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Stability and $\mathcal{L}_2$ Gain Analysis for a Class of Switched Symmetric Systems

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Abstract: In this paper, we study stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of a finite number of linear time-invariant symmetric subsystems. We focus our attention mainly on discrete-time systems. When all subsystems are Schur stable, we show that the switched system is exponentially stable under arbitrary switching. Furthermore, when all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we show that the switched system is exponentially stable and has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching. The key idea for both stability and $\mathcal{L}_2$ gain analysis in this paper is to establish a general Lyapunov function for all subsystems in the switched system.

Keywords: *Switched symmetric system; exponential stability; $\mathcal{L}_2$ gain; arbitrary switching; general Lyapunov function; linear matrix inequality (LMI).*

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1 Introduction

By a switched system, we mean a hybrid dynamical system that is composed of a family of continuous-time or discrete-time subsystems and as a rule orchestrating the switching among the subsystems. In the last two decades, there has been increasing interest in the stability analysis and controller design for such switched systems. The motivation for studying switched systems is from the fact that many practical systems are inherently multimodal in the sense that several dynamical subsystems are required to describe their behavior which may depend on various environmental factors [1], and that the methods of intelligent control design are based on the idea of switching among different controllers [2 − 5]. For recent progress and perspectives in the field of switched systems, see the survey papers [3, 6] and the references cited therein.

As also pointed out in [3, 6], there are three basic problems in stability and design of switched systems: (i) find conditions for stabilizability under arbitrary switching; (ii) identify the limited but useful class of stabilizing switching signals; and (iii) construct a stabilizing switching signal. There are many existing works concerning Problem (ii) and (iii). For example, references [7 − 10] considered Problem (ii) using piecewise Lyapunov functions, and references [11 − 13] considered Problem (ii) for switched systems with pairwise commutation or Lie-algebraic properties. References [14 − 15] considered Problem (iii) by dividing the state space associated with appropriate switching depending on state, and references [16, 17] considered quadratic stabilization, which belongs to Problem (iii), for switched systems composed of a pair of unstable linear subsystems by using a linear stable combination of unstable subsystems. However, we see very few dealing with the first problem, though it is desirable to require arbitrary switching in many real applications. Reference [18] showed that when all subsystems are stable and commutative pairwise, the switched system is stable under arbitrary switching. There are some other results concerning general Lyapunov functions for the subsystems in a switched system, but we do not find any explicit answer to Problem (i) except [18]. In this paper, rather than considering for a given switched system the condition for stabilizability under arbitrary switching, we are interested in the following question: *What kind of switched systems are stable under arbitrary switching? Specifically, is there a switched system whose subsystems are not commutative pairwise, yet it is stable under arbitrary switching?*

For switched systems, there are a few results concerning $\mathcal{L}_2$ gain analysis. Hespanha considered such a problem in his Ph.D. dissertation [19], by using a piecewise Lyapunov function approach. In [20], a modified approach has been proposed for more general switched systems and more exact results have been obtained. In that context, it has been shown that when all continuous-time subsystems are Hurwitz stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ_0, the switched system under an average dwell time scheme [7] achieves a weighted $\mathcal{L}_2$ gain γ_0, and the weighted $\mathcal{L}_2$ gain approaches normal $\mathcal{L}_2$ gain if the average dwell time is chosen sufficiently large. However, the results obtained in [19] and [20] are conservative, and it is supposed that the main reason is in the use of piecewise Lyapunov functions. Recently, reference [21] considered the computation of $\mathcal{L}_2$ gain for switched linear systems with large dwell time, and gave an algorithm by considering the separation between the stabilizing and antistabilizing solutions to a set of algebraic Riccati equations. Noticing that these papers deal with the class of switching signals with (average) dwell time, we are motivated to ask the following question: *Is there a switched system that preserves its subsystems' $\mathcal{L}_2$ gain properties under arbitrary switching?*

For the above questions concerning stability and $\mathcal{L}_2$ gain, we give a clear (though not complete) answer in this paper. More exactly, we will show that a class of switched systems, which are composed of a finite number of linear time-invariant symmetric subsystems and called shortly *switched symmetric systems*, will preserve their subsystems' stability and $\mathcal{L}_2$ gain properties under arbitrary switching. We take such symmetric systems into consideration since they appear quite often in many engineering disciplines (for example, RC or RL electrical networks, viscoelastic materials and chemical reactions) [22], and thus belong to an important class in control engineering.

Though our discussion is applicable also to continuous-time switched systems with some modifications, we focus our attention mainly on the following discrete-time switched system

$$\begin{aligned} x[k+1] &= A_{\sigma(k)}x[k] + B_{\sigma(k)}w[k], \quad x[k_0] = x_0, \\ z[k] &= C_{\sigma(k)}x[k] + D_{\sigma(k)}w[k], \end{aligned} \tag{1.1}$$

where $x[k] \in R^n$ is the state, $w[k] \in R^m$ is the input, $z[k] \in R^p$ is the output, $k_0 \geq 0$ is the initial point and x_0 is the initial state. $\sigma(k)\colon \mathcal{I}_+ \to \mathcal{I}_N = \{1, 2, \dots, N\}$ is a piecewise constant function, called a *switching signal*, which is assumed to be arbitrary. Here, $\mathcal{I}_+$ denotes the set of all nonnegative integers not less than k_0, and A_i, B_i, C_i, D_i $(i \in \mathcal{I}_N)$ are constant matrices of appropriate dimensions denoting the subsystems, $N > 1$ is the number of subsystems. Throughout this paper, we assume that all subsystems in (1.1) are symmetric in the sense of satisfying

$$A_i = A_i^{\mathrm{T}}, \quad B_i = C_i^{\mathrm{T}}, \quad D_i = D_i^{\mathrm{T}}, \quad \forall\, i \in \mathcal{I}_N\,. \tag{1.2}$$

It should be noted here that the assumption (1.2) does not cover all symmetric subsystems, which are usually defined in the form of transfer functions, and a more general definition is that $T_iA_i = A_i^{\mathrm{T}}T_i$, $T_iB_i = C_i^{\mathrm{T}}$, $D_i = D_i^{\mathrm{T}}$ holds for some nonsingular symmetric matrix T_i [22–24]. However, (1.2) represents an interesting class of symmetric systems [25], and for the benefit of this paper we are concentrated on such kind of symmetric systems.

We will say the switched system (1.1) is *exponentially stable* if $\|x[k]\| \leq \mu^{k-k_0}\|x_0\|$ with $0 < \mu < 1$ holds for any $k > k_0$ and any initial state x_0, and will say the switched system (1.1) *has an* $\mathcal{L}_2$ *gain* γ if $\sum\limits_{j=k_0}^{k} z^{\mathrm{T}}[j]z[j] \leq \gamma^2 \sum\limits_{j=k_0}^{k} w^{\mathrm{T}}[j]w[j]$ holds for any integer $k > k_0$ when $x_0 = 0$. These definitions are also valid for all the subsystems in (1.1).

This paper is organized as follows. In Section 2, assuming that all subsystems are Schur stable, we show that there exists a general Lyapunov function for all subsystems, and that the switched system is exponentially stable under arbitrary switching. In Section 3, assuming that all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we prove that there exists a general Lyapunov function for all subsystems in the sense of $\mathcal{L}_2$ gain, and that the switched system has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching. Finally we give some concluding remarks in Section 4.

2 Stability Analysis

In this section, we set $w[k] \equiv 0$ in the switched system (1.1) to consider stability of the system under arbitrary switching. We first give a preliminary result.

Lemma 2.1 *Consider the discrete-time symmetric system*

$$x[k+1] = Ax[k], \tag{2.1}$$

where $x[k] \in R^n$ is the state and A is a constant symmetric matrix. The system (2.1) is Schur stable if and only if

$$A^2 < I. \tag{2.2}$$

Proof Since A is a symmetric matrix, there exists a nonsingular matrix Q satisfying $Q^{\mathrm{T}} = Q^{-1}$ such that

$$Q^{\mathrm{T}} A Q = \mathrm{diag}\,\{\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_n\}, \tag{2.3}$$

where $\lambda_i,\ i = 1, 2, \ldots, n,$ are A's real eigenvalues (noticing that the symmetric matrix A has only real eigenvalues), and thus

$$Q^{\mathrm{T}} A^2 Q = \mathrm{diag}\,\{\lambda_1^2,\, \lambda_2^2,\, \ldots,\, \lambda_n^2\}. \tag{2.4}$$

The discrete-time system (2.1) is Schur stable if and only if $|\lambda_i| < 1,\ i = 1, 2, \ldots, n,$ which is equivalent to

$$Q^{\mathrm{T}} A^2 Q < I \tag{2.5}$$

according to (2.4). Since $QQ^{\mathrm{T}} = I,$ the inequality (2.5) is equivalent to (2.2). This completes the proof.

Remark 2.1 Lemma 2.1 implies that if all A_i's in (1.1) are Schur stable, there exists a general Lyapunov matrix $P = I$ for all A_i's, satisfying the LMI

$$A_i^{\mathrm{T}} P A_i - P < 0, \quad \forall\, i \in \mathcal{I}_N. \tag{2.6}$$

Hence, $V(x) = x^{\mathrm{T}} x$ serves as a general Lyapunov function for all subsystems in (1.1).

Now we state and prove the main result in this section.

Theorem 2.1 *When all subsystems in (1.1) are Schur stable, the switched symmetric system (1.1) is exponentially stable under arbitrary switching.*

Proof Since all subsystems in (1.1) are Schur stable, according to Lemma 2.1, the matrix inequality

$$A_i^2 < I \tag{2.7}$$

holds for all $i \in \mathcal{I}_N,$ and thus there exists a scalar $\epsilon \in (0,\, 1)$ such that

$$A_i^2 < (1 - \epsilon) I, \quad \forall i \in \mathcal{I}_N. \tag{2.8}$$

Now, we consider the Lyapunov function candidate

$$V(x) = x^{\mathrm{T}} x. \tag{2.9}$$

According to (2.8), we obtain for any integer $k > k_0$ that

$$\begin{aligned} V(x[k]) &= x^{\mathrm{T}}[k]\, x[k] \le (1 - \epsilon) x^{\mathrm{T}}[k-1] x[k-1] \\ &= (1 - \epsilon) V(x[k-1]) \end{aligned} \tag{2.10}$$

holds under arbitrary switching, and thus

$$V(x[k]) \le (1 - \epsilon)^{k - k_0} V(x[k_0]) \tag{2.11}$$

which means

$$\|x[k]\| \le (\sqrt{1 - \epsilon})^{k - k_0} \|x_0\|. \tag{2.12}$$

Since this inequality holds for any initial state x_0, the switched system (1.1) is exponentially stable.

Remark 2.2 For the continuous-time switched symmetric system

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{2.13}$$

where $A_i = A_i^{\mathrm{T}},\ i \in \mathcal{I}_N,$ are constant Hurwitz stable matrices, we easily see that $A_i < 0$ holds for all i, which implies that the general Lyapunov matrix $P = I$ satisfies the LMI

$$A_i^{\mathrm{T}} P + P A_i < 0, \quad \forall\, i \in \mathcal{I}_N, \tag{2.14}$$

and thus the switched system (2.13) is exponentially stable under arbitrary switching.

3 $\mathcal{L}_2$ Gain Analysis

In this section, we assume $x_0 = 0$ in the switched symmetric system (1.1) to study the $\mathcal{L}_2$ gain property of the system under arbitrary switching. First, we state and prove a lemma which plays an important role in the discussion of this section. We note that the idea of this lemma and its proof is motivated by Lemma 2 of [25], where continuous-time symmetric systems are dealt with.

Lemma 3.1 *Consider the discrete-time symmetric system*

$$\begin{aligned} x[k+1] &= Ax[k] + Bw[k] \\ z[k] &= Cx[k] + Dw[k], \end{aligned} \tag{3.1}$$

where $x[k] \in R^n$ is the state, $w[k] \in R^m$ is the input, $z[k] \in R^p$ is the output, and A, B, C, D are constant matrices of appropriate dimensions, satisfying $A = A^{\mathrm{T}},\ B = C^{\mathrm{T}}$, $D = D^{\mathrm{T}}$. The system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ if and only if

$$\begin{bmatrix} -I & A & B & 0 \\ A & -I & 0 & B \\ B^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & B^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.2}$$

Proof **Sufficiency** The condition (3.2) means that the matrix inequality

$$\begin{bmatrix} -P & PA & PB & 0 \\ A^{\mathrm{T}}P & -P & 0 & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & 0 & -\gamma I & D^{\mathrm{T}} \\ 0 & C & D & -\gamma I \end{bmatrix} < 0 \tag{3.3}$$

is satisfied with $P = I$. Hence, according to the Bounded Real Lemma [26] for discrete-time LTI system, the system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ.

Necessity Suppose that the system (3.1) is Schur stable and has an $\mathcal{L}_2$ gain smaller than γ . Then, there exists a matrix $P_0 > 0$ such that

$$\begin{bmatrix} -P_0 & P_0A & P_0B & 0 \\ AP_0 & -P_0 & 0 & B \\ B^{\mathrm{T}}P_0 & 0 & -\gamma I & D \\ 0 & B^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.4}$$

where C was replaced by B^{T}.

Now, we prove that $P = I$ is also a solution of the above matrix inequality (i.e., (3.4) holds when replacing P_0 with I). Since $P_0 > 0$, there always exists a nonsingular matrix U satisfying $U^{\mathrm{T}} = U^{-1}$ such that

$$\begin{gathered} U^{\mathrm{T}} P_0 U = \Sigma_0 = \operatorname{diag}\{\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_n\}, \\ \sigma_i > 0, \quad i = 1, 2, \ldots, n. \end{gathered} \tag{3.5}$$

Pre- and post-multiplying (3.4) by $\operatorname{diag}\{U^{\mathrm{T}},\, U^{\mathrm{T}},\, I,\, I\}$ and $\operatorname{diag}\{U, U, I, I\}$, respectively, we obtain

$$\begin{bmatrix} -\Sigma_0 & \Sigma_0 \bar{A} & \Sigma_0 \overline{B} & 0 \\ \bar{A}\Sigma_0 & -\Sigma_0 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_0 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.6}$$

where $\bar{A} = U^{\mathrm{T}} A U$, $\overline{B} = U^{\mathrm{T}} B$. Furthermore, pre- and post-multiplying the first and second rows and columns in (3.6) by Σ_0^{-1} leads to

$$\begin{bmatrix} -\Sigma_0^{-1} & \bar{A}\Sigma_0^{-1} & \overline{B} & 0 \\ \Sigma_0^{-1}\bar{A} & -\Sigma_0^{-1} & 0 & \Sigma_0^{-1}\overline{B} \\ \overline{B}^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}}\Sigma_0^{-1} & D & -\gamma I \end{bmatrix} < 0. \tag{3.7}$$

In (3.7), we exchange the first and second rows and columns, and then exchange the third and fourth rows and columns, to obtain

$$\begin{bmatrix} -\Sigma_0^{-1} & \Sigma_0^{-1}\bar{A} & \Sigma_0^{-1}\overline{B} & 0 \\ \bar{A}\Sigma_0^{-1} & -\Sigma_0^{-1} & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_0^{-1} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.8}$$

Since $\sigma_1 > 0$, there always exists a scalar λ_1 such that

$$0 < \lambda_1 < 1, \quad \lambda_1 \sigma_1 + (1 - \lambda_1)\sigma_1^{-1} = 1. \tag{3.9}$$

Then, by computing $\lambda_1 \times (3.6) + (1 - \lambda_1) \times (3.8)$, we obtain

$$\begin{bmatrix} -\Sigma_1 & \Sigma_1 \bar{A} & \Sigma_1 \overline{B} & 0 \\ \bar{A}\Sigma_1 & -\Sigma_1 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_1 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.10}$$

where

$$\begin{aligned} \Sigma_1 &= \operatorname{diag}\{\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1}, \lambda_1\sigma_2 + (1-\lambda_1)\sigma_2^{-1}, \ldots, \lambda_1\sigma_n + (1-\lambda_1)\sigma_n^{-1}\} \\ &\triangleq \operatorname{diag}\{1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n\} > 0. \end{aligned} \tag{3.11}$$

In the similar way to obtain (3.8), we can obtain from (3.10) that

$$\begin{bmatrix} -\Sigma_1^{-1} & \Sigma_1^{-1}\bar{A} & \Sigma_1^{-1}\overline{B} & 0 \\ \bar{A}\Sigma_1^{-1} & -\Sigma_1^{-1} & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_1^{-1} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.12}$$

Since $\bar{\sigma}_2 > 0$, there exists a scalar λ_2 such that

$$0 < \lambda_2 < 1, \quad \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1} = 1. \tag{3.13}$$

Then, the linear combination $\lambda_2 \times (3.10) + (1-\lambda_2) \times (3.12)$ results in

$$\begin{bmatrix} -\Sigma_2 & \Sigma_2\bar{A} & \Sigma_2\overline{B} & 0 \\ \bar{A}\Sigma_2 & -\Sigma_2 & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}}\Sigma_2 & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0, \tag{3.14}$$

where

$$\begin{aligned} \Sigma_2 &= \operatorname{diag}\{1,\, \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1},\, \ldots,\, \lambda_2\bar{\sigma}_n + (1-\lambda_2)\bar{\sigma}_n^{-1}\} \\ &\triangleq \operatorname{diag}\{1, 1, \ldots, \widetilde{\sigma}_n\} > 0. \end{aligned} \tag{3.15}$$

By repeating this process, we see that $\Sigma_n = I$ also satisfies (3.6), i.e.,

$$\begin{bmatrix} -I & \bar{A} & \overline{B} & 0 \\ \bar{A} & -I & 0 & \overline{B} \\ \overline{B}^{\mathrm{T}} & 0 & -\gamma I & D \\ 0 & \overline{B}^{\mathrm{T}} & D & -\gamma I \end{bmatrix} < 0. \tag{3.16}$$

Pre- and post-multiplying this matrix inequality by

$$\operatorname{diag}\{U, U, I, I\} \quad \text{and} \quad \operatorname{diag}\{U^{\mathrm{T}}, U^{\mathrm{T}}, I, I\},$$

respectively, we obtain (3.2). This completes the proof.

Now, we assume that all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ. Then, according to Lemma 3.1, we have

$$\begin{bmatrix} -I & A_i & B_i & 0 \\ A_i & -I & 0 & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}} & 0 & -\gamma I & D_i \\ 0 & C_i & D_i & -\gamma I \end{bmatrix} < 0 \tag{3.17}$$

for all $i \in \mathcal{I}_N$, which is equivalent to

$$\begin{bmatrix} A_i^2 + \dfrac{1}{\gamma}C_i^{\mathrm{T}}C_i - I & A_iB_i + \dfrac{1}{\gamma}C_i^{\mathrm{T}}D_i \\ B_i^{\mathrm{T}}A_i + \dfrac{1}{\gamma}D_iC_i & B_i^{\mathrm{T}}B_i + \dfrac{1}{\gamma}D_i^2 - \gamma I \end{bmatrix} < 0. \tag{3.18}$$

We compute the difference of the Lyapunov function candidate $V(x) = x^{\mathrm{T}}x$ along the trajectory of any subsystem to obtain

$$\begin{aligned} V(x[k+1]) - V(x[k]) &= x^{\mathrm{T}}[k+1]x[k+1] - x^{\mathrm{T}}[k]x[k] \\ &= (A_i x[k] + B_i w[k])^{\mathrm{T}}(A_i x[k] + B_i w[k]) - x^{\mathrm{T}}[k]\,x[k] \\ &= [\, x^{\mathrm{T}}[k] \;\; w^{\mathrm{T}}[k]\,] \begin{bmatrix} A_i^2 - I & A_i^{\mathrm{T}} B_i \\ B_i^{\mathrm{T}} A_i & B_i^{\mathrm{T}} B_i \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \\ &< -[\, x^{\mathrm{T}}[k] \;\; w^{\mathrm{T}}[k]\,] \begin{bmatrix} \frac{1}{\gamma} C_i^{\mathrm{T}} C_i & \frac{1}{\gamma} C_i^{\mathrm{T}} D_i \\ \frac{1}{\gamma} D_i C_i & \frac{1}{\gamma} D_i^2 - \gamma I \end{bmatrix} \begin{bmatrix} x[k] \\ w[k] \end{bmatrix} \\ &= -\frac{1}{\gamma}\left(z^{\mathrm{T}}[k]z[k] - \gamma^2 w^{\mathrm{T}}[k]w[k] \right), \end{aligned} \tag{3.19}$$

where (3.18) was used to obtain the inequality, and the trivial case of $x[k] = 0,\ w[k] = 0$ is excluded here.

For any piecewise constant switching signal and any given integer $k > k_0$, we let $k_1, \ldots, k_r$ $(r \geq 1)$ denote the switching points of $\sigma(k)$ over the interval $[k_0, k)$. Then, using the difference inequality (3.19), we obtain

$$\begin{aligned} V(x[k]) - V(x[k_r]) &< -\frac{1}{\gamma} \sum_{j=k_r}^{k-1} (z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j]) \\ V(x[k_r]) - V(x[k_{r-1}]) &< -\frac{1}{\gamma} \sum_{j=k_{r-1}}^{k_r-1} \left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j] \right) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ V(x[k_1]) - V(x[k_0]) &< -\frac{1}{\gamma} \sum_{j=k_0}^{k_1-1} \left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j] \right). \end{aligned} \tag{3.20}$$

We add all the above inequalities to get to

$$V(x[k]) - V(x[k_0]) < -\frac{1}{\gamma} \sum_{j=k_0}^{k-1} \left(z^{\mathrm{T}}[j]z[j] - \gamma^2 w^{\mathrm{T}}[j]w[j] \right). \tag{3.21}$$

Then, we use the assumption that $x[k_0] = x_0 = 0$ and the fact of $V(x[k]) \geq 0$ to obtain

$$\sum_{j=k_0}^{k-1} z^{\mathrm{T}}[j]z[j] < \gamma^2 \sum_{j=k_0}^{k-1} w^{\mathrm{T}}[j]w[j]. \tag{3.22}$$

We note that the above inequality holds for any $k > k_0$ including the case of $k \to \infty$, and that we did not add any limitation on the switching signal up to now.

We summarize the above discussion in the following theorem.

Theorem 3.2 *When all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ, the switched symmetric system (1.1) is exponentially stable and has an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching.*

Remark 3.1 From Lemma 3.1 and the proof of Theorem 3.1, we see that if all subsystems in (1.1) are Schur stable and have $\mathcal{L}_2$ gains smaller than γ, then there exists a general Lyapunov matrix $P = I$ for all subsystems, satisfying the LMI

$$\begin{bmatrix} -P & PA_i & PB_i & 0 \\ A_i^{\mathrm{T}}P & -P & 0 & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}}P & 0 & -\gamma I & D_i^{\mathrm{T}} \\ 0 & C_i & D_i & -\gamma I \end{bmatrix} < 0. \tag{3.23}$$

Hence, $V(x) = x^{\mathrm{T}}x$ serves as a general Lyapunov function for all subsystems in the sense of $\mathcal{L}_2$ gain.

Remark 3.2 Consider the continuous-time switched symmetric system

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}w(t) \\ z(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}w(t), \end{aligned} \tag{3.24}$$

where all the notations have the same meanings as in (1.1) except that the vectors $x(t)$, $w(t)$, $z(t)$ and the switching signal $\sigma(t)$ are with respect to the continuous time t . We assume that all subsystems in (3.24) are Hurwitz stable and have $\mathcal{L}_2$ gains smaller than γ. Then, the LMI

$$\begin{bmatrix} A_i^{\mathrm{T}}P + PA_i & PB_i & C_i^{\mathrm{T}} \\ B_i^{\mathrm{T}}P & -\gamma I & D_i^{\mathrm{T}} \\ C_i & D_i & -\gamma I \end{bmatrix} < 0 \tag{3.25}$$

has a general solution $P = I$ for all $i \in \mathcal{I}_N$. Using the same technique as in the proof of Theorem 3.1, we can prove that the switched symmetric system (3.24) is exponentially stable and has an $\mathcal{L}_2$ gain smaller than γ under arbitrary switching.

4 Concluding Remarks

In this paper, we have studied stability and $\mathcal{L}_2$ gain properties for a class of switched systems which are composed of a finite number of linear time-invariant symmetric subsystems. Assuming that all subsystems are Schur stable and have $\mathcal{L}_2$ gains smaller than a positive scalar γ, we have shown for both stability and $\mathcal{L}_2$ gain analysis that there exists a general Lyapunov function $V(x) = x^{\mathrm{T}}x$ for all subsystems, and that the switched system is exponentially stable and achieves an $\mathcal{L}_2$ gain smaller than the same γ under arbitrary switching.

We note finally that the result of the present paper can be extended to the switched symmetric systems in a more general sense. More precisely, if the equations $TA_i = A_i^{\mathrm{T}}T$, $TB_i = C_i^{\mathrm{T}}$, $D_i = D_i^{\mathrm{T}}$ are satisfied for a constant matrix $T > 0$, then we consider the similarity transformation $A_{\star i} = T^{1/2}A_iT^{-1/2}$, $B_{\star i} = T^{1/2}B_i$, $C_{\star i} = C_iT^{-1/2}$, $D_{\star i} = D_i$. Since stability and $\mathcal{L}_2$ gain properties of the system in this transformation do not change and we can easily confirm that $A_{\star i} = A_{\star i}^{\mathrm{T}}$, $B_{\star i} = C_{\star i}^{\mathrm{T}}$, we can apply the result we have obtained up to now for the system represented by the quadruplet $(A_{\star i}, B_{\star i}, C_{\star i}, D_{\star i})$ and thus derive corresponding result for the original switched system under arbitrary switching.

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