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Stabilization of a Class of Stochastic Nonlinear Time-Delay Systems*

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Abstract: In this paper, the stabilization problem is considered for a class of nonlinear continuous stochastic systems with state delays. The purpose of this problem is to design a state feedback controller such that the closed-loop system is exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic time-delay systems to be stable, and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness of the proposed design method.

Keywords: *Nonlinear systems; stochastic systems; time-delay; Lyapunov stability; algebraic matrix inequalities.*

Mathematics Subject Classification (2000): 93E15, 93B36, 93B55.

1 Introduction

Nonlinear stochastic control has long been an important research field that has attracted many researchers, and enormous results have been published in the literature. In particular, the fundamental nonlinear stochastic stabilization issue has received considerable research interests, and has found successful applications in control and communication problems, such as attitude control of satellites and missile control, macroeconomic system control, chemical process control, etc., see [8] for a survey.

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Recently, there have appeared many methods to tackle different kinds of nonlinear stochastic systems. For example, in [2], a minimax dynamic game approach has been developed for the controller design problem of the nonlinear stochastic systems that employ risk-sensitive performance criteria. The stabilization problem has been investigated in [3, 4] for nonlinear stochastic systems, and a stochastic counterpart of the input-to-state stabilization results has been provided. In [7], under an infinite-horizon risk-sensitive cost criterion, the problem of output feedback control design has been studied for a class of strict feedback stochastic nonlinear systems. In [16], the decentralized global stabilization problem has been dealt with by using a Lyapunov-based recursive design method. On the other hand, the dual nonlinear stochastic filtering problem has also been an active area for three decades [8], and a number of nonlinear filtering approaches have been proposed in the literature, such as extended Kalman filters, bound-optimal filters [13], exponentially bounded filters [14, 20], etc.

It is now a recognized fact that the time delay is frequently a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, and so on. Recently, increasing attention has been focused on robust and/or H_∞ control problems for linear systems with certain types of time-delays, see [1] for a survey. Within the stochastic framework, the stability analysis problem for linear time-delay systems has been studied by many authors. For example, in [11], the stability analysis problem for linear stochastic delay interval systems with Markovian switching has been considered. In [17], an LMI approach has been developed to cope with the robust H_∞ control problem for linear uncertain stochastic systems with state delay. As for nonlinear stochastic time-delay systems, the related results have been scattered, and most of the results have been concerned with the stability analysis issue, see e.g.[5, 9]. So far, the stabilization problem for general nonlinear time-delay systems has not been fully investigated and remains important.

In this paper, we will consider the stabilization problem for a class of nonlinear continuous stochastic systems with state delays. Such a class of systems have been intensively investigated in [18 − 20] for the nonlinear filtering problems. An effective algebraic matrix inequality approach is proposed to design the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness and effectiveness of the proposed design method.

Notation The notations in this paper are quite standard. R^n and $R^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimension. We let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of continuous functions φ from $[-\tau, 0]$ to R^n with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a matrix, denote by $\|A\|$ its operator norm, i.e., $\|A\| = \sup\{|Ax| \colon |x| = 1\} = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$ where $\lambda_{\max}(\cdot)$ (respectively, $\lambda_{\min}(\cdot)$) means the largest (respectively, smallest) eigenvalue of A. $l_2[0,\infty]$ is the space of square integrable vector. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}([-\tau,0];\,R^n)$ the family of all $\mathcal{F}_0$-measurable $C([-\tau,0];\,R^n)$-valued random variables $\xi=\{\xi(\theta)\colon\ -\tau\leq\theta\leq 0\}$ such that $\sup\limits_{-\tau\leq\theta\leq 0} E|\xi(\theta)|^p<\infty$ where $E\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2 Problem Formulation and Assumptions

Consider the following nonlinear continuous-time state delayed stochastic system in a fixed complete probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$:

$$dx(t)=[f(x(t),u(t))+g(x(t-\tau))]\,dt+Dx(t)\,dw(t), \tag{1}$$

$$x(t)=\varphi(t),\quad t\in[-\tau,0], \tag{2}$$

where $x(t)\in R^n$ is the state, $u(t)\in R^m$ is the deterministic input, $y(t)\in R^p$ is the measurement output, and $f(\cdot,\cdot)\in R^n$ and $g(\cdot)\in R^n$ are nonlinear vector functions. $\tau>0$ denotes the state delay and $\varphi(t)$ is a continuous vector valued initial function. Here, $w(t)=[w_1(t)\,w_2(t)\ \ldots\ w_m(t)]^{\mathrm{T}}\in R^m$ is an m-dimensional Brownian motion. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(t)$. D is a known constant matrices with appropriate dimensions.

Assumption 1 The nonlinear vector functions $f(\cdot,\cdot)$ and $g(\cdot)$ are assumed to satisfy $f(0,0)=0$, $g(0)=0$ and

$$\left|f(x(t),u(t))-[\,A\quad B\,]\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\right|\leq a_{11}\left|\begin{bmatrix}x(t)\\u(t)\end{bmatrix}\right|+a_{12}, \tag{3}$$

$$|g(x(t-\tau))-A_dx(t-\tau)|\leq a_{21}|x(t-\tau)|+a_{22}, \tag{4}$$

where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $A_d\in R^{n\times n}$ are known constant matrices, and $a_{11}>0$, $a_{12}\geq 0$, $a_{21}>0$ and $a_{22}\geq 0$ are known scalars.

Remark 1 The system (1)–(2) can be used to represent many important physical nonlinear systems subject to inherent state delays and stochastic exogenous noises with known statistics. Similar to [18–20], the nonlinear descriptions (3)–(4) quantify the maximum possible derivations from a linear model with (A,B,A_d) as its system parameter matrices, and are more general than those of [13], [14].

When a state feedback control law

$$u(t)=Kx(t) \tag{5}$$

is applied to the system (1)–(2), the closed-loop system is governed by

$$dx(t)=[f(x(t),\,Kx(t))+g(x(t-\tau))]\,dt+Dx(t)\,dw(t). \tag{6}$$

For notation convenience, we give the following definitions:

$$A_c=A+BK, \tag{7}$$

$$p(t)=f(x(t),Kx(t))-A_cx(t), \tag{8}$$

$$q(t)=g(x(t-\tau))-A_dx(t-\tau), \tag{9}$$

and then obtain from (6) that

$$dx(t) = [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)]\, dt + Dx(t)\, dw(t). \tag{10}$$

Now, let $x(t;\xi)$ denote the state trajectory from the initial data $x(\theta) = \xi(\theta)$ on $-\tau \le \theta \le 0$ in $L^2_{\mathcal{F}_0}([-\tau, 0];\, R^n)$. It is clear from Assumption 1 that the system (10) admits a trivial solution $x(t;0) \equiv 0$ corresponding to the initial data $\xi = 0$.

Furthermore, we introduce the following concepts for stability and boundedness in the mean square.

Definition 1 Consider the system (10). For every $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0];\, R^n)$,

(1) the trivial solution is exponentially stable in the mean square if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$E|x(t;\xi)|^2 \le \alpha x^{-\beta t} \sup_{-\tau \le \theta \le 0} E|\xi(\theta)|^2; \tag{11}$$

(2) the trivial solution is exponentially ultimately bounded in the mean square if there exist constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that

$$E|x(t;\xi)|^2 \le \alpha x^{-\beta t} \sup_{-\tau \le \theta \le 0} E|\xi(\theta)|^2 + \gamma. \tag{12}$$

The objective of this paper is to design a controller for the nonlinear time-delay system (1)−(2), such that the closed-loop systems is exponentially stable (or exponentially ultimately bounded) in the mean square. More specifically, we are interested in designing a controller parameter K such that:

(1) in the case of $a_{12} = 0$ and $a_{22} = 0$ (i.e., there are no bounded nonlinearities and uncertain disturbances), the solution of the system (10) is guaranteed to be exponentially stable;
(2) in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$ (i.e., there are bounded nonlinearities or uncertain disturbances), the solution of the system (10) is guaranteed to be exponentially ultimately bounded in the mean square.

3 Main Results and Proofs

In this section, the controller analysis problem will be considered firstly. Given a controller structure, we shall establish the conditions under which the system dynamics is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square. Then, we shall take the controller design problem into account, whose purpose is to derive the explicit expression for the expected controller gain in terms of the positive definite solution to an algebraic matrix inequality.

The following theorem will play an essential role in the design of the expected controllers. It reveals that the exponential stability (or exponential ultimate boundedness) of the controlled nonlinear time-delay stochastic system (10) can be guaranteed if a positive definite solution to a modified algebraic Riccati-like matrix inequality (quadratic matrix inequality) is known to exist.

Theorem 1 *Let the controller parameter K be given. If there exist positive scalars ε_1, ε_2, ε_3, ε_4 such that the following matrix inequality*

$$A_c^{\mathrm{T}} P + P A_c + D^{\mathrm{T}} P D + (\varepsilon_1 + \varepsilon_2) P^2 + 4\varepsilon_2^{-1} a_{11}^2 (I + K^{\mathrm{T}} K) + Q < 0 \tag{13}$$

where

$$Q = \varepsilon_1^{-1} A_d^{\mathrm{T}} A_d + 4\varepsilon_2^{-1} a_{21}^2 I \tag{14}$$

has a solution $P > 0$, then in the mean square, the system (10) is

(i) *exponentially stable in the case of $a_{12} = 0$ and $a_{22} = 0$;*
(ii) *exponentially ultimately bounded in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$.*

Proof Fix $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; R^n)$ arbitrarily and write $x(t;\xi) = x(t)$. For $(x(t), t) \in R^n \times R_+$, we define the Lyapunov function candidate

$$V(x(t), t) = x^{\mathrm{T}}(t) P x(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s) Q x(s)\, ds, \tag{15}$$

where P is the positive definite solution to the matrix inequality (13) and $Q > 0$ is defined in (14).

By Itô's formula (see, e.g., [10]), the stochastic derivative of V along a given trajectory is obtained as

$$\begin{aligned} dV(x(t), t) = {} & \big\{ x^{\mathrm{T}}(t) P [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)] \\ & + [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)]^{\mathrm{T}} P x(t) \\ & + x^{\mathrm{T}}(t) Q x(t) - x^{\mathrm{T}}(t-\tau) Q x(t-\tau) \\ & + x^{\mathrm{T}}(t) D^{\mathrm{T}} P D x(t) \big\} dt + 2 x^{\mathrm{T}}(t) P D x(t)\, dw(t) \\ = {} & \big\{ x^{\mathrm{T}}(t) [A_c^{\mathrm{T}} P + P A_c + D^{\mathrm{T}} P D + Q] x(t) \\ & + x^{\mathrm{T}}(t) P A_d x(t-\tau) + x^{\mathrm{T}}(t-\tau) A_d^{\mathrm{T}} P x(t) \\ & + x^{\mathrm{T}}(t) P [p(t) + q(t)] + [p(t) + q(t)]^{\mathrm{T}} P x(t) \\ & - x^{\mathrm{T}}(t-\tau) Q x(t-\tau) \big\} dt + 2 x^{\mathrm{T}}(t) P D x(t)\, dw(t). \end{aligned} \tag{16}$$

Let ε_1 and ε_2 be two positive scalars. Then the matrix inequality

$$\big[\varepsilon_1^{1/2} x^{\mathrm{T}}(t) P - \varepsilon_1^{-1/2} x^{\mathrm{T}}(t-\tau) A_d^{\mathrm{T}}\big] \big[\varepsilon_1^{1/2} x^{\mathrm{T}}(t) P - \varepsilon_1^{-1/2} x^{\mathrm{T}}(t-\tau) A_d^{\mathrm{T}}\big]^{\mathrm{T}} \geq 0$$

yields

$$\begin{aligned} & x^{\mathrm{T}}(t) P A_d x(t-\tau) + x^{\mathrm{T}}(t-\tau) A_d^{\mathrm{T}} P x(t) \\ & \qquad \leq \varepsilon_1 x^{\mathrm{T}}(t) P^2 x(t) + \varepsilon_1^{-1} x^{\mathrm{T}}(t-\tau) A_d^{\mathrm{T}} A_d x(t-\tau). \end{aligned} \tag{17}$$

In the sequel, we will use several times the following simple inequality

$$(u + v)^{\mathrm{T}} (u + v) \leq 2u^{\mathrm{T}} u + 2v^{\mathrm{T}} v,$$

where $u \in R^n$ and $v \in R^n$.

Noticing the Assumption 1 and the definitions (7)–(9), we have

$$\begin{aligned} p^{\mathrm{T}}(t)p(t) &= |f(x(t),Kx(t))-A_cx(t)|^2 \le \left\{a_{11}\left|\begin{bmatrix} x(t)\\ Kx(t)\end{bmatrix}\right|+a_{12}\right\}^2 \\ &\le 2a_{11}^2\left|\begin{bmatrix} x(t)\\ Kx(t)\end{bmatrix}\right|^2+2a_{12}^2 \le 2a_{11}^2x^{\mathrm{T}}(t)\big(I+K^{\mathrm{T}}K\big)x(t)+2a_{12}^2, \end{aligned} \tag{18}$$

$$\begin{aligned} q^{\mathrm{T}}(t)q(t) &= |g(x(t-\tau))-A_dx(t-\tau)|^2 \le \big\{a_{21}|x(t-\tau)|+a_{22}\big\}^2 \\ &\le 2a_{21}^2x^{\mathrm{T}}(t-\tau)x(t-\tau)+2a_{22}^2. \end{aligned} \tag{19}$$

Then, it follows from (18), (19) and

$$\Psi_1=\varepsilon_2^{1/2}x^{\mathrm{T}}(t)P-\varepsilon_2^{-1/2}[p(t)+q(t)]^{\mathrm{T}}, \quad \Psi_1\Psi_1^{\mathrm{T}}\ge 0$$

that

$$\begin{aligned} &x^{\mathrm{T}}(t)P[p(t)+q(t)]+[p(t)+q(t)]^{\mathrm{T}}Px(t) \\ &\qquad\le \varepsilon_2x^{\mathrm{T}}(t)P^2x(t)+\varepsilon_2^{-1}[p(t)+q(t)]^{\mathrm{T}}[p(t)+q(t)] \\ &\qquad\le \varepsilon_2x^{\mathrm{T}}(t)P^2x(t)+2\varepsilon_2^{-1}[p^{\mathrm{T}}(t)p(t)+q^{\mathrm{T}}(t)q(t)] \\ &\qquad= x^{\mathrm{T}}(t)[\varepsilon_2P^2+4\varepsilon_2^{-1}a_{11}^2(I+K^{\mathrm{T}}K)]x(t) \\ &\qquad\quad +4\varepsilon_2^{-1}a_{21}^2x^{\mathrm{T}}(t-\tau)x(t-\tau)+4\varepsilon_2^{-1}(a_{12}^2+a_{22}^2). \end{aligned} \tag{20}$$

For simplicity, we denote

$$\Pi=A_c^{\mathrm{T}}P+PA_c+D^{\mathrm{T}}PD+(\varepsilon_1+\varepsilon_2)P^2+4\varepsilon_2^{-1}a_{11}^2(I+K^{\mathrm{T}}K)+\varepsilon_1^{-1}A_d^{\mathrm{T}}A_d+4\varepsilon_2^{-1}a_{21}^2I, \tag{21}$$

and then (13) and (14) indicate that $\Pi<0$.

Substituting (14), (17) and (20) into (16) gives

$$dV(x(t),t)\le \big[x^{\mathrm{T}}(t)\Pi x(t)+4\varepsilon_2^{-1}(a_{12}^2+a_{22}^2)\big]dt+2x^{\mathrm{T}}(t)PDx(t)dw(t). \tag{22}$$

We are now in a position to show the expected exponential stability (or exponential ultimate boundedness) of the system (10), by using the the technique developed in [10]. Let $\beta>0$ be the unique root of the equation

$$\lambda_{\min}(-\Pi)-\beta\lambda_{\max}(P)-\beta\tau\lambda_{\max}(Q)x^{\beta\tau}=0 \tag{23}$$

where Π and Q are defined, respectively, in (21) and (14), P is the positive definite solution to (13), and τ is the time-delay.

We can obtain from (22) that

$$\begin{aligned} d\big[x^{\beta t}V(x(t),t)\big] &= x^{\beta t}\big[\beta V(x(t),t)dt+dV(x(t),t)\big] \\ &\le x^{\beta t}\left(-\big[\lambda_{\min}(-\Pi)-\beta\lambda_{\max}(P)\big]|x(t)|^2+\beta\lambda_{\max}(Q)\int_{t-\tau}^{t}|x(s)|^2ds\right)dt \\ &\quad +4\varepsilon_2^{-1}(a_{12}^2+a_{22}^2)x^{\beta t}dt+2x^{\beta t}x^{\mathrm{T}}(t)PDx(t)w(t)dt. \end{aligned}$$

Then, integrating both sides from 0 to $T > 0$ and taking the expectation result in

$$
\begin{aligned}
x^{\beta T} EV(x(T),T) \le \big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] &\sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 \\
&- \big[\lambda_{\min}(-\Pi) - \beta\lambda_{\max}(P)\big] E\int_0^{\mathrm{T}} x^{\beta t}|x(t)|^2 dt \\
&+ \beta\lambda_{\max}(Q) E\int_0^{\mathrm{T}} x^{\beta t}\int_{t-\tau}^{t} |x(s)|^2 ds dt + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}(x^{\beta T} - 1).
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_0^{\mathrm{T}} x^{\beta t}\int_{t-\tau}^{t} |x(s)|^2 ds dt &\le \int_{-\tau}^{\mathrm{T}} \Bigg(\int_{\max(s,0)}^{\min(s+\tau,T)} x^{\beta t} dt\Bigg) |x(s)|^2 ds \\
&\le \int_{-\tau}^{\mathrm{T}} \tau x^{\beta(s+\tau)}|x(s)|^2 ds \le \tau x^{\beta\tau}\int_0^{\mathrm{T}} x^{\beta t}|x(t)|^2 dt + \tau x^{\beta\tau}\int_{-\tau}^{0} |\xi(\theta)|^2 d\theta.
\end{aligned}
$$

Then, considering the definition of β in (23), we have

$$
\begin{aligned}
x^{\beta T} EV(x(T),T) \le \big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] &\sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 \\
&+ \beta\lambda_{\max}(Q)\tau^2 x^{\beta\tau} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 + 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}(x^{\beta T} - 1),
\end{aligned}
$$

and

$$
\begin{aligned}
E|x(T)|^2 \le \lambda_{\min}^{-1}(P)\Big(&\big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)\big] \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2 \\
&+ \beta\lambda_{\max}(Q)\tau^2 x^{\beta\tau} \sup_{-\tau\le\theta\le 0} E|\xi(\theta)|^2\Big) x^{-\beta T} \\
&+ 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}\lambda_{\min}^{-1}(P)(x^{\beta T} - 1)x^{-\beta T}.
\end{aligned}
$$

Notice that $(x^{\beta T} - 1)x^{-\beta T} < 1$ and let

$$
\alpha = \lambda_{\min}^{-1}(P)\big[\lambda_{\max}(P) + \tau\lambda_{\max}(Q)(1 + \beta\tau x^{\beta\tau})\big], \quad \gamma = 4\varepsilon_2^{-1}(a_{12}^2 + a_{22}^2)\beta^{-1}\lambda_{\min}^{-1}(P).
$$

Since $T > 0$ is arbitrary, the definition of exponential ultimate boundedness in (12) is then satisfied if $a_{12} \ne 0$ or $a_{22} \ne 0$. If $a_{12} = a_{12} = 0$, it is obvious that the definition of exponential stability in (11) is met. This completes the proof of Theorem 1.

Next, let us focus on deriving the *explicit* expression of expected controller gains by using an algebraic matrix inequality approach. It is worth mentioning that, in most literature concerning nonlinear stochastic stabilization problems, the solution has not been given as an explicit representation.

Based on Theorem 1, we can see that the controller design problem can be transformed into the following two-step problem: (i) find a necessary and sufficient condition for the existence of the positive definite matrix P such that there exists a controller gain K satisfying (13); and (ii) if the controller gain K exists, give the characterization of the set of expected controller gains in terms of the positive definite matrix P and some other free parameters.

Lemma 1 [6] *Let $X \in R^{m_1 \times n_1}$ and $Y \in R^{m_1 \times p_1}$ $(m_1 \leq p_1)$. There exists a matrix $U \in R^{n_1 \times p_1}$ which simultaneously satisfies $Y = XU$ and $UU^{\mathrm{T}} = I$ if and only if $XX^{\mathrm{T}} = YY^{\mathrm{T}}$.*

For presentation convenience, we define

$$\Gamma(\varepsilon_1, \varepsilon_2, P) = A^{\mathrm{T}}P + PA + D^{\mathrm{T}}PD + (\varepsilon_1 + \varepsilon_2)P^2 + 4\varepsilon_2^{-1}a_{11}^2 I + Q, \tag{24}$$

$$\begin{aligned}\Xi(\varepsilon_1, \varepsilon_2, P) = {} & A^{\mathrm{T}}P + PA + D^{\mathrm{T}}PD + P[(\varepsilon_1 + \varepsilon_2)I - 0.25\,\varepsilon_2 a_{11}^{-2}BB^{\mathrm{T}}]P \\ & + 4\varepsilon_2^{-1}(a_{11}^2 + a_{21}^2)I + \varepsilon_1^{-1}A_d^{\mathrm{T}}A_d,\end{aligned} \tag{25}$$

where Q is defined in (14).

The aforementioned two-step problem is solved in the following theorem.

Theorem 2 *There exist positive scalars ε_1, ε_2 and a positive definite matrix P such that the matrix inequality (13) has a solution K if and only if the following quadratic matrix inequality*

$$\Xi(\varepsilon_1, \varepsilon_2, P) < 0 \tag{26}$$

holds, where $\Xi(\varepsilon_1, \varepsilon_2, P)$ is defined in (25). Furthermore, if (26) is true, all gain matrices K satisfying the matrix inequality (13) can be parameterized by

$$K = (0.5\,a_{11}^{-1}\varepsilon_2^{1/2}\Lambda U - 0.25\,a_{11}^{-2}\varepsilon_2 PB)^{\mathrm{T}} \tag{27}$$

where $\Lambda \in R^{n \times m}$ is any matrix satisfying

$$\Lambda\Lambda^{\mathrm{T}} < -\Xi(\varepsilon_1, \varepsilon_2, P) \tag{28}$$

and $U \in R^{m \times m}$ is arbitrary orthogonal matrix (i.e., $UU^{\mathrm{T}} = I$).

Proof Rewrite the matrix inequality (13) as

$$K^{\mathrm{T}}B^{\mathrm{T}}P + PBK + 4\varepsilon_2^{-1}a_{11}^2 K^{\mathrm{T}}K + \Gamma(\varepsilon_1, \varepsilon_2, P) < 0, \tag{29}$$

where $\Gamma(\varepsilon_1, \varepsilon_2, P)$ is defined in (24).

In terms of the definition of $\Xi(\varepsilon_1, \varepsilon_2, P)$ in (25), we can rearrange (29) as

$$(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)^{\mathrm{T}} < -\Xi(\varepsilon_1, \varepsilon_2, P). \tag{30}$$

Obviously, there exists a controller gain matrix K such that the inequality (13) (or equivalently (30)) holds for some positive scalars ε_1, ε_2 and positive definite matrix P if and only if the right-hand side of (30) is positive definite, i.e., $-\Xi(\varepsilon_1, \varepsilon_2, P) > 0$ or (26) holds. The first part of this theorem is proved.

Assume now that (26) is true. Note that the dimension of the controller gain K is $m \times n$. From (30) and the definition of $\Lambda \in R^{n \times m}$ in (28), we could relate a Λ such that

$$(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)(2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\varepsilon_2^{1/2}a_{11}^{-1}PB)^{\mathrm{T}} = \Lambda\Lambda^{\mathrm{T}}. \tag{31}$$

It then follows from Lemma 1 that (31) holds if and only if

$$2\varepsilon_2^{-1/2}a_{11}K^{\mathrm{T}} + 0.5\,\varepsilon_2^{1/2}a_{11}^{-1}PB = \Lambda U, \tag{32}$$

where $U \in R^{m \times m}$ is an arbitrary orthogonal matrix. Therefore, the expression (27) follows immediately. This completes the proof of the theorem.

Finally, our main results can be summarized in the following corollary.

Corollary 1 *Consider the nonlinear discrete-time state delayed stochastic system (1)–(2) with the state feedback controller $u(t) = Kx(t)$. If there exist positive scalars ε_1, ε_2, and a positive definite matrix P such that the matrix inequality (26) holds, then the state feedback controller with its gain given in (27) will be such that the system (10) is exponentially stable in the case of $a_{12} = 0$ and $a_{22} = 0$; or exponentially ultimately bounded in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$, both in the mean square.*

Remark 2 Corollary 1 solves the addressed stabilization problem for the class of nonlinear time-delay stochastic systems in this paper. In implementation, we could first solve the quadratic matrix inequality (26), and then obtain the expected control parameters from (27) easily. Firstly, based on the algorithms provided in [15] and references therein, we may select appropriate positive scalar parameters ε_1 and ε_2 so as to reduce the conservatism that may have resulted from the inequalities (17) and (20). Then, (26) will be a standard quadratic matrix inequality (QMI) for P. For details concerning the *general* QMIs and relevant algorithms, we refer the reader to [12]. It can also be noticed that, there exists a lot of design freedom in our proposed procedure, such as the choices of matrices Λ and U, which could be used to achieve other expected performance specifications, e.g., reliability constraints.

4 Numerical Simulation

In this section, for the purpose of illustrating the usefulness and flexibility of the theory developed in this paper, we present a simulation example.

Assume that the nonlinear continuous-time stochastic state delayed system (1)–(2) is given by

$$\begin{aligned} dx_1(t) &= [-2x_1(t) - 0.1x_2(t) + 0.2\cos(x_1(t) + x_2(t)) \\ &\qquad + 0.1x_1(t-0.1) + 0.16\sin x_2(t) + 2.9u_1(t) + 0.2u_2(t)]\,dt + 0.2x_1\,dw(t), \\ dx_2(t) &= [-0.1x_1(t) + x_2(t) + 0.15\sin x_2(t) \\ &\qquad + 0.1x_2(t-0.1) + 0.15\cos x_1(t) + 0.1u_1(t) - 2.1u_2(t)]\,dt + 0.2x_2\,dw(t). \end{aligned}$$

Considering the system (1)–(2) with the constraints (3)–(4), we can obtain that

$$A = \begin{bmatrix} -2 & -0.1 \\ -0.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2.9 & 0.2 \\ 0.1 & -2.1 \end{bmatrix}, \quad A_d = 0.1I_2, \quad D = 0.2I_2,$$
$$d = 0.1, \quad a_{11} = 0.25; \quad a_{12} = 0.12; \quad a_{21} = 0; \quad a_{22} = 0.$$

We choose $\varepsilon_1 = 4.8$, $\varepsilon_1 = 8.2$, and solve (26) to obtain

$$P = \begin{bmatrix} 0.1287 & 0.0013 \\ 0.0013 & 0.2003 \end{bmatrix}.$$

Then, setting $\Lambda = 2I_2$ which meets (28) and considering two cases of $U = I_2$ and $U = -I_2$, we have two desired gain matrices as follows:

$$\text{Case 1:} \quad K_1 = \begin{bmatrix} -0.7938 & -0.7764 \\ -0.7580 & 25.2439 \end{bmatrix}, \qquad \text{Case 2:} \quad K_2 = \begin{bmatrix} -23.7023 & -0.7764 \\ -0.7580 & 2.3354 \end{bmatrix}.$$

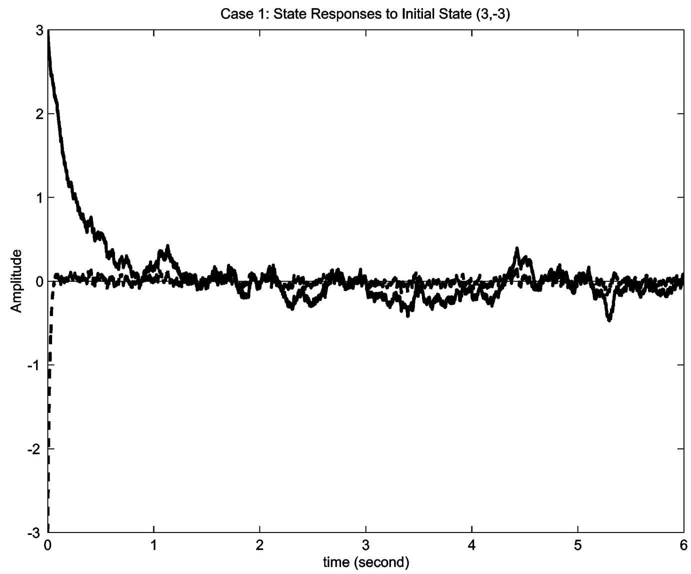

Figure 4.1. x_1 (solid), x_2 (dashed).

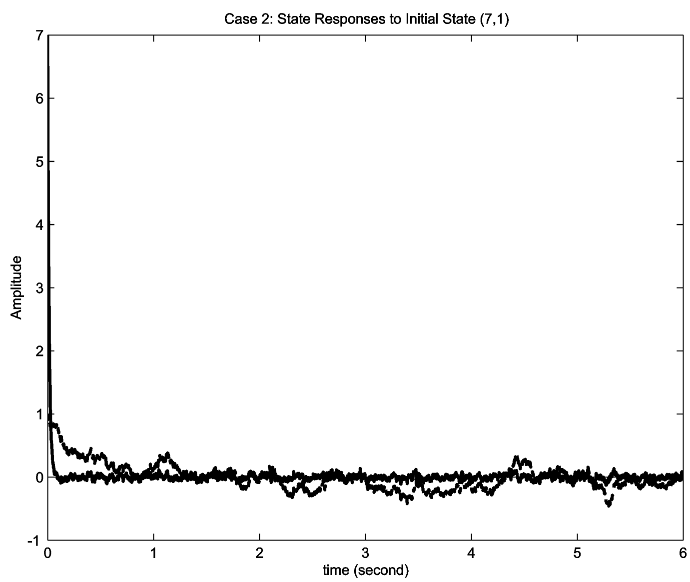

Figure 4.2. x_1 (solid), x_2 (dashed).

The responses of closed-loop system dynamics to initial conditions are shown in Figure 4.1 and Figure 4.2. The simulation results imply that the desired goal is well achieved, i.e., the closed-loop system is exponentially stable in the mean square.

5 Conclusions

In this paper, we have studied the stabilization problem for a class of nonlinear stochastic time-delay systems. The nonlinearities are assumed to have the similar form as those in [18 – 20]. We have developed an effective algebraic matrix inequality approach to designing the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We have investigated the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and have derived the explicit expression of the desired controller gains. A numerical simulation example has been provided to show the usefulness and effectiveness of the proposed design method.

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