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Robust $\mathcal{H}_\infty$ Analysis and Synthesis for Jumping Time-Delay Systems using Transformation Methods

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Abstract: A new transformation method is developed for the $\mathcal{H}_\infty$ analysis and synthesis of a class of uncertain time-delay systems with Markovian jump parameters. In these systems, the jumping parameters are modelled as a continuous-time, discrete-state Markov process and the parametric uncertainties are assumed to be real, time-varying and norm-bounded. The time-delay factor is constant. Complete results for delay dependent stochastic stability and stabilization criteria are developed for all admissible uncertainties. Then a dynamic output feedback controller is designed such that the closed-loop stochastic stability and a prescribed $\mathcal{H}_\infty$-performance are guaranteed. All the developed results are cast in the format of linear matrix inequalities

Keywords: *Time-delay systems; Markovian jump parameters; $\mathcal{H}_\infty$ analysis; $\mathcal{H}_\infty$ synthesis; uncertain parameters.*

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1 Introduction

It becomes increasingly apparent that delays occur in industrial and engineering systems due to various reasons including finite capabilities of information processing among different parts of the system, inherent phenomena like mass transport flow and recycling and/or by product of computational delays [12]. Considerable discussions on delays and their stabilization/destabilization effects in control systems have commanded the interests of numerous investigators in recent years, see [1, 6, 13] and their references. In the course of control design, it turns out that the design goals have to incorporate the impact of parameter shifting, component and interconnection failures which are frequently occurring in practical situations. It is thus appropriate to investigate control processes with the aid of stochastic models. One direction of investigation has been through piecewise deterministic systems or Markovian jump dynamical systems [2] in which the underlying dynamics are governed by different forms depending on the value of an associated finite-state Markov process thus offer a base model of combined continuous and discrete states. Research into this class of systems and their applications span several decades [5, 15]. When the plant modelling uncertainty or external disturbance uncertainty is of major concern in control systems, robust control theory provides tractable design tools using the time domain and the frequency domain. For Markov jumping linear continuous-time systems, the issue of robust stability and $\mathcal{H}_\infty$-control has been investigated in [4, 17] and their references. The class of time- delay systems with jump parameters have been recently considered in [1, 13] and for a modest coverage on the subject, see [2, 14].

The purpose of this paper is to extend the results of [1, 2, 13] further by developing new transformation methods that will help much in the study of stochastic stability and stabilization of a class of uncertain systems with Markovian jump parameters and distributed delays. In these systems, the jumping parameters are treated as continuous-time, discrete-state Markov process and the parametric uncertainties are assumed to be real, time-varying and norm-bounded. The time-delay factor is treated as a constant within a prespecified range. Complete results of delay-dependent stochastic stability criteria are developed for both the nominal and uncertain jumping distributed delay systems with $\mathcal{H}_\infty$ performance measure. Then we move to consider the $\mathcal{H}_\infty$ stabilization problem with instantaneous and delayed state feedback. Finally, we investigate the design of an $\mathcal{H}_\infty$ dynamic output feedback controller that ensures the close-loop stochastic stability. We establish that the $\mathcal{H}_\infty$ stability analysis and synthesis problems for the distributed-delay Markovian jump systems with and without uncertain parameters can be essentially solved in terms of the solutions of a finite set of coupled linear matrix inequalities. Several examples are presented to illustrate the theoretical analysis.

Notations and Facts: In the sequel, the Euclidean norm is used for vectors. We use W^t, W^{-1}, $\lambda(W)$ and $\|W\|$ to denote, respectively, the transpose of, the inverse of, the eigenvalues of and the induced norm of any square matrix W. We use $W > 0$ $(\geq, <, \leq 0)$ to denote a symmetric positive definite (positive semidefinite, negative, negative semidefinite matrix W with $\lambda_m(W)$ and $\lambda_M(W)$ being the minimum and maximum eigenvalues of W and I to denote the $n \times n$ identity matrix. The Lebesgue space $\mathcal{L}_2[0, T]$ consists of square-integrable functions on the interval $[0, T]$ equipped with the norm $\|\cdot\|_2$. $\mathbb{E}[\cdot]$ stands for mathematical expectation. Let $\mathcal{S} = \{1, 2, ..., s\}$ be a finite set, $\mathcal{C}[-\tau_j, 0]$ be the space of continuous functions on the interval $[-\tau_j, 0]$ and define $\bar{\mathcal{C}} \triangleq \bigcup_{j \in \mathcal{S}} \mathcal{C}[-\tau_j, 0] \times \{j\}$. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

Fact 1: For any real vectors β, ρ and any matrix $Q^t = Q > 0$ with appropriate dimensions, it follows that

$$-2\rho^t\beta \leq \rho^t Q\rho + \beta^t Q^{-1}\beta.$$

Fact 2: For any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions and $\Sigma_3^t\Sigma_3 \leq I$, it follows that

$$\Sigma_1\Sigma_3\Sigma_2 + \Sigma_2^t\Sigma_3^t\Sigma_1^t \leq \alpha^{-1}\Sigma_1\Sigma_1^t + \alpha\Sigma_2^t\Sigma_2, \quad \forall\, \alpha > 0.$$

Fact 3: Let Σ_1, Σ_2, Σ_3 and $0 < R = R^t$ be real constant matrices of compatible dimensions and $H(t)$ be a real matrix function satisfying $H^t(t)H(t) \le I$. Then for any $\rho > 0$ satisfying $\rho\Sigma_2^t\Sigma_2 < R$, the following matrix inequality holds:

$$(\Sigma_3 + \Sigma_1 H(t)\Sigma_2)R^{-1}(\Sigma_3^t + \Sigma_2^t H^t(t)\Sigma_1^t) \le \rho^{-1}\Sigma_1\Sigma_1^t + \Sigma_3(R - \rho\Sigma_2^t\Sigma_2)^{-1}\Sigma_3^t.$$

Fact 4 (Schur Complement): Given constant matrices Ω_1, Ω_2, Ω_3, where $\Omega_1 = \Omega_1^t$ and $0 < \Omega_2 = \Omega_2^t$ then $\Omega_1 + \Omega_3^t\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^t \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0 \qquad \text{or} \qquad \begin{bmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^t & \Omega_1 \end{bmatrix}.$$

2 Problem Statement

2.1 System description

Given a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where Ω is the sample space, $\mathcal{F}$ is the algebra of events and $\mathbf{P}$ is the probability measure defined on $\mathcal{F}$. Let the random form process $\{\eta_t, t \in [0, \mathcal{T}]\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories and taking values in a finite set $\mathcal{S} = \{1, 2, ..., s\}$ with generator $\Im = (\alpha_{ij})$ and transition probability from mode i at time t to mode j at time $t + \delta$, $i, j \in \mathcal{S}$:

$$\mathbf{p}_{ij} = Pr(\eta_{t+\delta} = j \mid \eta_t = i) = \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if} \quad i \ne j, \\ 1 + \alpha_{ij}\delta + o(\delta), & \text{if} \quad i = j \end{cases}, \tag{2.1}$$

with transition probability rates $\alpha_{ij} \ge 0$ for $i, j \in \mathcal{S}$, $i \ne j$ and

$$\alpha_{ii} = -\sum_{m=1,\, m\ne i}^{s} \alpha_{im}, \tag{2.2}$$

where $\delta > 0$ and $\lim\limits_{\delta\downarrow 0} o(\delta)/\delta = 0$. The set $\mathcal{S}$ comprises the various operational modes of the system under study. We consider a class of stochastic uncertain time-delay systems with Markovian jump parameters described over the space $(\Omega, \mathcal{F}, \mathbf{P})$ by:

$$\begin{aligned} (\Sigma_J): \quad \dot{x}(t) &= [A_o(\eta_t) + \Delta A_o(t, \eta_t)]x(t) + [A_d(\eta_t) + \Delta A_d(t, \eta_t)]x(t - \tau) + \Gamma(\eta_t)w(t), \\ &= A_{\Delta o}(t, \eta_t)x(t) + A_{\Delta d}(t, \eta_t)x(t - \tau) + \Gamma(\eta_t)w(t) \;\; t \ge 0, \end{aligned}$$

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i, \tag{2.3}$$

$$z(t) = G(\eta_t)x(t) + \Phi(\eta_t)w(t), \tag{2.4}$$

where $x(t) \in \Re^n$ is the state vector; $w(t) \in \Re^q$ is the disturbance input which belongs to $\mathcal{L}_2[0, \mathcal{T}]$; $y(t) \in \Re^p$ is the measured output; $z(t) \in \Re^r$ is the controlled output which belongs to $\mathcal{L}_2\big[(\Omega, \mathcal{F}, \mathbf{P}), [0, \mathcal{T}]\big]$ and $\tau \in [0, \tau^*]$ is a constant delay factor. For each possible value $\eta_t = i$, $i \in \mathcal{S}$, we will denote the system matrices of (Σ_J) associated with mode i by

$$\begin{aligned} &A_o(\eta_t) \triangleq A_o(i), \qquad \Gamma(\eta_t) \triangleq \Gamma(i), \qquad G(\eta_t) \triangleq G(i), \\ &A_d(\eta_t) \triangleq A_d(i), \qquad \Phi(\eta_t) \triangleq \Phi(i), \end{aligned} \tag{2.5}$$

where $A_o(i)$, $A_d(i)$, $G(i)$, $\Gamma(i)$ and $\Phi(i)$ are known real constant matrices of appropriate dimensions which describe the nominal system of (Σ_J). The matrices $\Delta A_o(t, \eta_t)$ and $\Delta A_d(t, \eta_t)$ are real, time-varying matrix functions representing the norm-bounded parameter uncertainties. For $\eta_t = i$, the admissible uncertainties are assumed to be modeled in the form:

$$[\Delta A_o(t,i) \quad \Delta A_d(t,i)] = M_a(i)\Delta(t,i)[N_a(i) \quad N_d(i)], \quad \|\Delta(t,i)\|_2 \le 1, \tag{2.6}$$

where $M_a(i) \in \Re^{n\times\alpha}$, $N_a(i) \in \Re^{\beta\times n}$ and $N_d(i) \in \Re^{\beta\times n}$ are known real constant matrices, with $\Delta(t,i) \in \Re^{\alpha\times\beta}$ being unknown, time-varying matrix function whose elements are Lebesgue measurable for any $i \in \mathcal{S}$.

Our purpose in this paper is to develop criteria for $\mathcal{H}_\infty$ analysis and synthesis for system (2.3) – (2.4). Initially, we focus on stochastic stability and $\mathcal{L}_2$-gain criterion and examine their robustness using the performance measure

$$\mathcal{J}(x) \triangleq \mathbb{E}\Big\{ \int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t)]\, dt \Big\}, \tag{2.7}$$

where $\gamma > 0$ is a desired level of disturbance attenuation.

2.2 Model transformation

For each possible value $\eta_t = i$, $i \in \mathcal{S}$, we introduce the following state transformation

$$\sigma(t) = x(t) + \int_{t-\tau}^{t} A_{\Delta d}(t,i)x(s)\, ds \tag{2.8}$$

into (2.3) to yield

$$\dot\sigma(t) = [A_{\Delta o}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t). \tag{2.9}$$

Given a sufficiently small scalar ε, we define the augmented state-vector

$$\zeta(t) = \begin{bmatrix} \sigma(t) \\ \varepsilon x(t) \end{bmatrix} \in \Re^{2n}. \tag{2.10}$$

By combining (2.3) and (2.8) – (2.10) and taking the limit $\varepsilon \to 0$, we obtain the transformed system

$$(\Sigma_T): \quad \dot\zeta(t) = \Lambda_\Delta(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon(i)\zeta(s)\, ds + \bar\Gamma(i)w(t),$$

$$\zeta(t) = \bar\phi(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, \tag{2.11}$$

$$z(t) = \bar G(i)\zeta(t) + \Phi(i)w(t), \tag{2.12}$$

where

$$\bar{\Gamma}(i) = \begin{bmatrix} \Gamma(i) \\ 0 \end{bmatrix}, \quad \bar{G}(i) = [0 \quad G(i)], \quad A_{od}(i) = A_o(i) + A_d(i),$$
$$\Lambda_\Delta(i) = \begin{bmatrix} 0 & A_{\Delta o}(t,i) + A_{\Delta d}(t,i) \\ -I & I \end{bmatrix}, \qquad \Upsilon(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{\Delta d}(t,i) \end{bmatrix}. \tag{2.13}$$

For convenience, we introduce the matrices for $i \in \mathcal{S}$

$$\Lambda_o(i) = \begin{bmatrix} 0 & A_{od}(i) \\ -I & I \end{bmatrix}, \qquad \bar{M}(i) = \begin{bmatrix} M_a \\ 0 \end{bmatrix}, \qquad \mathbb{P}(i) = \begin{bmatrix} P_\sigma(i) & 0 \\ P_d(i) & P_x(i) \end{bmatrix},$$
$$N_{ad}(i) = N_a(i) + N_d(i), \qquad \bar{N}_{ad}(i) = [0 \quad N_{ad}(i)], \qquad \bar{P}(i) = U\mathbb{P}(i), \tag{2.14}$$
$$U = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Remark 2.1 Some discussions on the model transformation are in order. On one hand, the σ-variable recovers the delay-dependent dynamics of system (Σ_J). On the other hand, the use of small scalar ε is meant to capture the slow-modes of the system. It is readily seen for absolutely continuous initial functions that systems (Σ_J) and (Σ_T) are equivalent. For single-mode systems $s = 1$, a different approach was developed in [6] based on description-type transformation. In the sequel, it will be shown that our transformation is more flexible.

For system (2.11)–(2.14), we provide the following definition.

Definition 2.1 System (Σ_T) is said to be *delay dependent robustly stochastically stable (DDRSS) with disturbance attenuation* $\gamma > 0$ if for zero initial vector function $\phi \equiv 0$ defined on the interval $[-\tau, 0]$ and initial mode $\eta_o \in \mathcal{S}$

$$\|z(t)\|_{E_2} := \mathbb{E}\left[\int_0^\infty z^t(t)z(t)\,dt\right]^{1/2} < \gamma\|w(t)\|_2$$

for all $0 \neq w(t) \in \mathcal{L}_2[0,\infty)$ and for all admissible uncertainties satisfying (2.6).

3 $\mathcal{L}_2$-Gain Analysis

The theorem and corollaries established in the sequel show that the stability behavior of system Σ_T (or equivalently Σ_J) is related to the existence of a positive definite solution of a family of linear matrix inequalities (LMIs) thereby providing a clear key to designing the feedback controller.

Theorem 3.1 *System Σ_T is DDRSS with disturbance attenuation $\gamma > 0$ if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $0 < P_\sigma(i)$, $P_d(i)$, $P_x(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_2(i) & \Pi_{21}(i) & \Pi_{22}(i) & \Pi_{23}(i) & \Pi_{24}(i) \\ \Pi_{21}^t(i) & -\varepsilon_1(i)I & 0 & 0 & 0 \\ \Pi_{22}^t(i) & 0 & -\tau\varepsilon_2(i)I & 0 & 0 \\ \Pi_{23}^t(i) & 0 & 0 & -\tau Q_x(i) + \tau\varepsilon_2(i)N_d(i)N_d^t(i) & 0 \\ \Pi_{24}^t(i) & 0 & 0 & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) \end{bmatrix} < 0,$$
$$\begin{bmatrix} -Q_x(i) & N_d(i) \\ N_d^t(i) & -\varepsilon_2(i)I \end{bmatrix} < 0, \qquad \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{3.1}$$

where

$$\Pi_2(i) = \begin{bmatrix} -P_d(i) - P_d^t(i) + \sum_{m=1}^{s} \alpha_{im} P_\sigma(m) & -P_x(i) + P_d^t(i) + P_\sigma^t(i) A_{od}(i) \\ -P_x^t(i) + P_d(i) + A_{od}^t(i) P_\sigma(i) & \begin{matrix} P_x(i) + P_x^t(i) + \tau Q_x(i) \\ +G^t(i)G(i) + \rho(i)\tau^2 \sum_{m=1}^{s} \alpha_{im} Q_x(m) \\ +\varepsilon_1(i) \bar{N}_{ad}^t(i) \bar{N}_{ad}(i) \end{matrix} \end{bmatrix}, \quad (3.2)$$

$$\Pi_{21}(i) = \begin{bmatrix} P_\sigma^t(i) E_1 M_a(i) \\ 0 \end{bmatrix}, \quad \Pi_{22}(i) = \begin{bmatrix} \tau P_\sigma^t(i) E_1 M_a(i) & \tau P_d^t(i) E_1 M_a(i) \\ 0 & \tau P_x^t(i) E_1 M_a(i) \end{bmatrix}, \quad (3.3)$$

$$\Pi_{23}(i) = \begin{bmatrix} \tau P_d^t(i) \\ \tau P_x^t(i) \end{bmatrix}, \qquad \Pi_{24}(i) = \begin{bmatrix} P_\sigma^t(i)\Gamma(i) \\ G^t(i)\Phi(i) \end{bmatrix}. \quad (3.4)$$

Proof Let $\mathbf{x}_s(t) \stackrel{\triangle}{=} x(s+t)$, $t-\tau \le s \le t$ and define the process $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ over the state space $\bar{\mathcal{C}}$. It should be observed that $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ is strong Markovian [9] so is the process $\{(\zeta(t), \eta_t),\ t \ge 0\}$. Now for $\eta_t = i \in \mathcal{S}$, and given $Q(i) = Q^t(i) > 0$, let the Lyapunov functional $V(\cdot)\colon \Re^n \times \Re_+ \times \mathcal{S} \to \Re_+$ of the transformed system be selected as

$$V(t,\zeta,i) = \zeta^t(t)\bar{P}(i)\zeta(t) + \int_{t-\tau}^{t} \int_{\theta}^{t} \zeta^t(s) E_2 Q_x(i) E_2^t \zeta(s)\, ds d\theta. \quad (3.5)$$

The weak infinitesimal operator $\Im_1^\zeta[\cdot]$ of the process $\{\zeta(t), i, t \ge 0\}$ for system (2.11)–(2.14) at the point $\{t, x, i\}$ is given by [5, 9]:

$$\Im_1^\zeta[V] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \zeta}\dot{\zeta}(t)\Big|_{\eta_t = i} + \sum_{m=1}^{s} \alpha_{im} V(t,\zeta,i,m). \quad (3.6)$$

Using (2.9)–(2.14) we get:

$$\begin{aligned} \frac{\partial V}{\partial \zeta}\dot{\zeta}(t) &= 2\zeta^t(t) U \mathbb{P}^t(i)\dot{\zeta}(t) = 2\sigma^t(t) P_\sigma^t(i)\dot{\sigma}(t) = 2\zeta^t(t)\mathbb{P}^t(i) \begin{bmatrix} \dot{\sigma}(t) \\ 0 \end{bmatrix} \\ &= 2\zeta^t(t)\mathbb{P}^t(i) \begin{bmatrix} A_{\Delta o}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t) \\ -\sigma(t) + x(t) + \int_{t-\tau}^{t} A_{\Delta d}(t,i)x(s)\, ds \end{bmatrix} \\ &= 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_\Delta(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) \\ &\quad + 2\int_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\, d\theta. \end{aligned} \quad (3.7)$$

Hence, it follows from (3.6) – (3.7) that

$$\begin{aligned} \Im_1^{\zeta}[V] &= \zeta^t(t)\left[\Lambda_\Delta^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_\Delta(i) + \sum_{m=1}^{s}\alpha_{im}\mathbb{P}(m)\right]\zeta(t) \\ &+ 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta + \int_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta \\ &- \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta + \sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,dsd\theta. \end{aligned} \tag{3.8}$$

Since for some $\rho(i) > 0$, $i \in \mathcal{S}$

$$\sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,dsd\theta \le \tau^2\rho(i)\zeta^t(t)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t\zeta(t) \tag{3.9}$$

and by **Fact 1**, we have

$$\begin{aligned} 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta &= 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)x(\theta)\,d\theta \\ &\le \tau\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}^t(i)\zeta(\theta) + \int_{t-\tau}^{t}x^t(s)Q_x(i)x(s)\,ds \\ &= \tau\zeta^t(t)\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}(i)\zeta(t) + \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta. \end{aligned} \tag{3.10}$$

Now, it follows from (3.8) – (3.10) that

$$\begin{aligned} \Im_1^{\zeta}[V] &\le \zeta^t(t)\Bigg[\Lambda_\Delta^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_\Delta(i) + \sum_{m=1}^{s}\alpha_{im}\mathbb{P}(m) \\ &+ \rho(i)\tau^2E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\mathbb{P}^t(i)E_2A_{\Delta d}(t,i)Q_x^{-1}(i)A_{\Delta d}^t(t,i)E_2^t\mathbb{P}(i)\Bigg]\zeta(t) \\ &+ \tau E_2Q_x(i)E_2^t + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t). \end{aligned} \tag{3.11}$$

Application of **Facts 2 – 3** to (3.11) yields:

$$\begin{aligned} \Im_1^{\zeta}[V] &\le \zeta^t(t)\Bigg[\Lambda_o^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_o(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m) \\ &+ \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}_{ad}^t(i)\bar{N}_{ad}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) \\ &+ \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) \\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\Bigg]\zeta(t) \\ &+ 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) = \zeta^t(t)\Pi_1\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) \end{aligned} \tag{3.12}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. By taking $w(t) \equiv 0$, the robust stability of system (2.10) readily follows from (3.12) when $\Pi_1 < 0$. Thus we conclude that $\Im_1^\zeta[V] < 0$ for all $\zeta \neq 0$ and $\Im_1^\zeta[V] \leq 0$ for all ζ. By Dynkin's formula [9], one has $\mathbb{E}\Big[\int_0^\infty \Im_1^\zeta[V]\,dt\Big] = \mathbb{E}[V(t,x,i)|_{t=\infty}] - V(t,\zeta,i)|_{t=0} \geq 0$. With some manipulations using (2.10) and (3.12), we obtain:

$$
\begin{aligned}
\mathcal{J}(x) &= \mathbb{E}\Big\{\int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_1^\zeta[V] - \Im_1^\zeta[V]]dt\Big\} \\
&\leq \mathbb{E}\Big\{\int_0^\infty [z^t(t)z(t) - \gamma^2 w^t(t)w(t) + \Im_1^\zeta[V]]dt\Big\} \\
&\leq \mathbb{E}\Big\{\int_0^\infty \zeta^t(t)\Big[\Lambda_o^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_o(i) + \sum_{m=1}^s \alpha_{im}\bar{P}(m) \\
&\quad + \tau E_2 Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}_{ad}^t(i)\bar{N}_{ad}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) \\
&\quad + \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) \\
&\quad + \tau^2\rho(i)E_2\sum_{m=1}^s \alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i) \\
&\quad + [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)]\Big]\zeta(t)\Big\}.
\end{aligned}
\tag{3.13}
$$

By using (3.1) − (3.4) and **Fact 4**, it follows from inequality (3.13) that $\mathcal{J}(x) < 0$ and hence system (2.11) − (2.12) is *DDRSS with disturbance attenuation* $\gamma > 0$.

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 3.1 *Consider the nominal jump system*

$$
(\Sigma_{Tn}): \quad \dot{\zeta}(t) = \Lambda_o(i)\zeta(t) + \int_{t-\tau}^t \Upsilon(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),
$$

$$
\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \tag{3.14}
$$

$$
z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t). \tag{3.15}
$$

System Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ if given matrix sequence $Q(i) = Q^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $P(i) = P^t(i) > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs

$$
\begin{bmatrix} \Pi_{20}(i) & \Pi_{23}(i) & \Pi_{24}(i) \\ \Pi_{23}^t(i) & -\tau Q_x(i) & 0 \\ \Pi_{24}^t(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) \end{bmatrix} < 0, \qquad \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \quad i \in \mathcal{S}, \tag{3.16}
$$

where

$$
\Pi_{20}(i) = \begin{bmatrix} -P_d(i) - P_d^t(i) + \sum_{m=1}^s \alpha_{im}P_\sigma(m) & -P_x(i) + P_d^t(i) + P_\sigma^t(i)A_{od}(i) \\ & P_x(i) + P_x^t(i) + \tau Q_x(i) \\ -P_x^t(i) + P_d(i) + A_{od}^t(i)P_\sigma(i) & +G^t(i)G(i) + \rho(i)\tau^2 \sum_{m=1}^s \alpha_{im}Q_x(m) \end{bmatrix}.
$$

Remark 3.1 In the foregoing analysis, τ is assumed to be known and constant. If it turns out to be known, the largest value can be computed by solving a generalized eigenvalue problem of the form:

$$\begin{array}{ll} \text{Maximize} & \tau \\ \text{subject to} & \begin{array}{c} P_\sigma(i) > 0,\ P_d(i),\ P_x(i), \\ \varepsilon_1(i) > 0,\ \varepsilon_2(i) > 0,\ \rho(i) > 0,\ \gamma > 0 \quad i \in \mathcal{S}. \end{array} \end{array}$$

This problem can be readily solved using the LMI toolbox.

3.1 Example 1

In order to illustrate Theorem 3.1, we consider a pilot-scale multi-reach water quality system [11] which can fall into the type (2.3)−(2.6). Let the Markov process governing the mode switching has generator

$$\Im = \begin{bmatrix} -4 & 3 & 1 \\ 2 & -6 & 4 \\ 4 & 4 & -8 \end{bmatrix}.$$

For the three operating conditions (modes), the associated date are:

Mode 1:

$$A_o(1) = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \qquad A_d(1) = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad \Gamma(1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

$$G(1) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad \Phi(1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad M_a(1) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix},$$

$$N_a(1) = [0.2 \quad 0.4], \qquad N_d(1) = [0.1 \quad 0.3].$$

Mode 2:

$$A_o(2) = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \qquad A_d(2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \Gamma(2) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G(2) = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad \Phi(2) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, \qquad M_a(2) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},$$

$$N_a(2) = [0.2 \quad 0.2], \qquad N_d(2) = [0.1 \quad 0.2].$$

Mode 3:

$$A_o(3) = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_d(3) = \begin{bmatrix} -0.9 & 0 \\ -1 & -1.1 \end{bmatrix}, \qquad \Gamma(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$G(3) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \qquad \Phi(3) = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad M_a(3) = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$$N_a(3) = [0.3 \quad 0.3], \qquad N_d(3) = [0.2 \quad 0.1].$$

Invoking the software environment [7], we solve inequalities (3.1) subject to (3.2) – (3.4) for $i = 1, 2, 3$. The feasible solutions obtained for

$$\begin{aligned} &\varepsilon_1(1) = 0.7825, \quad &&\varepsilon_2(1) = 1.5634, \quad &&\rho(1) = 3.2312,\\ &\varepsilon_1(2) = 1.2671, \quad &&\varepsilon_2(2) = 3.3451, \quad &&\rho(2) = 2.7645,\\ &\varepsilon_1(3) = 4.2355, \quad &&\varepsilon_2(3) = 0.6673, \quad &&\rho(3) = 4.4436 \end{aligned}$$

show water quality system is DDRSS with a disturbance attenuation level of $\gamma = 1.25$ for any constant time delay $\tau \le 0.6715$.

4 Robust $\mathcal{H}_\infty$ Stabilization

In this section, we consider the control uncertain jumping system with $\eta_t = i \in \mathcal{S}$:

$$\begin{aligned} (\Sigma_{JC}):\quad &\dot{x}(t) = A_{\Delta o}(t,i)x(t) + A_{\Delta d}(t,i)x(t-\tau) + B_{\Delta o}(t,i)u(t) + \Gamma(i)w(t), \quad t \ge 0,\\ &x(t) = \phi(t), \quad t \in [-\tau, 0], \quad \eta_o = i, \qquad (4.1)\\ &z(t) = G(i)x(t) + \Phi(i)w(t), \qquad (4.2) \end{aligned}$$

where $u(t) \in \Re^r$ is the control input and

$$B_{\Delta o}(t,i) = B_o(t,i) + M_a(i)\Delta(t,i)N_b(i) \tag{4.3}$$

with $N_b(i) \in \Re^{\beta \times r}$. We will examine two distinct case of state feedback stabilization: instantaneous feedback and delayed feedback.

4.1 Instantaneous state feedback

In this case we use the control law for $\eta_t = i \in \mathcal{S}$

$$u(t) = K(i)x(t), \qquad i \in \mathcal{S} \tag{4.4}$$

such that the use of (2.8) and (4.4) into (4.1) yields for $\eta_t = i$:

$$\begin{aligned} \dot{\sigma}(t) &= [A_{\Delta k}(t,i) + A_{\Delta d}(t,i)]x(t) + \Gamma(i)w(t),\\ A_{\Delta k}(t,i) &= A_{\Delta o}(t,i) + B_{\Delta o}(t,i)K(i). \end{aligned} \tag{4.5}$$

In this case the transformed system becomes

$$\begin{aligned} (\Sigma_{TK}):\quad &\dot{\zeta}(t) = \Lambda_{\Delta k}(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),\\ &\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, \qquad (4.6)\\ &z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t), \qquad (4.7) \end{aligned}$$

where

$$\Lambda_{\Delta k}(i) = \begin{bmatrix} 0 & A_{\Delta k}(t,i) + A_{\Delta d}(t,i) \\ -I & I \end{bmatrix}. \tag{4.8}$$

Taking into consideration the standard result

$$\mathbb{P}^{-1}(i) = \begin{bmatrix} X_\sigma(i) & 0 \\ X_d(i) & X_x(i) \end{bmatrix}, \qquad X_\sigma(i) = P_\sigma^{-1}(i), \quad X_x(i) = P_x^{-1}(i), \quad X_d(i) = -X_x P_d(i) X_\sigma \tag{4.9}$$

we define the following matrices for $i \in \mathcal{S}$:

$$\begin{aligned} \Lambda_{ok}(i) &= \begin{bmatrix} 0 & A_{od}(i) + B_o(i)K(i) \\ -I & I \end{bmatrix}, \quad \bar{B}_o(i) = \begin{bmatrix} B_o(i) \\ 0 \end{bmatrix}, \quad Z(i) = \begin{bmatrix} 0 \\ X_\sigma(i) \end{bmatrix}, \\ \bar{A}_{od}^t(i) &= [A_{od}^t(i) \quad I], \quad N_{kd}(i) = N_{ad}(i) + N_b(i)K(i), \quad \bar{N}_{kd}(i) = [0 \quad N_{kd}(i)], \\ Y(i) &= [X_d(i) \quad X_x(i)], \quad H(i) = [H_2(i) \quad H_1(i)], \quad N_{dk}(i) = N_d(i) + N_b(i)K_d(i), \\ \Omega(\tau, i) &= G^t(i)G(i) + \tau E_2 Q_x(i) E_2^t + \rho(i)\tau^2 E_2 \sum_{m=1}^{s} \alpha_{im} Q_x(m) E_2^t + \varepsilon_1(i) N_{ad}^t(i) N_{ad}(i). \end{aligned} \tag{4.10}$$

The following theorem establish the main result:

Theorem 4.1 *System Σ_{TK} is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.3) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $H(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_3(i) & \bar{M}(i) & \tau E_1 M_a(i) & \tau E_2 A_d(i) & \begin{matrix} \bar{\Gamma}(i) \\ + Y^t(i)G(i)\Phi(i) \end{matrix} & \mathcal{R}(i) \\ \bar{M}^t(i) & -\varepsilon_1(i) I & 0 & 0 & 0 & 0 \\ M_a^t(i)E_1^t & 0 & -\tau\varepsilon_2(i)I & 0 & 0 & 0 \\ \tau A_d^t(i)E_2^t & 0 & 0 & \begin{matrix} -\tau Q_x(i) \\ + \tau\varepsilon_2(i)N_d(i)N_d^t(i) \end{matrix} & 0 & 0 \\ \begin{matrix} \bar{\Gamma}^t(i) \\ + \Phi^t(i)G^t(i)Y(i) \end{matrix} & 0 & 0 & 0 & \begin{matrix} -\gamma^2 I \\ + \Phi^t(i)\Phi(i) \end{matrix} & 0 \\ \mathcal{R}^t(i) & 0 & 0 & 0 & 0 & -\mathcal{Y}(i) \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q_x(i) & N_d(i) \\ N_d^t(i) & -\varepsilon_2(i) I \end{bmatrix} < 0, \qquad \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{4.11}$$

where

$$\begin{aligned} \Pi_3(i) &= Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1(i)Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)H(i) \\ &\quad + H^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega(\tau, i)Y(i) + \alpha_{ii}E_1 Z^t(i) E_2 \\ &\quad + \varepsilon_1(i)Y^t(i)N_{ad}^t(i)N_b(i)E_1^t L(i) + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_b(i)E_1^t L(i) \\ &\quad + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_{ad}(i)Y(i) \\ \mathcal{Y}(i) &= \mathrm{diag}\big[E_1 Z^t(1)E_2 \ldots E_1 Z^t(i-1)E_2 \quad E_1 Z^t(i+1)E_2 \ldots E_1 Z^t(s)E_2\big], \\ \mathcal{R}(i) &= \big[\sqrt{\alpha_{i1}} E_1 Z^t(1)E_2 \ldots \sqrt{\alpha_{is}} E_1 Z^t(s)E_2\big], \end{aligned} \tag{4.12}$$

and the state-feedback gain is given by $K(i) = H_1(i)[Y(i)E_1]^{-1}$.

Proof Again, let $\mathbf{x}_s(t) \stackrel{\triangle}{=} x(s+t)$, $t-\tau \le s \le t$ and define the process $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ over the state space $\bar{\mathcal{C}}$. It should be observed that $\{(\mathbf{x}(t), \eta_t),\ t \ge 0\}$ is strong

Markovian [9] so is the process $\{(\zeta(t), \eta_t),\ t \geq 0\}$. Now for $\eta_t = i \in \mathcal{S}$, and given $Q(i) = Q^t(i) > 0$, let the Lyapunov functional $V(\cdot)\colon \Re^n \times \Re_+ \times \mathcal{S} \to \Re_+$ as given by (3.5) and hence the weak infinitesimal operator $\Im_2^{\zeta}[\cdot]$ of the process $\{\zeta(t), \eta_t,\ t \geq 0\}$ for system (4.6) – (4.9) at the point $\{t, x, \eta_t\}$ is given by (3.6). It is easy to see that:

$$\frac{\partial V}{\partial \zeta}\,\dot{\zeta}(t) = 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_{\Delta k}(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) + 2\int_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta. \tag{4.13}$$

Hence, it follows from (3.6) and (4.13) that

$$\Im_2^{\zeta}[V] = \zeta^t(t)\Big[\Lambda_{\Delta k}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{\Delta k}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m)\Big]\zeta(t) \tag{4.14}$$
$$+ 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon(i)\zeta(\theta)\,d\theta + \int_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta$$
$$- \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta + \sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2\zeta(s)\,ds d\theta.$$

By making use of (3.9) – (3.10) into (4.14) and applying Facts 2 – 3, we get

$$\begin{aligned}\Im_2^{\zeta}[V] \leq \zeta^t(t)\Big[&\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m) + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i)\\ &+ \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) + \tau E_2Q_x(i)E_2^t\\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\Big]\zeta(t)\\ &+ \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t)\end{aligned} \tag{4.15}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. By similarity to Theorem 3.1 the robust stability of system Σ_{TK} is guaranteed readily follows from (3.12) and Definition 2.1. Thus we conclude that $\Im_2^{\zeta}[V] < 0$ for all $\zeta \neq 0$ and $\Im_2^{\zeta}[V] \leq 0$ for all ζ. Also, by Dynkin's formula [9], one has $\mathbb{E}[\int_0^{\infty}\Im_2^{\zeta}[V]dt] = \mathbb{E}[V(t,x,i)|_{t=\infty}] - V(t,\zeta,i)|_{t=0} \geq 0$. With some manipulations using (4.7) and (4.15), it is readily seen that:

$$\begin{aligned}\mathcal{J}(x) &\leq \mathbb{E}\Big\{\int_0^{\infty}[z^t(t)z(t) - \gamma^2w^t(t)w(t) + \Im_2^{\zeta}[V]]dt\Big\} \qquad (4.16)\\ &\leq \mathbb{E}\Big\{\int_0^{\infty}\zeta^t(t)\Big[\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m)\\ &\quad + \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i)\\ &\quad + \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i)\\ &\quad + \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i)\\ &\quad + [\bar{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)]\Big]\zeta(t)\Big\}.\end{aligned}$$

In line of Theorem 3.1, it follows from inequality (4.16) that $\mathcal{J}(x) < 0$ is guaranteed if the following inequality

$$\begin{aligned}\Lambda_{ok}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{ok}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m) + \tau E_2 Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i)\\ + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)E_1\bar{M}(i)\bar{M}^t(i)E_1^t\mathbb{P}(i)\\ + \tau\mathbb{P}^t(i)E_2A_d(i)[Q_x(i) - \varepsilon_2(i)N_d(i)N_d^t(i)]^{-1}A_d^t(i)E_2^t\mathbb{P}(i)\\ + \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i)\\ + [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)] < 0\end{aligned} \tag{4.17}$$

holds. Premultiplying (4.17) by $\mathbb{P}^{-t}(i)$, postmultiplying by $\mathbb{P}^{-1}(i)$, using (4.9)–(4.10) and manipulating with the help of Fact 3, we obtain the LMI (4.11). It follows that system (4.6)–(4.7) is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.4).

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 4.1 *The nominal jump system Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ under the control law (4.4) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $H(i)$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{gathered}\begin{bmatrix} \Pi_{30}(i) & \tau E_2 A_d(i) & \bar{\Gamma}(i) + Y^t(i)G(i)\Phi(i) & \mathcal{R}(i) \\ \tau A_d^t(i)E_2^t & -\tau Q_x(i) & 0 & 0 \\ \bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) & 0 \\ \mathcal{R}^t(i) & 0 & 0 & -\mathcal{Y}(i)\end{bmatrix} < 0,\\ \begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S},\end{gathered} \tag{4.18}$$

where

$$\begin{aligned}\Pi_{30}(i) &= Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)H(i)\\ &\quad + H^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega_o(\tau,i)Y(i) + \alpha_{ii}E_1Z^t(i)E_2,\\ \Omega_o(\tau,i) &= G^t(i)G(i) + \tau E_2Q_x(i)E_2^t + \rho(i)\tau^2E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t,\end{aligned}$$

and the state-feedback gain is given by $K(i) = H_1(i)[Y(i)E_1]^{-1}$.

4.2 Delayed state feedback

In this case we use the control law for $\eta_t = i \in \mathcal{S}$ as

$$u(t) = K_d(i)x(t-\tau), \qquad i \in \mathcal{S}, \tag{4.19}$$

along with the following state transformation

$$\sigma(t) = x(t) + \int_{t-\tau}^{t} [A_{\Delta d}(t,i) + B_{\Delta o}(t,i)K_d(i)]x(s)\,ds \tag{4.20}$$

such that the use of (4.19) − (4.20) into (4.1) with (2.13) − (2.14) yields for $\eta_t = i \in \mathcal{S}$:

$$\begin{aligned} \dot{\sigma}(t) &= [A_{\Delta o}(t,i) + A_{\Delta kd}(t,i)]x(t) + \Gamma(i)w(t), \\ A_{\Delta kd}(t,i) &= A_{\Delta d}(t,i) + B_{\Delta o}(t,i)K_d(i). \end{aligned} \tag{4.21}$$

Simple algebra yields the transformed system:

$$(\Sigma_{TD}): \quad \dot{\zeta}(t) = \Lambda_{\Delta d}(i)\zeta(t) + \int_{t-\tau}^{t} \Upsilon_k(i)\zeta(s)\,ds + \bar{\Gamma}(i)w(t),$$

$$\zeta(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \tag{4.22}$$

$$z(t) = \bar{G}(i)\zeta(t) + \Phi(i)w(t), \tag{4.23}$$

where

$$\Lambda_{\Delta d}(i) = \begin{bmatrix} 0 & A_{\Delta o}(t,i) + A_{\Delta kd}(t,i) \\ -I & I \end{bmatrix}, \qquad \Upsilon_k(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{\Delta kd}(t,i) \end{bmatrix}. \tag{4.24}$$

Define

$$\begin{gathered} A_{okd}(i) = A_{od}(i) + B_o(i)K_d(i), \qquad A_{kd}(i) = A_d(i) + B_o(i)K_d(i), \\ L(i) = [L_2(i) \quad L_1(i)], \\ \Lambda_{od}(i) = \begin{bmatrix} 0 & A_{okd}(i) \\ -I & I \end{bmatrix}, \qquad N_{dr}(i) = N_d(i) + N_b(i)L(i)NR(i). \end{gathered} \tag{4.25}$$

Taking into account the matrices of (4.9) − (4.10), we establish the following theorem:

Theorem 4.2 *System Σ_{TD} is DDRSS with disturbance attenuation $\gamma > 0$ under the control law (4.19) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $L(i)$, $R(i)$, $i \in \mathcal{S}$ and scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$, $\gamma > 0$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix} \Pi_4(i) & \bar{M}(i) & \tau E_1 M_a(i) & \begin{matrix}\tau E_2 A_d(i) \\ + \tau E_2 B_o(i) \\ \times L(i) E_2 R(i)\end{matrix} & \begin{matrix}\bar{\Gamma}(i) + Y^t(i) \\ \times G(i)\Phi(i)\end{matrix} & \mathcal{R}(i) \\ \bar{M}^t(i) & -\varepsilon_1(i)I & 0 & 0 & 0 & 0 \\ M_a^t(i)E_1^t & 0 & -\tau\varepsilon_2(i)I & 0 & 0 & 0 \\ \begin{matrix}\tau A_d^t(i)E_2^t \\ + \tau R^t(i)E_2^t L^t(i) B_o^t(i) E_2^t\end{matrix} & 0 & 0 & \begin{matrix}-\tau Q_x(i) + \tau\varepsilon_2(i) \\ \times N_{dr}(i)N_{dr}^t(i)\end{matrix} & 0 & 0 \\ \bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & 0 & 0 & \begin{matrix}-\gamma^2 I \\ + \Phi^t(i)\Phi(i)\end{matrix} & 0 \\ \mathcal{R}^t(i) & 0 & 0 & 0 & 0 & -\mathcal{Y}(i) \end{bmatrix} < 0,$$

$$\begin{bmatrix} -Q_x(i) & N_{dr}(i) \\ N_{dr}^t(i) & -\varepsilon_2(i)I \end{bmatrix} < 0, \qquad \begin{bmatrix} -Y(i)E_1 & I \\ I & -R(i) \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad i \in \mathcal{S}, \tag{4.26}$$

where

$$\begin{aligned}\Pi_4(i) &= Y^t(i)\bar{A}^t_{od}(i) + \bar{A}_{od}(i)Y(i) - E_1 Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)L(i)\\ &\quad + L^t(i)\bar{B}^t_o(i) + Y^t(i)\Omega(\tau,i)Y(i) + \alpha_{ii}E_1 Z^t(i)E_2\\ &\quad + \varepsilon_1(i)Y^t(i)N^t_{ad}(i)N_b(i)E_1^t L(i) + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_b(i)E_1^t L(i)\\ &\quad + \varepsilon_1(i)L^t(i)E_1 N_b^t(i)N_{ad}(i)Y(i)\end{aligned} \tag{4.27}$$

and the delayed-feedback gain is given by $K_d(i) = L(i)E_1 R(i)$.

Proof By similarity to Theorem 3.1 and letting the Lyapunov functional $V(\cdot)$ be given by (3.5), the weak infinitesimal operator $\Im_3^\zeta[\cdot]$ of the process $\{\zeta(t), \eta_t,\ t \geq 0\}$ for system (4.22)–(4.23) at the point $\{t, x, \eta_t\}$ is given by (3.6). Hence, it is easy to see that:

$$\begin{aligned}\frac{\partial V}{\partial \zeta}\,\dot{\zeta}(t) &= 2\zeta^t(t)\mathbb{P}^t(i)\Lambda_{\Delta d}(i)\zeta(t) + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t)\\ &\quad + 2\int_{t-\tau}^{t} \zeta^t(t)\mathbb{P}^t(i)\Upsilon_k(i)\zeta(\theta)\,d\theta.\end{aligned} \tag{4.28}$$

Hence, it follows from (3.6) and (4.27) that

$$\begin{aligned}\Im_3^\zeta[V] &= \zeta^t(t)\Big[\Lambda^t_{\Delta d}(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{\Delta d}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m)\Big]\zeta(t)\\ &\quad + 2\zeta^t(t)\mathbb{P}^t(i)\bar{\Gamma}(i)w(t) + 2\int_{t-\tau}^{t}\zeta^t(t)\mathbb{P}^t(i)\Upsilon_k(i)\zeta(\theta)\,d\theta\\ &\quad + \int_{t-\tau}^{t}\zeta^t(t)E_2Q_x(i)E_2^t\zeta(t)\,d\theta - \int_{t-\tau}^{t}\zeta^t(\theta)E_2Q_x(i)E_2^t\zeta(\theta)\,d\theta\\ &\quad + \sum_{m=1}^{s}\alpha_{im}\int_{t-\tau}^{t}\int_{\theta}^{t}\zeta^t(s)E_2Q_x(m)E_2^t\zeta(s)\,ds d\theta.\end{aligned} \tag{4.29}$$

Following parallel developments to Theorem 4.1, we applying Facts 2 – 3, use (3.9), (4.7), (4.10) and (4.24) – (4.25) and manipulate, we get

$$\begin{aligned}\mathcal{J}(x) \leq \mathbb{E}\Big\{\int_0^\infty \zeta^t(t)\Big[&\Lambda^t_{od}(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{od}(i) + \sum_{m=1}^{s}\alpha_{im}\bar{P}(m)\\ &+ \tau E_2Q_x(i)E_2^t + \varepsilon_1(i)\bar{N}^t_{kd}(i)\bar{N}_{kd}(i) + \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i)\\ &+ \tau\mathbb{P}^t(i)E_2A_{kd}(i)[Q_x(i) - \varepsilon_2(i)N_{dk}(i)N^t_{dk}(i)]^{-1}A^t_{kd}(i)E_2^t\mathbb{P}(i)\\ &+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i)\\ &+ [\bar{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)]\Big]\zeta(t)\Big\}\end{aligned} \tag{4.30}$$

for some scalars $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i) > 0$. It follows from inequality (4.30) that $\mathcal{J}(x) < 0$ is guaranteed if the following inequality

$$\begin{aligned}
\Lambda_{od}^t(i)\mathbb{P}(i) + \mathbb{P}^t(i)\Lambda_{od}(i) + \sum_{m=1}^{s} \alpha_{im}\bar{P}(m) + \tau E_2 Q_x(i) E_2^t + \varepsilon_1(i)\bar{N}_{kd}^t(i)\bar{N}_{kd}(i) \\
+ \varepsilon_1^{-1}(i)\mathbb{P}^t(i)\bar{M}(i)\bar{M}^t(i)\mathbb{P}(i) \\
+ \tau\mathbb{P}^t(i)E_2 A_{kd}(i)[Q_x(i) - \varepsilon_2(i)N_{dk}(i)N_{dk}^t(i)]^{-1}A_{kd}^t(i)E_2^t\mathbb{P}(i) \\
+ \tau^2\rho(i)E_2\sum_{m=1}^{s}\alpha_{im}Q_x(m)E_2^t + \tau\varepsilon_2^{-1}(i)\mathbb{P}^t(i)E_1 M_a(i)M_a^t(i)E_1^t\mathbb{P}(i) + \bar{G}^t(i)\bar{G}(i) \\
+ [\mathbb{P}^t(i)\bar{\Gamma}(i) + \bar{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t(i)\mathbb{P}(i) + \Phi^t(i)\bar{G}(i)] < 0
\end{aligned} \tag{4.31}$$

holds. Premultiplying (4.17) by $\mathbb{P}^{-t}(i)$, postmultiplying by $\mathbb{P}^{-1}(i)$, using (4.27) and manipulating with the help of Fact 3, we obtain the LMI (4.26). It follows that system (4.22)–(4.23) is DDRSS with disturbance attenuation $\gamma > 0$ under the state-delayed control law (4.19).

The following corollary can be readily derived as special case of Theorem 3.1:

Corollary 4.2 *The nominal jump system Σ_{Tn} is delay dependent stochastically stable (DDSS) with disturbance attenuation $\gamma > 0$ under the control law (4.19) if given matrix sequence $Q_x(i) = Q_x^t(i) > 0$, $i \in \mathcal{S}$, there exist matrices $Y(i)$, $Z(i)$, $L(i)$, $R(i)$, $i \in \mathcal{S}$, satisfying the system of LMIs*

$$\begin{bmatrix}
\Pi_{40}(i) & \tau E_2[A_d(i) + B_o(i)L(i)E_2R(i)] & \bar{\Gamma}(i) + Y^t(i)G(i)\Phi(i) & \mathcal{R}(i) \\
\tau A_d^t(i)E_2^t & -\tau Q_x(i) & 0 & 0 \\
\bar{\Gamma}^t(i) + \Phi^t(i)G^t(i)Y(i) & 0 & -\gamma^2 I + \Phi^t(i)\Phi(i) & 0 \\
\mathcal{R}^t(i) & 0 & 0 & -\mathcal{Y}(i)
\end{bmatrix} < 0,$$

$$\begin{bmatrix} -\gamma^2 I & \Phi^t(i) \\ \Phi(i) & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} -Y(i)E_1 & I \\ I & -R(i) \end{bmatrix} \geq 0, \qquad i \in \mathcal{S}, \tag{4.32}$$

where

$$\begin{aligned}
\Pi_{40}(i) = Y^t(i)\bar{A}_{od}^t(i) + \bar{A}_{od}(i)Y(i) - E_1 Z^t(i) - Z(i)E_1^t + \bar{B}_o(i)L(i) \\
+ L^t(i)\bar{B}_o^t(i) + Y^t(i)\Omega_o(\tau, i)Y(i) + \alpha_{ii}E_1 Z^t(i)E_2
\end{aligned} \tag{4.33}$$

and the delayed-feedback gain is given by $K_d(i) = L(i)NR(i)$.

4.2 Example 2

We use the data of Example 1 in addition to

$$B_o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_o(2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B_o(3) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$N_b(1) = [0.1 \quad 0.3], \qquad N_b(2) = [0.2 \quad 0.2], \qquad N_b(3) = [0.3 \quad 0.1]$$

and the level of disturbance attenuation $\gamma = 1.35$. For the data under consideration and in view of Theorem 4.1, the feasible solutions of LMIs (4.11) using the software LMILab [7] yields the gain matrices

$$K(1) = \begin{bmatrix} 0.8532 & 0.9260 \\ -1.4317 & -1.2628 \end{bmatrix}, \qquad K(2) = \begin{bmatrix} 0.9145 & -0.6128 \\ 0.5844 & 1.9912 \end{bmatrix},$$
$$K(3) = \begin{bmatrix} 1.1425 & 0.6603 \\ -0.3123 & 0.4912 \end{bmatrix}$$

for

$$\begin{aligned} &\varepsilon_1(1) = 1.3345, && \varepsilon_2(1) = 0.9144, && \rho(1) = 2.4367, \\ &\varepsilon_1(2) = 2.3567, && \varepsilon_2(2) = 2.5433, && \rho(2) = 1.5321, \\ &\varepsilon_1(3) = 5.2355, && \varepsilon_2(3) = 0.6673, && \rho(3) = 2.3226, \end{aligned}$$

and $\tau \le 0.4772$.

On the other hand, considering Theorem 4.2 we solve the LMIs (4.26) to get the gain matrices

$$K_d(1) = \begin{bmatrix} 0.0454 & -0.9231 \\ 0.0422 & 0.9123 \end{bmatrix}, \qquad K_d(2) = \begin{bmatrix} -0.1636 & 0.2628 \\ -0.5628 & 1.2182 \end{bmatrix},$$
$$K_d(3) = \begin{bmatrix} 0.3144 & 1.1268 \\ -0.7435 & -0.8655 \end{bmatrix}$$

for

$$\begin{aligned} &\varepsilon_1(1) = 3.4225, && \varepsilon_2(1) = 0.7428, && \rho(1) = 1.3452, \\ &\varepsilon_1(2) = 1.7111, && \varepsilon_2(2) = 1.6655, && \rho(2) = 3.0987, \\ &\varepsilon_1(3) = 4.0205, && \varepsilon_2(3) = 0.0876, && \rho(3) = 4.2247 \end{aligned}$$

and $\tau \le 0.4653$.

5 $\mathcal{H}_\infty$-Output Feedback Controller

In this section, we consider the design of an $\mathcal{H}_\infty$-output feedback controller for the jumping system for $\eta = i \in \mathcal{S}$

$$\begin{aligned} \dot{x}(t) &= A_{\Delta o}(t,i)x(t) + A_{\Delta d}(t,i)x(t-\tau) + B_{\Delta o}(t,i)u(t) + \Gamma(i)w(t), \\ x(t) &= \phi(t), \quad t \in [-\tau, 0], \quad t \ge 0, \end{aligned} \tag{5.1}$$
$$y(t) = C_o(i)x(t) + D_o(i)w(t), \tag{5.2}$$
$$z(t) = G(i)x(t) + \Phi(i)w(t), \tag{5.3}$$

where $y(t) \in \Re^p$ is the measured output and the matrices $C_o(i)$, $D_o(i)$ are constant with appropriate dimensions. Note that system (5.1)–(5.3) is more general (2.3)–(2.4) for control design purposes. A dynamic output feedback controller for $i \in \mathcal{S}$, has the form:

$$\begin{aligned} \dot{x}_C(t) &= A_C(i)x_C(t) + B_C(i)[y(t) - C_o(i)x_C(t)], \\ u(t) &= C_C(i)x_C(t), \end{aligned} \tag{5.4}$$

where $x_C(t) \in \Re^n$ is the state of the controller and the matrices $A_C(i) \in \Re^{n\times n}$, $B_C(i) \in \Re^{n\times p}$, $C_C(i) \in \Re^{m\times n}$ are controller matrices to be determined. Combining (5.1) – (5.4) for $i \in \mathcal{S}$, we obtain the closed-loop system

$$\begin{aligned} \dot{\xi}(t) &= A_{JC\Delta}(t,i)\xi(t) + A_{JCd\Delta}(t,i)\xi(t-\tau(t)) + \Gamma_{JC\Delta}(t,i)w(t), \quad t \geq 0, \\ \xi(t) &= \phi_{JC}(t), \quad t \in [-\tau^*, 0], \\ z(t) &= \bar{G}(i)\xi(t) + \Phi(i)w(t), \end{aligned} \tag{5.5}$$

where

$$\xi(t) = \begin{bmatrix} x(t) \\ x_C(t) \end{bmatrix} \in \Re^{2n},$$

$$A_{JCd\Delta}(t,i) = \bar{A}_d(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCd}(i), \tag{5.6}$$

$$A_{JC\Delta}(t,i) = \begin{bmatrix} A_{\Delta o}(i) & B_{\Delta o}(i)C_C(i) \\ B_C(i)C_o(i) & A_C(i) - B_C(i)C_o(i) \end{bmatrix} = A_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCa}(i),$$

$$\Gamma_{JC\Delta}(t,i) = \begin{bmatrix} \Gamma(i) \\ B_C D_o(i) \end{bmatrix} = \Gamma_{JCo}(i) + \bar{M}_a(i)\Delta_a\bar{N}_d(i)$$

and

$$\begin{aligned} A_{JCo}(i) &= \begin{bmatrix} A_o(i) & B_o(i)C_C(i) \\ B_C(i)C_o(i) & A_C(i) - B_C(i)C_o(i) \end{bmatrix}, \\ \bar{M}_{JC}(i) &= \begin{bmatrix} 0 \\ \bar{M}_a \end{bmatrix}, \quad \bar{N}_{JCd} = [0 \quad \hat{N}_d], \\ \bar{M}_a(i) &= \begin{bmatrix} M_a(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{N}_a(i) = \begin{bmatrix} N_a(i) & 0 \\ 0 & 0 \end{bmatrix}, \\ \hat{N}_d(i) &= \begin{bmatrix} N_d(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_d(i) = \begin{bmatrix} A_d(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_{JCo}(i) = \begin{bmatrix} \Gamma(i) \\ B_C D_o(i) \end{bmatrix}. \end{aligned} \tag{5.7}$$

Now for each possible value $\eta_t = i$, $i \in \mathcal{S}$, we introduce the following state transformation

$$\mu(t) = \xi(t) + \int_{t-\tau}^{t} A_{JCd\Delta}(t,i)\xi(s)\,ds \tag{5.8}$$

into (5.5) to yield

$$\dot{\mu}(t) = [A_{JC\Delta}(t,i) + A_{JCd\Delta}(t,i)]\xi(t) + \bar{\Gamma}_{JCo}(i)w(t). \tag{5.9}$$

Define the augmented state-vector

$$\omega(t) = \begin{bmatrix} \mu(t) \\ \xi(t) \end{bmatrix} \in \Re^{4n}. \tag{5.10}$$

By combining (5.1) and (5.8) – (5.10), we obtain the transformed system

$$\dot{\omega}(t) = \Lambda_{JC\Delta}(i)\omega(t) + \int_{t-\tau}^{t} \Upsilon_{JC\Delta}(i)\omega(s)\,ds + \Gamma_{JCo}(i)w(t),$$

$$\omega(t) = \bar{\phi}(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \geq 0, \tag{5.11}$$

$$z(t) = \hat{G}(i)\omega(t) + \Phi(i)w(t), \tag{5.12}$$

where

$$\Lambda_{JC\Delta}(i) = \begin{bmatrix} 0 & A_{JC\Delta}(t,i) + A_{JCd\Delta}(t,i) \\ -I & I \end{bmatrix} = \Lambda_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCe}(i),$$

$$\Upsilon_{JC\Delta}(i) = \begin{bmatrix} 0 & 0 \\ 0 & A_{JCd\Delta}(t,i) \end{bmatrix} = \Upsilon_{JCo}(i) + \bar{M}_{JC}(i)\Delta(t,i)\bar{N}_{JCd}, \tag{5.13}$$

$$\Lambda_{JCo}(i) = \begin{bmatrix} 0 & A_{JCo}(i) + \bar{A}_d(i) \\ -I & I \end{bmatrix}, \quad \Upsilon_{JCo}(i) = \begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_d(i) \end{bmatrix},$$

$$\bar{\Gamma}_{JCo}(i) = \begin{bmatrix} \Gamma_{JCo}(i) \\ 0 \end{bmatrix}, \quad \bar{N}_{JCe} = [\hat{N}_d(i) + \hat{N}_a(i) \quad 0], \quad \hat{G}(i) = [0 \quad \bar{G}(i)].$$

Given matrices

$$0 < \mathcal{P}_\mu(i) \in \Re^{2n}, \quad \mathcal{P}_d(i) \in \Re^{2n}, \quad \mathcal{P}_\xi(i) \in \Re^{2n}, \quad i \in \mathcal{S},$$
$$\mathcal{P}(i) = \begin{bmatrix} \mathcal{P}_\mu(i) & 0 \\ \mathcal{P}_d(i) & \mathcal{P}_\xi(i) \end{bmatrix} \in \Re^{4n} \tag{5.14}$$

such that for $i \in \mathcal{S}$

$$\mathcal{P}^{-1}(i) = \begin{bmatrix} \mathcal{X}_\mu(i) & 0 \\ \mathcal{X}_d(i) & \mathcal{X}_\xi(i) \end{bmatrix}, \quad \mathcal{X}_\mu(i) = \begin{bmatrix} \mathcal{X}_{\mu 1}(i) & 0 \\ 0 & \mathcal{X}_{\mu 2} \end{bmatrix},$$

$$\mathcal{X}_\xi(i) = \begin{bmatrix} \mathcal{X}_{\xi 1}(i) & 0 \\ 0 & \mathcal{X}_{\xi 2} \end{bmatrix}, \quad \mathcal{X}_d(i) = \begin{bmatrix} \mathcal{X}_{d1}(i) & 0 \\ 0 & \mathcal{X}_{d2} \end{bmatrix}, \tag{5.15}$$

$$\mathcal{X}_\mu(i) = \mathcal{P}_\mu^{-1}(i), \quad \mathcal{X}_d(i) = -\mathcal{X}_\mu(i)\mathcal{P}_d(i)\mathcal{X}_\xi(i), \quad \mathcal{X}_\xi(i) = \mathcal{P}_\xi^{-1}(i)$$

and define the matrices:

$$\Sigma(i) = [\mathcal{X}_\mu(i) \quad \mathcal{X}_\xi(i)], \quad \bar{A}^t{}_{JCod}(i) = [A^t_{JCo}(i) + \bar{A}^t_d(i) \quad I], \quad \Xi(i) = \begin{bmatrix} 0 \\ \mathcal{X}_\mu(i) \end{bmatrix},$$

$$\Theta(\tau,i) = \tau E_2 I\!\!R(i) E_2^t + \varepsilon_1(i)\bar{N}^t_{JCd}(i)\bar{N}_{JCd}(i) \tag{5.16}$$
$$+ \bar{G}^t(i)\bar{G}(i) + \tau^2\rho(i)E_2 \sum_{m=1}^{s} \alpha_{im} I\!\!R(m) E_2^t.$$

It follows from Theorem 3.1 that given matrix sequence $0 < I\!\!R(i) = I\!\!R^t(i)$, $i \in \mathcal{S}$ the transformed system (5.11)–(5.12) is DDRSS with disturbance attenuation $\gamma > 0$ if the algebraic inequality:

$$\Sigma^t(i)\Lambda^t_{JCod}(i) + \Lambda_{JCod}(i)\Sigma(i) - E_1\Xi^t(i) - \Xi(i)E_1^t$$
$$+ E_1\Xi^t(i)E_2\left(\sum_{m=1}^{s} \alpha_{im}[E_2^t\Xi(m)]^{-1}\right)E_2^t\Xi(i)E_1^t + \varepsilon_1^{-1}(i)\bar{M}_a(i)E_1\bar{M}_a^t(i)E_1^t$$
$$+ \tau\varepsilon_2^{-1}(i)\bar{M}_{JC}(i)\bar{M}^t_{JC}(i) + \Sigma^t(i)\Theta(\tau,i)\Sigma(i)(i)$$
$$+ \tau E_2\bar{A}_d(i)[I\!\!R(i) - \varepsilon_2(i)\bar{N}_{JCd}(i)\bar{N}^t_{JCd}(i)]^{-1}\bar{A}^t_d(i)E_2^t \tag{5.17}$$
$$+ [\bar{\Gamma}_{JCo}(i) + \mathcal{X}^t(i)\hat{G}^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[\bar{\Gamma}^t_{JCo}(i) + \Phi^t(i)\hat{G}(i)\mathcal{X}]$$
$$\triangleq I\!\!M(\tau,i) = \begin{bmatrix} I\!\!M_\mu(\tau,i) & I\!\!M_c(\tau,i) \\ I\!\!M_c^t(\tau,i) & I\!\!M_\xi(\tau,i) \end{bmatrix} < 0$$

is satisfied for some positive scalars $\varepsilon_1(i)$, $\varepsilon_2(i)$, $\rho(i)$, $i \in \mathcal{S}$, where

$$\begin{aligned}
I\!\!M_\mu(\tau,i) &= \begin{bmatrix} I\!\!M_{\mu1}(\tau,i) & I\!\!M_{\mu3}(\tau,i) \\ I\!\!M^t_{\mu3}(\tau,i) & I\!\!M_{\mu2}(\tau,i) \end{bmatrix}, \qquad (5.18)\\
I\!\!M_c(\tau,i) &= \begin{bmatrix} I\!\!M_{c1}(\tau,i) & I\!\!M_{c3}(\tau,i) \\ I\!\!M_{c4}(\tau,i) & I\!\!M_{c2}(\tau,i) \end{bmatrix},\\
I\!\!M_\xi(\tau,i) &= \begin{bmatrix} I\!\!M_{\xi1}(\tau,i) & 0 \\ 0 & I\!\!M_{\xi2}(\tau,i) \end{bmatrix},\\
\Omega_\mu(\tau,i) &= \tau I\!\!R(i) + \epsilon_1(i)[N_a(i)+N_d(i)][N^t_a(i)+N^t_d(i)] + G^t(i)G(i)\\
&\quad + \tau^2\rho(i)\sum_m \alpha_{im} I\!\!R(m),\\
I\!\!M_{\mu1}(\tau,i) &= [A_o(i)+A_d(i)]\mathcal{X}_{d1}(i) + \mathcal{X}^t_{d1}(i)[A^t_o(i)+A^t_d(i)]\\
&\quad + \mathcal{X}^t_{\mu1}(i)\sum_m \mathcal{X}^{-1}_{\mu1}(m)\mathcal{X}_{\mu1}(i) + \epsilon^{-1}M_a(i)M^t_a(i) + \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Gamma^t(i),\\
I\!\!M_{\mu3}(\tau,i) &= B_o(i)C_C(i)\mathcal{X}_{d2}(i) + \mathcal{X}^t_{d1}(i)C^t_o(i)B^t_C(i)\\
&\quad + \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}[D^t_o(i)B^t_C(i) + \Phi^t(i)G(i)\mathcal{X}_{d2}(i)],\\
I\!\!M_{\mu2}(\tau,i) &= [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) + \mathcal{X}^t_{d2}[A^t_C(i) - C^t_o(i)B^t_C(i)]\\
&\quad + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\sum_m \alpha_{im}\mathcal{X}^{-1}_{\mu2}(m)\mathcal{X}_{\mu2}(i),\\
I\!\!M_{c1}(\tau,i) &= -\mathcal{X}^t_{\mu1}(i) + \mathcal{X}^t_{d1}(i) + [A_o(i)+A_d(o)]\mathcal{X}^t_{\mu1}(i),\\
I\!\!M_{c2}(\tau,i) &= -\mathcal{X}^t_{\mu2}(i) + \mathcal{X}^t_{d2}(i) + [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{\mu2}(i)\\
&\quad + [B_C(i)D_o(i) + \mathcal{X}^t_{d2}(i)G^t(i)\Phi(i)][\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}_{\mu2}(i)\\
&\quad + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\sum_m \alpha_{im}\mathcal{X}^{-1}_{\mu1}(m)\mathcal{X}_{\xi2}(i),\\
I\!\!M_{c4}(\tau,i) &= B_C(i)C_o(i)\mathcal{X}^t_{\mu1}(i),\\
I\!\!M_{c3}(\tau,i) &= \Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}^t_{\mu2}(i) + B_o(i)C_C(i)\mathcal{X}^t_{\mu2}(i),\\
I\!\!M_{\xi1}(\tau,i) &= \mathcal{X}_{\mu1}(i) + \mathcal{X}^t_{\mu1}(i) + \tau\epsilon_2^{-1}M_a(i)M^t_a(i) + \tau A_d(i)[I\!\!R(i) - \epsilon_2 N_{od}N^t_{od}]^{-1}A^t_d(i),\\
I\!\!M_{\xi2}(\tau,i) &= \mathcal{X}_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i) + \mathcal{X}^t_{\mu2}(i)\Omega_\mu(\tau,i)\mathcal{X}_{\mu2}(i)\\
&\quad + \mathcal{X}^t_{\mu2}(i)G^t(i)\Phi(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]^{-1}\Phi^t(i)G(i)\mathcal{X}_{\mu2}(i).
\end{aligned}$$

Our objective is to develop conditions that can be used for computing the gains of the dynamic output feedback controller. The following theorem summarizes the main solvability conditions for controller (5.4) guaranteeing that the closed-loop system (5.11) – (5.12) is delay-dependent robustly stochastically stable with disturbance attenuation γ.

Theorem 5.1 *Consider the closed-loop system (5.11) – (5.12) with matrices described in (5.6) – (5.7) and (5.13) – (5.16). Given scalars $\gamma > 0$, $\varepsilon_1(i) > 0$, $\varepsilon_2(i) > 0$, $\rho(i)$, $i \in \mathcal{S}$, there exists a dynamic output feedback controller of the type (5.4) such that the closed-loop system (5.11) – (5.12) is DDRSS with a disturbance attenuation γ if there exist matrices $\mathcal{X}_{\mu1}(i)$, $\mathcal{X}_{\mu2}(i)$, $\mathcal{X}_{\xi1}(i)$, $\mathcal{X}_{\xi2}(i)$, $\mathcal{X}_{d1}(i)$, $\mathcal{X}_{d2}(i)$, $i \in \mathcal{S}$ satisfying the following system of simultaneous matrix inequalities and equations*

$$\begin{bmatrix} \mathcal{X}_{\xi 1}(i) + \mathcal{X}_{\xi 1}^t(i) & \tau M_a(i) & \tau A_d(i) \\ \tau M_a^t(i) & -\epsilon_2 I & 0 \\ \tau A_d^t(i) & 0 & -[I\!R - \epsilon_2 N_{od} N_{od}^t] \end{bmatrix} < 0, \tag{5.19}$$

$$\begin{bmatrix} \begin{matrix}[A_o(i) + A_d(i)]\mathcal{X}_{d1}(i) \\ + \mathcal{X}_{d1}^t(i)[A_o(i) + A_d(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 1}^t(i)\end{matrix} & M_a(i) & \Gamma(i) & \mathcal{W}_1(i) \\ M_a^t(i) & -\epsilon_1 I & 0 & \\ \Gamma^t(i) & 0 & -[\gamma^2 I - \Phi^t(i)\Phi(i)] & 0 \\ \mathcal{W}_1^t(i) & 0 & 0 & -\mathcal{V}_1(i) \end{bmatrix} < 0, \tag{5.20}$$

$$\begin{bmatrix} \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\mu 2}^t(i) + \mathcal{X}_{\mu 2}^t(i)\Omega_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) & \mathcal{X}_{\mu 2}^t(i) G^t(i)\Phi(i) \\ \Phi^t(i) G(i)\mathcal{X}_{\mu 2}(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] \end{bmatrix} < 0, \tag{5.21}$$

$$\begin{bmatrix} \begin{matrix}[A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) \\ + \mathcal{X}_{d2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 2}^t(i)\end{matrix} & \mathcal{X}_{\mu 2}^t(i) & \mathcal{W}_2(i) \\ \mathcal{X}_{\mu 2}(i) & -\Omega_\mu(\tau, i) & 0 \\ \mathcal{W}_2^t(i) & 0 & -\mathcal{V}_2(i) \end{bmatrix} < 0, \tag{5.22}$$

$$\mathcal{X}_{d1}(i) - \mathcal{X}_{\mu 1}(i) + \mathcal{X}_{\xi 1}^t(i)[A_o(i) + A_d(i)]^t = 0, \tag{5.23}$$

$$\mathcal{X}_{d2}(i) - \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\xi 2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \mathcal{X}_{\xi 2}(i)\Omega_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) = 0. \tag{5.24}$$

Then the associated controller matrices are given by:

$$\begin{aligned} A_C(i) &= A_o(i), \\ B_C(i) &= -\mathcal{X}_{d2}^t(i) G^t(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]\Phi(i) D_o^\dagger(i), \\ C_C(i) &= B_o^\dagger(i)\Gamma(i)[\gamma^2 I - \Phi^t(i)\Phi(i)]\Phi(i) G(i), \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} \mathcal{V}_1(i) &= \operatorname{diag}\big[\mathcal{X}_{\mu 1}^t(1)\dots\mathcal{X}_{\mu 1}^t(i-1)\ \mathcal{X}_{\mu 1}^t(i+1)\dots\mathcal{X}_{\mu 1}^t(s)\big], \\ \mathcal{V}_2(i) &= \operatorname{diag}\big[\mathcal{X}_{\mu 2}^t(1)\dots\mathcal{X}_{\mu 2}^t(i-1)\ \mathcal{X}_{\mu 2}^t(i+1)\dots\mathcal{X}_{\mu 2}^t(s)\big], \\ \mathcal{W}_1(i) &= \big[\sqrt{\alpha_{i1}}\mathcal{X}_{\mu 1}^t(1)\dots\sqrt{\alpha_{is}}\mathcal{X}_{\mu 1}^t(s)\big], \\ \mathcal{W}_2(i) &= \big[\sqrt{\alpha_{i1}}\mathcal{X}_{\mu 1}^t(1)\dots\sqrt{\alpha_{is}}\mathcal{X}_{\mu 1}^t(s)\big] \end{aligned}$$

and $B_o^\dagger(i)$ and $D_o^\dagger(i)$ are the pseudo-inverse of $D_o(i)$ and $B_o(i)$, respectively.

Proof We start from matrix inequality (5.17) and using (5.18) with standard algebraic manipulations, it follows that the choice of the controller matrices (5.25) subject to inequalities (5.19)–(5.24) ensures that $I\!M(\tau, i) < 0$, $i \in \mathcal{S}$ and hence guarantees that system (5.11)–(5.12) is DDRSS with a disturbance attenuation γ and the proof is completed.

In the absence of uncertainties, the closed-loop system (5.11)–(5.12) reduces to

$$\begin{aligned} \dot\omega(t) &= \Lambda_{JCo}(i)\omega(t) + \int_{t-\tau}^{t} \Upsilon_{JCo}(i)\omega(s)\,ds + \Gamma_{JCo}(i)w(t), \\ \omega(t) &= \bar\phi(t), \quad t \in [-2\tau, 0], \quad \eta_o = i, \quad t \ge 0, \end{aligned} \tag{5.26}$$

$$z(t) = \hat G(i)\omega(t) + \Phi(i)w(t) \tag{5.27}$$

and for which the following corollary holds:

Corollary 5.1 *Consider the closed-loop system (5.26) – (5.27) with matrices described in (5.6) – (5.7) and (5.13) – (5.16). Given scalars $\rho(i) > 0$, $i \in calS$, $\gamma > 0$, there exists a dynamic output feedback controller of the type (5.4) such that the closed-loop system (5.26) – (5.27) is DDRSS with a disturbance attenuation γ if there exist matrices $\mathcal{X}_{\mu 1}(i)$, $\mathcal{X}_{\mu 2}(i)$, $\mathcal{X}_{\xi 1}(i)$, $\mathcal{X}_{\xi 2}(i)$, $\mathcal{X}_{d1}(i)$, $\mathcal{X}_{d2}(i)$, $i \in \mathcal{S}$ satisfying the following system of simultaneous matrix inequalities and equations*

$$\begin{bmatrix} \mathcal{X}_{\xi 1}(i) + \mathcal{X}_{\xi 1}^t(i)(i) & \tau A_d(i) \\ \tau A_d^t(i) & -I\!R \end{bmatrix} < 0, \tag{5.28}$$

$$\begin{bmatrix} \begin{matrix} [A_o(i) + A_d(i)]\mathcal{X}_{d1}(i) \\ + \mathcal{X}_{d1}^t(i)[A_o(i) + A_d(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 1}^t(i) \end{matrix} & \Gamma(i) & \mathcal{W}_1(i) \\ \Gamma^t(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] & 0 \\ \mathcal{W}_1^t(i) & 0 & -\mathcal{V}_1(i) \end{bmatrix} < 0, \tag{5.29}$$

$$\begin{bmatrix} \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\mu 2}^t(i) + \mathcal{X}_{\mu 2}^t(i)\bar{\Omega}_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) & \mathcal{X}_{\mu 2}^t(i)G^t(i)\Phi(i) \\ \Phi^t(i)G(i)\mathcal{X}_{\mu 2}(i) & -[\gamma^2 I - \Phi^t(i)\Phi(i)] \end{bmatrix} < 0, \tag{5.30}$$

$$\begin{bmatrix} \begin{matrix} [A_C(i) - B_C(i)C_o(i)]\mathcal{X}_{d2}(i) \\ + \mathcal{X}_{d2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \alpha_{ii}\mathcal{X}_{\mu 2}^t(i) \end{matrix} & \mathcal{X}_{\mu 2}^t(i) & \mathcal{W}_2(i) \\ \mathcal{X}_{\mu 2}(i) & -\bar{\Omega}_\mu(\tau, i) & 0 \\ \mathcal{W}_2^t(i) & 0 & -\mathcal{V}_2(i) \end{bmatrix} < 0, \tag{5.31}$$

$$\mathcal{X}_{d1}(i) - \mathcal{X}_{\mu 1}(i) + \mathcal{X}_{\xi 1}^t(i)[A_o(i) + A_d(i)]^t = 0, \tag{5.32}$$

$$\mathcal{X}_{d2}(i) - \mathcal{X}_{\mu 2}(i) + \mathcal{X}_{\xi 2}^t(i)[A_C(i) - B_C(i)C_o(i)]^t + \mathcal{X}_{\xi 2}(i)\bar{\Omega}_\mu(\tau, i)\mathcal{X}_{\mu 2}(i) = 0, \tag{5.33}$$

where

$$\bar{\Omega}_\mu(\tau, i) = \tau I\!R(i) + G^t(i)G(i) + \tau^2\rho(i)\sum_m \alpha_{im} I\!R(m) \tag{5.34}$$

and the associated controller matrices are given by (5.25).

5.1 Example 3

We consider the multi-reach water quality system with the data given in Examples 1 and 2 in addition to the following

$$C_o(1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C_o(2) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C_o(3) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$D_o(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_o(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_o(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

With the aid of the LMILab [7], the feasible solutions of LMIs (5.19) – (5.24) yields the controller matrices:

$$A_C(1) = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.09 \end{bmatrix}, \quad A_C(2) = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_C(3) = \begin{bmatrix} -1.9 & 0 \\ 0 & -1 \end{bmatrix},$$
$$B_C(1) = \begin{bmatrix} 0.7854 & -1.3246 \\ 0.2234 & -2.0045 \end{bmatrix}, \quad B_C(2) = \begin{bmatrix} -1.1157 & 0.8006 \\ 0.7256 & -1.7654 \end{bmatrix},$$

$$B_C(3) = \begin{bmatrix} 0.3423 & -1.0206 \\ -0.5494 & 3.1145 \end{bmatrix},$$

$$C_C(1) = \begin{bmatrix} 0.2238 & 0.0912 \\ 0.5412 & 0.7644 \end{bmatrix}, \qquad C_C(2) = \begin{bmatrix} 0.3458 & 0.9442 \\ -0.1244 & -0.4564 \end{bmatrix},$$

$$C_C(3) = \begin{bmatrix} -0.8121 & 0.8724 \\ 0.8126 & -0.6944 \end{bmatrix}$$

for $\tau \leq 0.6545$.

6 Conclusion

This paper has introduced a new transformation method for the $\mathcal{H}_\infty$ analysis and synthesis of a class of uncertain time-delay systems with Markovian jump parameters. It has been established that the new method exhibits the delay-dependence properties of the uncertain jumping system and therefore provides a tractable methodology for stability analysis, stabilization and output feedback control. All the developed results have been cast into the format of linear matrix inequalities and several examples have been worked out to illustrate the theory.

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