \frac{dc}{2\mu c - 4g^m c^2}, \tag{4.9}$$

where $\varkappa_1 < \varkappa$, $\varkappa_1 = \frac{1}{2}\,r^2$, $\varkappa = \frac{1}{2}\,(r_0+\sigma)^2$. Estimate (4.9) implies

$$\tau \le \frac{1}{2\mu}\,\ln\left|\frac{(r_0+\sigma)^2}{r^2}\,\frac{(r^2-r_0^2)}{2r_0\sigma+\sigma^2}\right|.$$

Similarly we estimate the interval of time sufficient for solutions starting in the domain $r^* - \sigma \ge 0$, where $r^* = \left(\frac{\mu}{g^M}\right)^{1/2}$, to get to the moving surface $r^2 = \frac{\mu}{g(r,\theta,\alpha)}$.

Note that the function $g(r,\theta,\alpha)$, $\alpha \in \mathcal{S} \subseteq R^d$, is not assumed continuously differentiable, therefore equation (4.6) is efficiently studied by qualitative technique whereas its immediate integration is difficult.

5 Concluding Remarks

This investigation of uncertain system dynamics contributes to the well-known results for this class of equations in several directions. First, it is shown that under certain conditions the problem of qualitative analysis of solutions to the uncertain system is reduced to the investigation of the property of having a fixed sign of special matrices estimating the matrix-valued function and its total derivative along solutions of the system under consideration. Second, non-smooth and non-differentiable functions may be used as the elements of the matrix-valued function. Note also that our results possess a considerable potential for their extension to new classes of equations modelling the dynamics of uncertain systems and in particular uncertain controlled systems.

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