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Imaginary Axis Eigenvalues of a Delay System with Applications in Stability Analysis

Z. Zahreddine

[1] *Department of Basic Sciences, College of Arts & Sciences, University of Sharjah, Sharjah, P.O. Box 27272,*

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Abstract: We present a matrix method for determining the imaginary axis eigenvalues of a delay differential system. Both neutral and retarded delay systems are considered. We produce a second order polynomial matrix which is singular for all imaginary axis eigenvalues of the delay system leading to the recovery of eigenvectors associated with imaginary axis eigenvalues. The use of Kronecker products is emphasized in the proofs. Examples are given to illustrate the applicability of the new results in stability analysis.

Keywords: *Delay systems; stability analysis; Kronecker products; imaginary axis eigenvalues.*

Mathematics Subject Classification (2000): 34A25, 34C20; 34A09, 93D99.

1 Introduction

Consider a linear delay differential equation of the form

$$x'(t) + Ax'(t-h) = Bx(t) + Cx(t-h),$$

where A, B and $C \in \mathbf{R}^{n\times n}$, $\mathbf{R}$ being the set of real numbers. When $A = 0$, we get a retarded delay system, otherwise the system is neutral.

The purpose of this work is to present a matrix method for determining the imaginary axis eigenvalues of the above equation. Such eigenvalues occupy a special place in the theory of delay equations. They can be used to give the frequencies of oscillating solutions, and detect the onset of Hopf bifurcations. Although the idea of the approach is taken from the theory of quadratic functionals for delay equations [8], but the proofs will be quite direct, with strong emphasis on Kronecker products. In addition, we shall look at the matrix single delay case, and we propose a $2n^2 \times 2n^2$ matrix having spectrum containing all imaginary axis eigenvalues of the delay system. Our technique works equally well for both neutral and retarded delay systems. Therefore, we produce a polynomial matrix which is second order in s, and is singular for all values of s which are imaginary axis eigenvalues of the delay system, and in so doing, we can often directly recover the eigenvectors associated with imaginary axis eigenvalues.

Cooke and Grossman [4] made use of imaginary axis eigenvalues in their paper on stability switching for retarded systems. Also, Marshall [7] gave a polynomial elimination method for finding imaginary axis eigenvalues.

Imaginary axis eigenvalues were studied by the author in stability contexts in [10], where such eigenvalues represented singularities arising in the extended Routh array [9], which generalizes the well-known and historic Routh array to the complex domain. In Section 2, we introduce the notations and basic definitions. The main results are given in Section 3. Examples illustrating the applicability of the new results in stability analysis will be given in Section 4.

2 Notations and Basic Definitions

In this section we introduce the basic ideas and the terminology that we shall be using in the remaining sections. The ideas that we present here, were primarily motivated by the quadratic energy functionals pioneered by Repin [8], and later promoted by Infante and Castelan [3, 5, 6]. In the following, we will be converting matrix ordinary differential equations to vector form and visa versa. In order to do this, we will be making use of the elementary transformation $\Phi\colon \mathbf{C}^{n\times n} \to \mathbf{C}^{n^2}$, $\mathbf{C}$ being the set of complex numbers, which transforms elements $E = [\mathbf{e}_1, \dots, \mathbf{e}_n]^{\mathrm{T}}$ of $\mathbf{C}^{n\times n}$ into $\Phi(E) = [\mathbf{e}_1^{\mathrm{T}}, \dots, \mathbf{e}_n^{\mathrm{T}}]^{\mathrm{T}}$.

For any two complex matrices M and N, we recall the Kronecker product $M \otimes N$ and we note the identity $\Phi(MPN) = (M \otimes N^{\mathrm{T}})\Phi(P)$. We shall use this identity to conveniently move between matrix equations and vector equations. On several occasions in this paper, we shall be using certain basic facts about Kronecker products, and for that we refer the reader to Brewer [1, 2].

We now introduce the ordinary differential equation which motivates our work. If A, B and C are as defined above, we consider the system of ordinary differential equations:

$$\begin{aligned} X'(t) + AY'(t) &= BX(t) + CY(t), \\ X'(t)A^{\mathrm{T}} + Y'(t) &= -X(t)C^{\mathrm{T}} - Y(t)B^{\mathrm{T}}. \end{aligned} \tag{1}$$

Let $\mathbf{C}_1$ denote the vector space $\mathbf{C}_1 = \mathbf{C}^{n\times n} \times \mathbf{C}^{n\times n}$, and Ψ the operator on $\mathbf{C}_1$ defined by

$$\Psi \begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} Q + AR \\ QA^{\mathrm{T}} + R \end{bmatrix}$$

for $Q, R \in \mathbf{C}^{n\times n}$. Also, we let Ω be given by

$$\Omega \begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} BQ + CR \\ -QC^{\mathrm{T}} - RB^{\mathrm{T}} \end{bmatrix}.$$

With

$$Z(t) = \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

we write the matrix differential equation (1) as $\Psi Z'(t) = \Omega Z(t)$.

In order to write equation (1) in vector coordinates, let $x = \Phi(X)$, $y = \Phi(Y)$, $z = [x,\, y]^{\mathrm{T}}$ and consider the matrices

$$A_0 = \begin{bmatrix} I\otimes I & A\otimes I \\ I\otimes A & I\otimes I \end{bmatrix}, \qquad B_0 = \begin{bmatrix} B\otimes I & C\otimes I \\ -I\otimes C & -I\otimes B \end{bmatrix}.$$

In this way, (1) can be written as the following vector differential equation:

$$A_0 z'(t) = B_0 z(t). \tag{2}$$

Suppose now that $\mathbf{Z} = (\mathbf{X}, \mathbf{Y}) = (X_0\mathbf{e}^{st}, Y_0\mathbf{e}^{st})$ is a matrix solution of the differential equation (1). When differentiating with respect to t, we get

$$\begin{gathered} (sI - B)\mathbf{X} + (sA - C)\mathbf{Y} = 0, \\ \mathbf{X}(sA^{\mathrm{T}} + C^{\mathrm{T}}) + \mathbf{Y}(sI + B^{\mathrm{T}}) = 0. \end{gathered} \tag{3}$$

For every complex number s, we define $\Gamma = \Gamma(s)$ to be the operator $\Gamma = s\Psi - \Omega$ taking $\mathbf{C}_1$ into $\mathbf{C}_1$, so that

$$\Gamma \begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} (sI - B)Q + (sA - C)R \\ Q(sA^{\mathrm{T}} + C^{\mathrm{T}}) + R(sI + B^{\mathrm{T}}) \end{bmatrix} \tag{4}$$

for $Q, R \in \mathbf{C}^{n\times n}$.

Therefore (3) can be written as $\Gamma\mathbf{Z} = 0$. With $\mathbf{z} = \Phi(\mathbf{Z})$, we have

$$(sA_0 - B_0)\mathbf{z} = 0.$$

In order to investigate the behavior of Γ, we try to solve

$$\Gamma \begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} Q_0 \\ R_0 \end{bmatrix} \tag{5}$$

for Q and R.

By combining (4) and (5), we get

$$\begin{bmatrix} (sI - B)Q + (sA - C)R \\ Q(sA^{\mathrm{T}} + C^{\mathrm{T}}) + R(sI + B^{\mathrm{T}}) \end{bmatrix} = \begin{bmatrix} Q_0 \\ R_0 \end{bmatrix}. \tag{6}$$

If in (6), we right multiply the upper equation by $sI + B^{\mathrm{T}}$, and left multiply the lower equation by $sA - C$, then subtract, we get

$$(sI - B)Q(sI + B^{\mathrm{T}}) - (sA - C)Q(sA^{\mathrm{T}} + C^{\mathrm{T}}) = Q_0(sI + B^{\mathrm{T}}) - (sA - C)R_0.$$

Similarly, if in (6), we right multiply the upper equation by $sA^{\mathrm{T}} + C^{\mathrm{T}}$, and left multiply the lower equation by $sI - B$, then subtract, we get

$$(sI - B)R(sI + B^{\mathrm{T}}) - (sA - C)R(sA^{\mathrm{T}} + C^{\mathrm{T}}) = (sI - B)R_0 - Q_0(sA^{\mathrm{T}} + C^{\mathrm{T}}).$$

The latter equation can better be expressed in concise operator language in the following way: For every complex s, let $\Gamma^+ = \Gamma^+(s)\colon\ \mathbf{C}_1 \to \mathbf{C}_1$ be the operator defined by

$$\Gamma^+ \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U(sI + B^{\mathrm{T}}) - (sA - C)V \\ -U(sB^{\mathrm{T}} + C^{\mathrm{T}}) + (sI - B)V \end{bmatrix}$$

for $U, V \in \mathbf{C}^{n\times n}$, and let $\lambda = \lambda(s)$ be the operator on $\mathbf{C}^{n\times n}$ given by

$$\lambda W = (sI - B)W(sI + B^{\mathrm{T}}) - (sA - C)W(sA^{\mathrm{T}} + C^{\mathrm{T}})$$

for $W \in \mathbf{C}^{n\times n}$. It is clear that for each complex s, we have

$$\Gamma^+\Gamma \begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} \lambda Q \\ \lambda R \end{bmatrix}. \tag{7}$$

Again using the map Φ defined at the beginning of this section, we have natural associations of matrices with operators $\Gamma(s)$, $\Gamma^+(s)$, and $\lambda(s)$. With $q = \Phi(Q)$, $r = \Phi(R)$, $u = \Phi(U)$, $v = \Phi(V)$ we can write $H[\,q,\, r\,]^{\mathrm{T}}$ for $\Gamma[\,Q,\, R\,]^{\mathrm{T}}$ and $H^+[\,u,\, v\,]^{\mathrm{T}}$ for $\Gamma^+[\,U,\, V\,]^{\mathrm{T}}$, where

$$H = H(s) = \begin{bmatrix} (sI - B)\otimes I & (sA - C)\otimes I \\ I\otimes(sA + C) & I\otimes(sI + B) \end{bmatrix} = sA_0 - B_0,$$
$$H^+ = H^+(s) = \begin{bmatrix} I\otimes(sI + B) & -(sA - C)\otimes I \\ -I\otimes(sA + C) & (sI - B)\otimes I \end{bmatrix}.$$

Similarly, with $w = \Phi(W)$, we can write Λw for λW, where

$$\Lambda = \Lambda(s) = (sI - B)\otimes(sI + B) - (sA - C)\otimes(sA + C).$$

In particular, we note that (7) is written as $H^+H[\,q,\, r\,]^{\mathrm{T}} = [\,\Lambda q,\, \Lambda r\,]^{\mathrm{T}}$, i.e.

$$H^+H = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix}.$$

It follows that the determinant of H is closely related to that of Λ, i.e., $|H^+H| = |\Lambda|^2$. The following theorem makes it evident that $H(s)$ and $H^+(s)$ have the same determinant.

Theorem 2.1 *For every complex s, we have $|H(s)| = |H^+(s)|$.*

Proof Write $H = \begin{bmatrix} \alpha & \beta \\ \chi & \delta \end{bmatrix}$, where $\alpha = a(s)\otimes I$, $\beta = b(s)\otimes I$, $\chi = I\otimes c(s)$, $\delta = I\otimes d(s)$, and $a(s) = sI - B$, $b(s) = sA - C$, $c(s) = sA + C$, $d(s) = sI + B$. With $m = n^2$ row interchanges, we get

$$|H| = (-1)^m \begin{vmatrix} \chi & \delta \\ \alpha & \beta \end{vmatrix}$$

and after the same number of column interchanges, we find that

$$|H| = \begin{vmatrix} \delta & \chi \\ \beta & \alpha \end{vmatrix}. \tag{8}$$

We consider two cases:

Case 1 If A is non-singular, then neither $|c(s)|$ nor $|d(s)|$ are uniformly zero. Consider following identity which holds for all but a finite number of complex s,

$$|H| = |\delta| \cdot |\alpha - \beta\delta^{-1}\chi| = |\delta| \cdot |\alpha\chi^{-1} - \beta\delta^{-1}| \cdot |\chi|. \tag{9}$$

If we use formula (8) in (9), we get the following identity which also holds for all but a finite number of complex s,

$$|H^+| = \begin{vmatrix} \delta & -\beta \\ -\chi & \alpha \end{vmatrix} = |\delta| \cdot |\alpha - \chi\delta^{-1}\beta| = |\delta| \cdot |\chi| \cdot |\chi^{-1}\alpha - \delta^{-1}\beta|. \tag{10}$$

Using the Kronecker product identities $(I \otimes M)(N \otimes I) = (N \otimes I)(I \otimes M)$ and $(N \otimes M)^{-1} = N^{-1} \otimes M^{-1}$, we see that $\alpha\chi^{-1} = \chi^{-1}\alpha$ and $\beta\delta^{-1} = \delta^{-1}\beta$ for all but a finite number of complex s, so that $|H(s)| = |H^+(s)|$ for all but a finite number of complex s as well. Since $|H(s)|$ and $|H^+(s)|$ are both polynomials, we conclude that $|H(s)| = |H^+(s)|$ for all $s \in \mathbf{C}$.

Case 2 If A is singular, let $A(\varepsilon) \to A$ as $\varepsilon \to 0$, with $A(\varepsilon)$ being non-singular for each nonzero ε in a neighborhood of zero. Define H_ε, H_ε^+ in the same way as H and H^+ with $A(\varepsilon)$ replacing A, we get $|H_\varepsilon(s)| = |H_\varepsilon^+(s)|$ for all $s \in \mathbf{C}$. By continuity of the determinant, we have $|H(s)| = |H^+(s)|$ for all $s \in \mathbf{C}$, and the proof is complete.

Corollary 2.1 *For all $s \in \mathbf{C}$, we have $|H(s)|^2 = |\Lambda(s)|^2$.*

Proof This follows immediately from $|H^+(s)H(s)| = |\Lambda(s)|^2$ and $|H(s)| = |H^+(s)|$.

Corollary 2.1 is enough for our purposes, since it makes it clear that the matrices $H(s)$ and $\Lambda(s)$ are either both singular, or both non-singular. But, it must be noted that with some intricate argument involving Kronecker products, the matrices $H(s)$ and $\Lambda(s)$ themselves can be shown to have equal determinants for all complex s.

3 Main Results

In this section, we consider the delay equation introduced at the beginning of Section 1

$$x'(t) + Ax'(t-h) = Bx(t) + Cx(t-h) \tag{11}$$

and we show that all imaginary axis eigenvalues of this equation are zeros of $|\Lambda(s)|$, and hence also of $|H(s)|$, so that they are generalized eigenvalues of the matrix pair (B_0, A_0) of Section 2.

Theorem 3.1 *Let A, B and $C \in \mathbf{R}^{n\times n}$. Then all imaginary axis eigenvalues of the delay differential equation (11) are zeros of $|\Lambda(s)|$, and therefore also of $|sA_0 = B_0|$. If s is an imaginary axis eigenvalue of (11) with associated eigenvector v, then $\mathbf{v} = \Phi(vv^*)$, is contained in the kernel of the matrix $\Lambda(s)$.*

Proof If $s = iw$ is an imaginary axis eigenvalue of the system (11), then we have

$$|s(I + Ae^{-sh}) - B - Ce^{-sh}| = 0 = |sI - B + e^{-sh}(sB - C)|.$$

We know that for every associated eigenvector v of (11), we have

$$(sI - B)v = -e^{-sh}(sA - C)v. \tag{12}$$

By applying the conjugation and the transposition operations, we get

$$v^*(-sI - B^{\mathrm{T}}) = -e^{sh}v^*(-sA^{\mathrm{T}} - C^{\mathrm{T}}),$$

from which we conclude that

$$v^*(sI + B^{\mathrm{T}}) = -e^{sh}v^*(sA^{\mathrm{T}} + C^{\mathrm{T}}). \tag{13}$$

Multiplying the left-hand side of (12) by the left-hand side of (13), and similarly for the right-hand sides, we get

$$(sI - B)vv^*(sI + B^{\mathrm{T}}) = (sA - C)vv^*(sA^{\mathrm{T}} + C^{\mathrm{T}}).$$

Let $\mathbf{v} = \Phi(vv^*)$, where the map Φ is as defined in Section 2, then

$$((sI - B) \otimes (sI + B) - (sA - C) \otimes (sA + C))\mathbf{v} = 0,$$

or in other words

$$\Lambda(s)\mathbf{v} = 0,$$

from which it follows that $|\Lambda(s)| = 0$.

By Corollary 2.1, it follows immediately that $0 = |H(s)| = |sA_0 - B_0|$ and that completes the proof.

A immediate corollary of Theorem 3.1 is the following.

Corollary 3.1 *Let A, B and $C \in \mathbf{R}^{n\times n}$. If $s = iw$ is an imaginary axis eigenvalue of the delay differential equation (11), then the operators $\Gamma(s)$ and $\lambda(s)$ defined in Section 2 are both singular at s.*

An interesting situation arises when, for an imaginary axis eigenvalue s of the delay differential equation (11), the kernel of $\Lambda(s)$ has dimension 1. In this case, we can easily find an eigenvector of (11), associated with s, directly from $\ker(\Lambda(s))$.

Corollary 3.2 *Let s be an imaginary axis eigenvalue of the delay differential equation (11), and suppose that $\ker(\Lambda(s))$ has dimension 1. Let $\mathbf{x}$ be any eigenvector of $\Lambda(s)$, and let $v = [\alpha_1, \ldots, \alpha_n]^{\mathrm{T}}$ be any eigenvector of (11) associated with s. Let $V = vv^* = [V_{jk}]$, $\Phi(V) = \mathbf{v}$, $\Phi(X) = \mathbf{x}$ and $X = [X_{jk}]$. Then $X_{jk} = 0$ if and only if $V_{jk} = 0$. If $X_{jj} \neq 0$, then the complex vector having $\overline{X}_{jk}/\overline{X}_{jj}$ for the k-th entry is the eigenvector v/α_j of (11), having 1 for j-th entry.*

Proof Since $\mathbf{v}, \mathbf{x} \in \ker(\Lambda(s))$, we have $\mathbf{x} = a\mathbf{v}$ for some nonzero complex a. Therefore $X = aV$. Now $X_{jk} = aV_{jk} = a\alpha_j\overline{\alpha}_k$, and if $X_{jj} \neq 0$, we have $X_{jk}/X_{jj} = \alpha_k/\alpha_j$, and the proof is complete.

It is worthwhile to mention the following remark.

Remark 3.1 Whenever we have $|A_0| \neq 0$, the zeros of $|sA_0 - B_0|$ coincide with those of $|sI - A_0^{-1}B_0|$, so that the pure imaginary eigenvalues of the delay system (11) of

Theorem 3.1 are also eigenvalues of the matrix $A_0^{-1}B_0$. In fact, there is quite a simple criterion for the condition that A_0 is singular. By [1], it is known that $(I \otimes A)(A \otimes I) = A \otimes A$, therefore $|A_0| = |I \otimes I - (I \otimes A)(A \otimes I)| = |I \otimes I - A \otimes A|$. Since $\operatorname{Eig}(R \otimes S)$ is the set product of $\operatorname{Eig}(R)$ with $\operatorname{Eig}(S)$ for any square matrices R and S, then A_0 is singular if and only if $1 = \mu\delta$ for some μ, δ in $\operatorname{Eig}(A)$.

4 Examples

We now give some examples to verify the applicability of Theorem 3.1 in stability computation.

Example 4.1. The first stability interval of a neutral system

Consider the neutral delay equation

$$x'(t) + Ax'(t-h) = Bx(t) + Cx(t-h)$$

where

$$A = \begin{bmatrix} 0.75 & 0.25 \\ -0.25 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.5 & 0.25 \\ -0.25 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} -3.5 & -0.5 \\ 0.5 & -3 \end{bmatrix}.$$

First note that $\operatorname{Eig}(A) \approx \{0.655, 0.095\}$, and therefore A is an asymptotically stable discrete matrix. Note that with zero delay, we have $(I + A)x'(t) = (B + C)x(t)$, and since $I + A$ is invertible, we can write this as $x'(t) = (I + A)^{-1}(B + C)x(t)$. A MATLAB computation shows that the eigenvalues of $(I + A)^{-1}(B + C)$ are -1.000 and -1.1379, and we have asymptotic stability of the above neutral delay system with zero delay.

If we wish to determine the smallest h for which the system is not asymptotically stable, we first have to find all possible imaginary axis eigenvalues of the system. Therefore, we form the matrices

$$A_0 = \begin{bmatrix} I \otimes I & A \otimes I \\ I \otimes A & I \otimes I \end{bmatrix}, \quad B_0 = \begin{bmatrix} B \otimes I & C \otimes I \\ -I \otimes C & -I \otimes B \end{bmatrix}.$$

Since $1 \notin \operatorname{Eig}(A)\cdot\operatorname{Eig}(A)$, we could compute the eigenvalues of $A_0^{-1}B_0$. Using MATLAB, we can also obtain the zeros of $|sA_0 - B_0|$ by determining the generalized eigenvalues of the matrix pair (B_0, A_0) directly. We then find that $\operatorname{Eig}(B_0, A_0) \approx \{\pm 4.1654i, \pm 2.3834i, \pm 1.4524 \pm 2.4206i\}$. Let $\Omega = \{\pm 4.1654, \pm 2.3834\}$, and note that $i\Omega$ contains all possible imaginary axis eigenvalues of the system. Next, we note that for real w, the matrix $T = iwI - B + e^{-iwh}(iwA - C)$ is singular if and only if $z = e^{-iwh}$ is a unit magnitude generalized eigenvalue of the matrix pair $(B - iwI, iwA - C)$. Checking all numbers of Ω, we find from MATLAB that for $w = 4.1654$, we have $\operatorname{Eig}(B - iwI, iwA - C) = \{-0.3987 - 0.9171i, 0.5369 - 1.3327i\}$, and $z = -0.3987 - 0.9171i$ lies on the unit circle. With $w = -4.1654$ we get the set of conjugates for generalized eigenvalues, and $z = -0.3987 + 0.9171i$ lies on the unit circle. With $w = 2.3834$ the generalized eigenvalues of the associated matrix pair are $0.0485 - 0.7281i$, $0.6286 - 0.7778i$, and we again have the conjugates for generalized eigenvalues with $w = -2.3834$. We have magnitude one for $z = 0.6268 \pm 0.7778i$.

Now for $w = 4.1654$, $z = -0.3987 - 0.9171i$ the smallest value of h with $\varepsilon^{-iwh} = z$, is $h_1 = 0.4756$. With $w = 2.3834$, $z = 0.6286 - 0.7778i$, the smallest value of h with $e^{-iwh} = z$ is $h_2 = 0.3739$. For the other two values of w, we obtain by symmetry the same two values h_1 and h_2. Therefore we have imaginary axis eigenvalues for the delay equation under consideration with $h \approx 0.3739$, and for all smaller nonnegative values of h the system is asymptotically stable.

Example 4.2. Stability switching in a retarded system

Consider the scalar delay equation

$$x''(t) + x'(t) - x'(t-h) + 4x(t) - 2x(t-h) = 0$$

which has (11) as matrix counterpart with

$$A = 0, \quad B = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

With zero delay, this system is a pure oscillator having eigenvalues $\pm i\sqrt{2}$. To determine the other imaginary axis eigenvalues, we note that A_0 here is the $2n^2 \times 2n^2$ identity matrix, $n = 2$, and we use MATLAB to find the eigenvalues of B_0. Then we find $\mathrm{Eig}(B_0) \approx \{\pm 2.4495i, \pm 1.4142i, \pm 0.5000 \pm 1.9365i\}$, and the set of imaginary axis eigenvalues of B_0 is $i\Omega$, where $\Omega \approx \{\pm 2.4495, \pm 1.4142\}$. Now with $T = iwI - B - e^{-iwh}C$, we know T is singular if and only if $z = e^{-iwh}$ is a generalized eigenvalue of the pair $(iwI - B, C)$.

Now, for any $w \in \Omega$, we let $B_w = iwI - B$, and we note that the generalized eigenvalues of the pair (B_w, C) are the solutions z of the equation $|B_w - zC| = 0$. Writing B_w and C in terms of their rows as

$$B_w = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ c \end{bmatrix},$$

we have

$$0 = |B_w - zC| = |B_w| - z \begin{vmatrix} b_1 \\ c \end{vmatrix}.$$

With $w = 2.4495$, we get $z = 0.2000 + 0.9798i$, which has unit magnitude. With $w = 1.4142$, we get $z = 1 + 0i$, and we again obtain conjugate eigenvalues with opposite values of w. With $w = w_1 = 2.4495$, the smallest h making $e^{-iwh} = z = 0.2000 + 0.9798i$ is given by $w_1 h = 2\pi - \arctan(0.9798/0.2000)$ i.e. $h \approx 2.006$. Adding positive integer multiples of $2\pi/w_1$ provides the other corresponding values of h for which the delay system has imaginary axis eigenvalues. Similarly, with $w = w_2 = 1.4142$, the smallest h making $e^{-iwh} = z = 1$ is zero, and adding natural multiples of $2\pi/w_2$ again provides the others. The conjugate frequencies $w = -w_1$, $w = -w_2$ give us the same values of h. The first few values of h are

$$h_0 = 0, \quad h_1 \approx 2.006, \quad h_2 \approx 4.443, \quad h_3 \approx 4.571, \quad h_4 \approx 7.136.$$

Example 4.3. Using matrix polynomials for finding imaginary axis eigenvalues of a delay system

We reconsider the above examples to show how one can determine imaginary axis eigenvalues using the matrix polynomial $\Lambda(s)$ defined in Section 2. For the system (11), we recall that $\Lambda(s) = (sI - B) \otimes (sI + B) - (sA - C) \otimes (sA + C)$. $\Lambda(s)$ can also be written as $\Lambda(s) = D_0 s^2 + D_1 s + D_2$ where

$$\begin{aligned} D_0 &= I \otimes I - A \otimes A, \\ D_1 &= I \otimes B - B \otimes I + C \otimes A - A \otimes C, \\ D_2 &= C \otimes C - B \otimes B. \end{aligned}$$

If $\Lambda(s) = D_0 s^2 + D_1 s + D_2$ is put in the context of Example 4.1, then D_0 is invertible, so that the problem becomes that of looking for the s-values where $D_0^{-1}\Lambda(s) = E_0 s^2 + E_1 s + E_2$ is singular, with $E_0 = I \otimes I = I_0$, $E_1 = D_0^{-1} D_1$, $E_2 = D_0^{-1} D_2$. Now, these are the eigenvalues of the matrix

$$F = \begin{bmatrix} 0 & I_0 \\ -E_2 & -E_1 \end{bmatrix}.$$

With MATLAB computation we find that $\mathrm{Eig}(F) \approx \{\pm 4.1654i, \pm 2.3834i, \pm 1.4524 \pm 2.4206i\}$, just as before.

If $\Lambda(s) = D_0 s^2 + D_1 s + D_2$ is put in the context of Example 4.2, where $A = 0$, we get $D_0 = I \otimes I = I_0$, $D_1 = I \otimes B - B \otimes I$. Here, the s-values making $\Lambda(s)$ singular are the eigenvalues of

$$F = \begin{bmatrix} 0 & I_0 \\ -D_2 & -D_1 \end{bmatrix},$$

and with MATLAB computation we get $\mathrm{Eig}(F) \approx \{\pm 2.4495i, \pm 1.4142i, \ \pm 0.5000 \pm 1.9365i\}$, as in Example 4.2.

Example 4.4

This example is designed to show the practical simplicity of finding associated eigenvectors using matrix polynomials. Again we return to Example 4.1 and we show how the kernel of $\Lambda(s)$ can be used to obtain eigenvectors associated with delay equation imaginary axis eigenvalues. From these eigenvectors we can immediately find the value of $z = e^{-sh}$. Beginning with $s = 4.1654i$, we have $\Lambda(s) = D_0 s^2 + D_1 s + D_2$, and with MATLAB computation we find that this matrix has exactly one zero eigenvalue, with associated eigenvector

$$\mathbf{x} \approx \begin{bmatrix} 0.8476 + 0.3654i \\ -0.1357 - 0.2295i \\ -0.2600 + 0.0588i \\ 0.0707 + 0.0305i \end{bmatrix}.$$

With $\Phi(X) = \mathbf{x}$, we have

$$X \approx \begin{bmatrix} 0.8476 + 0.3654i & -0.1357 - 0.2295i \\ -0.2600 + 0.0588i & 0.0707 + 0.0305i \end{bmatrix}.$$

Since $X_{11} \neq 0$, we know from Corollary 3.2 that the eigenvector $v = [\,1 \;\; \alpha_2\,]^{\mathrm{T}}$ associated with s is given by $\alpha_2 = \overline{X}_{12}/\overline{X}_{11} \approx -0.2335+0.1701i$. We now return to the characteristic equation $(sI-B)v = -z(sA-C)v$ of the delay system (11), with $z = e^{-sh}$. Evaluating at $s = 4.1654i$, we write $(sI-B)v = m = [\,m_1 \;\; m_2\,]^{\mathrm{T}}$ and $(sA-C)v = r = [\,r_1 \;\; r_2\,]^{\mathrm{T}}$. Then $z = -m_1/r_1 = -m_2/r_2 \approx -0.3987 - 0.9171i$, as before, and z has unit magnitude. Now, with the case $s = 2.3834i$, we obtain $v = [\,1 \;\; -3.9177 - 1.5938i\,]^{\mathrm{T}}$, $z = 0.6286 - 0.7778i$, $|z| = 1$, and z is as previously found. As usual, the values of z obtained from $s = -4.1654i$, $s = -2.3834i$ are conjugate to those obtained from the first two.

It is possible that an eigenvector v of the delay equation (11) simultaneously satisfies $(sI - B)v = 0 = (sA - C)v$ for its associated eigenvalue s. In this case, it is possible to use the generalized eigenvalue approach displayed in Examples 4.1 and 4.2 to find the value of $z = e^{-sh}$.

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