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Optimal Control of Nonlinear Systems with Controlled Transitions

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Abstract: This paper studies the optimum stochastic control problem with piecewise deterministic dynamics. The controls enter through the system dynamics as well as the transitions for the underlying Markov chain process, and are allowed to depend on both the continuous state and the current state of the Markov chain. The paper shows that the feedback optimal control relies on the viscosity solutions of a finite set of coupled Hamilton-Jacobi-Bellman (HJB) equations. Explicit control structures are provided by using the concept of subdifferential of a continuous function.

Keywords: *Markov process; Hamilton–Jacobi–Bellman equations; viscosity solutions; β-stochastically stabilizable.*

Mathematics Subject Classification (2000): 49J15, 49J52, 49J55, 93E03.

1 Introduction

In this paper, we consider a dynamical system which is nonlinear in the state and linear in the piecewise continuous control u_1:

$$\begin{aligned} \frac{dx}{dt}(t) &= f(x(t),\theta(t)) + B(x(t),\theta(t))u_1(t), \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $x \in R^n$, x_0 is a fixed (known) initial state, u_1 is a control, taking values in a bounded set $U_1 \subset R^r$, and $\theta(t)$ is a controlled, continuous time Markov process, taking values in a finite state space $\mathcal{S}$, of cardinality s. Transitions from state $i \in \mathcal{S}$ to $j \in \mathcal{S}$ occur at a rate controlled by a second controller, who chooses at time t an action $u_2(t)$

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from a finite set $U_2(i)$ of actions available at state i. Let $U_2 = \cup_{i\in\mathcal{S}} U_2(i)$. The controlled rate matrix (of transitions within $\mathcal{S}$) is

$$\Lambda = \{\lambda_{i,a,j}\}, \quad i, j \in \mathcal{S}, \quad a = u_2(t) \in U_2(i), \tag{1.2}$$

where henceforth we drop the "commas" in the subscripts of λ. The λ_{iaj}'s are real numbers such that for any $i \neq j$, and $a \in U_2(i)$, $\lambda_{iaj} \geq 0$, and for all $a \in U_2(i)$ and $i \in \mathcal{S}$, $\lambda_{iai} = -\sum_{j\neq i} \lambda_{iaj}$. Fix some initial state i_0 of the controlled Markov chain $\mathcal{S}$, and the final time t_f (which may be infinite). We consider the class of policies $\mu_k \in \mathcal{U}_k$ for controller $(k = 1, 2)$, whose elements (taking values in U_k) are of the form

$$u_k(t) = \mu_k(t, x(t), \theta(t)), \quad t \in [0, t_f). \tag{1.3}$$

For the finite-horizon case, μ_k is taken to be piecewise continuous in the first argument and local Lipschitz in second argument and measurable in the third argument. In the infinite-horizon case, the dependence of μ_k on t is dropped, but otherwise it is defined the same way. Define $\mathcal{X} = R^n \times \mathcal{S}$ to be the combined state space of the system and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$ to be the class of admissible multi-strategies $\mu = (u_1, u_2)$, appropriately defined depending on whether t_f is finite or infinite. Let $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product. Define a running cost $L\colon \mathcal{X} \times \mathcal{U}_1 \to [0, \infty)$ as

$$L(x, i, u_1) = Q(x, i) + \langle u_1, R(x, i)u_1\rangle, \tag{1.4}$$

where the definitions of Q and R will be made precise later in Section 2.1.

To any fixed initial state (x_0, i_0) and a multi-strategy $\mu \in \mathcal{U}$, there corresponds a unique probability measure $P^{\mu}_{x_0,i_0}$ on the canonical probability space of the states and actions of the players, equipped with the standard Borel σ-algebra. Denote by $E^{\mu}_{x_0,i_0}$ the expectation operator corresponding to $P^{\mu}_{x_0,i_0}$.

For each fixed initial state (x_0, i_0), multi-strategy $\mu \in \mathcal{U}$, and a finite horizon of duration t_f, the discounted (expected) cost function is defined as

$$J_\beta(0; x_0, i_0, \mu; t_f) = E^{\mu}_{x_0,i_0}\left\{ g(x(t_f), \theta(t_f))e^{-\beta t_f} + \int_0^{t_f} e^{-\beta t} L(x(t), \theta(t), u_1(t))\, dt \right\}, \tag{1.5}$$

where g is a terminal cost function whose condition will be specified in next section, $\beta \geq 0$ is the discount factor, and the expectation is over the joint process $\{x, \theta\}$. For t_f infinite, a corresponding discounted cost function is defined as:

$$J_\beta(0; x_0, i_0, \mu) = E^{\mu}_{x_0,i_0}\left\{ \int_0^{\infty} e^{-\beta t} L(x(t), \theta(t), u_1(t))\, dt \right\}. \tag{1.6}$$

We further denote the *cost-to-go* from any time-state pair $(t; x, i)$, under a multi-strategy $\mu \in \mathcal{U}$ by

$$J_\beta(t; x, i, \mu; t_f) = E^{\mu}_{x,i}\left\{ g(x(t_f), \theta(t_f))e^{-\beta(t_f - t)} + \int_t^{t_f} e^{-\beta(\tau - t)} L(x(\tau), \theta(\tau), u_1(\tau))\, d\tau \right\} \tag{1.7}$$

and

$$J_\beta(t;x,i,\mu) = E^{\mu}_{x,i}\left\{ \int_t^\infty e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau \right\}, \tag{1.8}$$

for the finite-horizon and infinite-horizon cases, respectively. The optimal *value functions* are then defined by, respectively,

$$V(t;x,i;t_f) = \inf_{\mu_1\in\mathcal{U}_1}\inf_{\mu_2\in\mathcal{U}_2} J_\beta(t;x,i,\mu;t_f) \tag{1.9}$$

and

$$V(x,i) = \inf_{\mu_1\in\mathcal{U}_1}\inf_{\mu_2\in\mathcal{U}_2} J_\beta(t;x,i,\mu) \tag{1.10}$$

for $i = 1,2,\dots,s$. Dynamic programming arguments (for background on the approach that can be used here, see [2, 3]), lead to the two coupled HJB equations (2.11) and (2.12), corresponding to the finite and infinite-horizon cases, respectively. More precisely, if these equations admit unique *viscosity solutions* on R^n, then $V(t;x,i;t_f)$ and $V(t;x)$ thus defined constitute the optimal value functions for the finite-horizon and infinite-horizon cases, respectively. Xiao and Başar have showed, under the assumptions given in next section, that the coupled HJB equations (for finite and infinite-horizon, respectively) admit viscosity solutions, and moreover the viscosity solutions are unique if $\beta > 0$ [4].

In this paper, we study the structures of the optimal controllers for the nonlinear system (1.1). The major challenge here is that this is not a standard optimal control problem in which only a single HJB equation is considered (f.g. see [6−8]). The optimal control considered in this paper is related to a system of coupled HJB equations. Based on the results obtained in [4], we show that the optimal feedback control is determined by the subdifferential of the viscosity solutions of (2.11) in finite horizon case, and (2.12) in infinite horizon case. Explicit expression of the optimal feedback controller is obtained in terms of the subdifferential of viscosity solutions of (2.11) or (2.12).

The remainder of this paper is structured as follows. In Section 2, we provide the necessary assumptions for the system (1.1), and the definitions of viscosity solutions of the coupled HJB equations (2.11) and (2.12). In Section 3, we give a detail discussion of the structure of the optimal controller by using the concept of viscosity solution and the concept of subdifferential of a continuous function. The pair of optimal feedback controller, one is the control which enters through the system dynamics, and the another is the control of transitions for the underlying Markov chain process, is explicitly given in this section. As an illustration, in Section 4 we apply the result to the linear quadratic case. The paper ends with the concluding remarks.

2 Assumptions and Definitions

2.1 Assumptions

Assumption 1 For each i, f is an n-vector function, and there exists a constant $L_f \geq 0$ such that

$$\sup_i\{|f(x,i) - f(y,i)|\} \leq L_f|x-y|, \quad x \in R^n,$$

where $|\cdot|$ stands for the Euclidean norm of R^n.

Assumption 2 For each i, $B(x,i)$ is an $n \times r$ matrix, and

$$\sup_i \{\|B(x,i)\|\} \le C_{b1}, \quad \sup_i \{\|B(x,i) - B(y,i)\|\} \le C_{b2}|x-y|, \quad \forall\, x, y \in R^n,$$

for some positive constants C_{b1} and C_{b2}, where $\|\cdot\|$ stands for the matrix's norm.

Assumption 3 For each i, $Q(\cdot, i)\colon\ R^n \to [0, +\infty)$, with

$$0 \le \sup_i \{Q(x,i)\} \le C_q |x|^2, \quad \forall\, x \in R^n,$$

for some $C_q > 0$.

Assumption 4 For each i, $R(x,i)$ is an $n \times n$ matrix with $R(x,i) = R(x,i)^{\mathrm{T}} > 0$, for all $x \in R^n$, and

$$\sup_i \{\|R(x,i)\|\} \le C_r, \quad \sup_i \{\|R(x,i) - R(y,i)\|\} \le C'_r |x-y|, \quad x, y \in R^n,$$

for some $C_r, C'_r > 0$, and there exists $L_R > 0$ such that

$$\sup_i \{\|R^{-1}(x,i) - R^{-1}(y,i)\|\} \le L_R |x-y|, \quad \forall\, x, y \in R^n.$$

Assumption 5 For $i \ne j$, $0 \le \lambda_{iaj} \le C_\lambda$, where C_λ is a positive constant, and

$$\lambda_{iai} + \sum_{j \ne i} \lambda_{iaj} \equiv 0, \quad 1 \le i \le s.$$

Assumption 6 For each i and any $g(\cdot, i)\colon\ R^n \to [0, \infty)$,

$$\sup_i \{|g(x,i)|\} \le (1 + C_g)|x|^2,$$
$$\sup_i \{|g(x,i) - g(y,i)|\} \le C'_g (1 + |x| + |y|)|x-y|$$

for all $x, y \in R^n$, where C_g, C'_g are positive constants.

Assumption 7 β is a positive real number.

Assumption 8 For any $z \in R^n$, there exists a nondecreasing function $\omega\colon\ \{0\} \cup R^+ \to \{0\} \cup R^+$ such that $\omega(0) = 0$, $\lim_{\rho \to +\infty} \omega(\rho)/\rho = +\infty$ and

$$\langle z,\ B(x,i) R^{-1}(x,i) B(x,i)^{\mathrm{T}} z \rangle \ge \omega(|z|), \quad \forall\, x \in R^n, \quad \forall\, i \in \mathcal{S},$$

where $B(x,i)^{\mathrm{T}}$ represents the transpose of $B(x,i)$.

Throughout the paper, the following conventions will be adopted, unless otherwise indicated:

(1) u_2 and a are used interchangeably to denote the second control;
(2) by an abuse of notation $\mu_1(t)$ will be used to denote $\mu_1(x(t), \theta(t))$.

2.2 Two coupled Hamilton–Jacobi–Bellman (HJB) equations

Let Ω be a nonempty open set of R^n. We here introduce two coupled Hamilton-Jacobi-Bellman (HJB) equations.

(I) Finite horizon:

$$\begin{gathered}\beta V(t,x,i) + \sup_{u_1\in U_1, u_2\in U_2(i)} \Big[-\mathcal{A}^{(u_1,u_2)} V(t,x,i) - L(x,i,u_1)\Big] = 0 \ \text{ in } \ (0,t_f]\times\Omega; \\ V(t_f,x,i) = g(x,i) \ \text{ on } \ \Omega, \quad i=1,2,\dots,s, \quad \text{where } s \text{ is a positive integer,}\end{gathered} \tag{2.11}$$

and $U_1 \subset R^r$, $U_2(i)$ is a finite set for each $i\in\mathcal{S}=\{1,2,\dots,s\}$.

(II) Infinite horizon:

$$\beta V(x,i) + \sup_{u_1\in R^r, u_2\in U_2(i)} \Big[-G^{(u_1,u_2)} V(x,i) - L(x,i,u_1)\Big] = 0 \ \text{ in } \ \Omega \tag{2.12}$$

for $i=1,2,\dots,s$, where again $U_1\subset R^r$ and $U_2(i)$ is a finite set for each $i\in\mathcal{S}$.

Here, the operators $\mathcal{A}$ and G are defined, respectively, as follows for each $u_1\in R^r$, $a\in U_2(i)$, $i\in\mathcal{S}$:

$$\mathcal{A}^{(u_1,a)}V(t,x,i) = \frac{\partial V(t,x,i)}{\partial t} + [D_x V(t,x,i)]\cdot F(x,u_1,i) + \sum_{j\in\mathcal{S}} \lambda_{iaj} V(t,x,j) \tag{2.13}$$

and

$$G^{(u_1,a)}V(x,i) = [D_x V(x,i)]\cdot F(x,u_1,i) + \sum_{j\in\mathcal{S}} \lambda_{iaj} V(x,j), \tag{1.14}$$

with

$$F(x,u_1,i) = f(x,i) + B(x,i)u_1, \quad L(x,i,u_1) = Q(x,i) + \langle u_1, R(x,i)u_1\rangle. \tag{2.15}$$

Definition 2.1 Let $\overline{V}$ be a vector function

$$\overline{V} = (V(\cdot,\cdot,1),\, V(\cdot,\cdot,2),\, \dots,\, V(\cdot,\cdot,s))\colon \ ([0,t_f]\times\Omega)^s \to R^n.$$

We say that

(1) $\overline{V}$ is a *viscosity subsolution* of (2.11), if for any i, $V(\cdot,\cdot,i)$ is upper semi-continuous and

$$\begin{gathered}\beta\Phi(t_0,x_0,i) + \sup_{(u_1,u_2)} \Big[-\mathcal{A}^{(u_1,u_2)}\Phi(t_0,x_0,i) - L(x_0,i,u_1)\Big] \le 0 \quad \text{on} \quad \Omega, \\ \Phi(t_f,x,i) \le g(t_f,x,i) \quad \text{on} \quad \Omega,\end{gathered}$$

whenever $\Phi(\cdot,\cdot,i)\in C^1([0,t_f]\times\Omega)$ is such that $V(t,x,i)-\Phi(t,x,i)$ attains a local maximum at (t_0,x_0) with $\Phi(t_0,x_0,j) = V(t_0,x_0,j)$ for each $j\in\mathcal{S}$, and $(u_1,u_2)\in U_1\times U_2$;

(2) $\overline{V}$ is a *viscosity supersolution* of (2.11) if for any i, $V(\cdot,\cdot,i)$ is lower semi-continuous and

$$\beta\Phi(t_0,x_0,i)+\sup_{(u_1,u_2)}\Big[-\mathcal{A}^{(u_1,u_2)}\Phi(t_0,x_0,i)-L(x_0,i,u_1)\Big]\geq 0 \quad \text{on} \quad \Omega,$$
$$\Phi(t_f,x,i)\geq g(t_f,x,i),$$

whenever $\Phi(\cdot,\cdot,i)\in C^1([0,t_f]\times\Omega)$ is such that $V(t,x,i)-\Phi(t,x,i)$ attains a local minimum at (t_0,x_0) with $\Phi(t_0,x_0,j)=V(t_0,x_0,j)$ for each $j\in\mathcal{S}$, and $(u_1,u_2)\in U_1\times U_2$;

(3) $\overline{V}$ is a *viscosity solution* of (2.11) if $\overline{V}$ is both a viscosity supersolution and a viscosity subsolution.

The notion of a viscosity solution for (2.12) can be introduced analogously. The following theorem is from [4].

Theorem 2.1 *Let the control space $\mathcal{U}_1$ be bounded. Under the Assumptions 1 – 8 given above, the coupled Hamilton–Jacobi–Bellman (HJB) equations (2.11) (resp. (2.12)) admit unique viscosity solutions on $[0,t_f]\times\Omega$ (resp. Ω). The viscosity solutions are the optimal value functions given by (1.9) (resp. (1.10)).*

3 Construction of the Optimum Stochastic Control

We discuss in this section the derivation of the optimal control law of the system (1.1). We first introduce the notations of a superdifferential and a subdifferential of a continuous function.

Definition 3.1 Let $V\in C([0,t_f]\times R^n)$ and $(t,x)\in[0,t_f]\times R^n$. Then

(1) the superdifferential, $D^+V(t,x)$, of V at (t,x) is

$$D^+V(t,x)=\left\{(q,p)\in R^{n+1}\colon \limsup_{(s,y)\to(t,x)}\frac{V(s,y)-V(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|x-y|}\leq 0\right\};$$

(2) The subdifferential, $D^-V(t,x)$, of V at (t,x) is

$$D^-V(t,x)=\left\{(q,p)\in R^{n+1}\colon \limsup_{(s,y)\to(t,x)}\frac{V(s,y)-V(t,x)-q(s-t)-p\cdot(y-x)}{|s-t|+|x-y|}\geq 0\right\}.$$

Remark 3.1 It is easy to see that when V is differentiable at (t,x), we have

$$D^+V(t,x)=D^-V(t,x)=\left(\frac{\partial}{\partial t}V(t,x),D_xV(t,x)\right).$$

Lemma 3.1 *The following propositions hold*

(1) $\{V(\cdot\,;\cdot\,,i;t_f)\}_{i=1}^{s}$ *is a viscosity subsolution of (2.11) in* $[0,t_f]\times\Omega$ *if and only if for each* $i\in\mathcal{S}$

$$-q+\beta V(t;x,i;t_f)-\inf_{\mu_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(t;x,i;t_f)\Big\}+H_i(x,p)\le 0 \tag{3.16}$$

for any $(q,p)\in D^{+}V(t;x,i;t_f)$,

(2) $\{V(\cdot\,;\cdot\,,i;t_f)\}_{i=1}^{s}$ *is a viscosity supersolution of (2.11) in* $[0,t_f]\times\Omega$ *if and only if for each* $i\in\mathcal{S}$

$$-q+\beta V(t;x,i;t_f)-\inf_{\mu_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(t;x,i;t_f)\Big\}+H_i(x,p)\ge 0 \tag{3.17}$$

for any $(q,p)\in D^{-}V(t;x,i;t_f)$.

In both cases, H_i *is defined to be where*

$$H_i(x,p)=-\langle p,\,f(x,i)\rangle-Q(x,i)+\frac{1}{4}\langle p,\,B(x,i)R^{-1}(x,i)B(x,i)^{\mathrm{T}}p\rangle.$$

Proof We prove the first part, as the proof of the second part is similar.

Suppose that (3.16) holds. Let $\varphi(\cdot,\cdot,i)\subset C^1([0,t_f]\times R^n)$ be such that (t_0,x_0) is a local maximizer of $V(\cdot\,;\cdot,i;t_f)-\varphi(\cdot,\cdot,i)$ for some i with $V(t_0;x_0,k;t_f)=\varphi(t_0,x_0,k)$, $k=1,2,\dots,s$. Since $\varphi(\cdot,\cdot,i)\in C^1([0,t_f]\times R^n)$ for each i, it yields

$$\begin{aligned}\varphi(t,x,i)&=\varphi(t_0,x_0,i)+\varphi_t(t_0,x_0,i)(t-t_0)+\varphi_x(t_0,x_0,i)(x-x_0)\\&\quad+\,o(|t-t_0|)+o(|x-x_0|).\end{aligned}$$

Hence for (t,x) sufficiently close to (t_0,x_0), that (t_0,x_0) is a local maximizer of $V(\cdot\,;\cdot,i;t_f)-\varphi(\cdot,\cdot,i)$ leads to

$$\begin{aligned}V(t;x,i;t_f)&\le\varphi(t_0,x_0,i)+\varphi_t(t_0,x_0,i)(t-t_0)+\varphi_x(t_0,x_0,i)(x-x_0)\\&\quad+\,o(|t-t_0|)+o(|x-x_0|).\end{aligned}$$

Now let $p=\varphi_x(t_0,x_0,i)$ and $q=\varphi_t(t_0,x_0,i)$. Then (3.16) implies that

$$-\varphi_t(t_0,x_0,i)+\beta\varphi(t_0;x_0,i;t_f)-\inf_{\mu_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}\varphi(t_0;x_0,i;t_f)\Big\}+H_i\left(x_0,\varphi_x(t_0,x_0,i)\right)\le 0.$$

Thus $\{V(\cdot\,;\cdot,i;t_f)\}_{i=1}^{s}$ is a viscosity subsolution of (2.11).

Conversely, let $(q,p)\in D^{+}V(t;x,i;t_f)$. When (s,y) is sufficiently close to (t,x), according to the definition of superdifferential, we have

$$V(s;y,i;t_f)\le V(t;x,i;t_f)+q(s-t)+p\cdot(y-x)+o(|s-t|)+o(|y-x|).$$

Introduce test functions

$$\varphi(s,y,i)=V(t;x,i;t_f)+q(s-t)+p\cdot(y-x)+g_1(|s-t|)+g_2(|x-y|)$$

for $i=1,2,\dots,s$, and $g_j\colon[0,\infty)\to[0,\infty)$, $j=1,2,$ are nondecreasing functions such that $g_j(r)=o(r)$ and $\left.\dfrac{d}{dr}g_j(r)\right|_{r=0}=0$ (for construction of such functions, see [1] or [2]). Hence by such choice of g_1, g_2, one can see that in fact (t,x) is a local strict maximizer of $V(\cdot\,;\cdot,\cdot;i)-\varphi(\cdot,\cdot,i)$ and

$$\varphi_x(t,x,i)=p,\quad \varphi_t(t,x,i)=q,$$

as a result of which (3.16) holds by the definition of viscosity subsolution.

Definition 3.2 An admissible feedback controller $u_1(t) = \mu_1(t,x)$ for system (1.1) is a nonlinear mapping $F\colon\ R^+ \times R^n \to R^r$ such that $\mu_1(t,x) = F(t,x)$ and the following system

$$\begin{aligned} &\frac{dx}{dt}(t) = f(x(t),\theta(t)) + B(x(t),\theta(t))F(t,x(t)) \\ &x(0) = x_0 \\ &\theta(0) = i_0 \end{aligned} \tag{3.18}$$

has at least one solution defined on $(0,\infty)$ in the Carathéodory sense, i.e., absolutely continuous functions verifying (3.18) almost everywhere.

Definition 3.3 We say that the system (1.1) is β-stochastically stabilizable if, for all finite $x_0 \in R^n$ and $i_0 \in \mathcal{S}$ with $x(0) = x_0$, $\theta(0) = i_0$, there exists an admissible feedback control $\mu(t,x(t),\theta(t))$ such that

$$\lim_{t_f\to\infty} E_{x,i} \int_0^{t_f} \left[e^{-\beta t} L(x(t,\theta(t),\mu),\theta(t),\mu)\right] dt < \infty.$$

Now we are in the position for construction of the optimal control.

Theorem 3.1 *Assume that (2.11) has a viscosity solution $\{V(\cdot;\cdot,i;t_f)\}_{i=1}^s$. Suppose that for each i there exists a p with $(q,p) \in D^- V(t;x,i;t_f)$ such that*

$$\mu_1^*(x,i) = -\frac{1}{2} R^{-1}(x,i) B^{\mathrm{T}}(x,i) p(t,x,i) \tag{3.19}$$

is an admissible feedback controller for system (1.1). Let $\mu_2^*(t,x,i) = f(t,x,i)$ be the argument such that $\Big\{\sum_{j\in\mathcal{S}} \lambda_{iaj} V(t;x,i;t_f)\Big\}$ reaches its infimum (which always exists because the set U_2 is a finite set and U_1 is a bounded set). Then (μ_1^*,μ_2^*) is an optimal feedback control for system (1.1) under the cost function

$$J_\beta(t;x,i,\mu;t_f) = E^\mu_{x,i}\Bigg\{g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau\Bigg\}. \tag{3.20}$$

Proof According to Theorem 2.1, we know that

$$V(t;x,i;t_f) = \inf_{\mu\in\mathcal{U}} E^\mu_{x,i}\Bigg\{g(x(t_f),\theta(t_f))e^{-\beta(t_f-t)} + \int_t^{t_f} e^{-\beta(\tau-t)} L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau\Bigg\} \tag{3.21}$$

and thus $V(t;x,i;t_f)$ is absolutely continuous with respect to t for each fixed x. For $(q,p) \in D^-V(t;x,i;t_f)$, $(t,x) \in [0,t_f] \times \Omega$, by the definition of subdifferential, one can see that

$$V(s;y,i;t_f) \ge V(t;x,i;t_f) + q(s-t) + p\cdot(y-x) + o(|s-t|) + o(|y-x|) \tag{3.22}$$

when (s, y) is sufficiently close to (t, x). Similar to the proof of Lemma 3.1, we define a C^1 function

$$\psi(s, y, i) = V(t; x, i; t_f) + q(s - t) + p \cdot (y - x) + g_1(|s - t|) + g_2(|y - x|) \tag{3.23}$$

where $g_j\colon [0, \infty) \to [0, \infty)$, $j = 1, 2$, are nondecreasing functions such that $g_j(r) = o(r)$ and $dfracddrg(r)|_{r=0} = 0$. It is now ready to see that $V(s; y, i; t_f) - \psi(s, y, i)$ has a local strict minimizer at (t, x). The definition of viscosity subsolution leads to

$$-\psi_t(t, x, i) + \beta\psi(t, x, i) - \inf_{\mu_2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{iaj} \psi(t, x, i) \right\} + H_i(x, \psi_x(t, x, i)) \geq 0. \tag{3.24}$$

According to the definition of H_i, the above inequality can be written as

$$\begin{aligned} &\psi_t(t, x, i) + L(x, i, u_1) - \left| R^{1/2}(x, i) u_1 + \frac{1}{2} R^{-1/2} B^{\mathrm{T}}(x, i) p(t, x, i) \right|^2 \\ &\psi_x(t, x, i)[f(x, i) + B(x, i) u_1] - \beta\psi(t, x, i) + \inf_{\mu_2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{iaj} \psi(t, x, i) \right\} \leq 0. \end{aligned} \tag{3.25}$$

According to Dynkin's formula (see [2]), we know that

$$E_{x,i} e^{-(\beta t_f - t)} \psi(t_f, x(t_f), \theta(t_f)) - \psi(t, x, i) = \int\limits_t^{t_f} \mathcal{A}^{(\mu_1, \mu_2)} e^{-\beta(\tau - t)} \psi(\tau, x(\tau), \theta(\tau))\, d\tau.$$

Integrating (3.25) from t to t_f one obtains

$$\begin{aligned} V(t; x, i; t_f) \geq E^{\mu}_{x,i} &\left\{ g(x(t_f), \theta(t_f)) e^{-\beta(t_f - t)} + \int\limits_t^{t_f} e^{-\beta(\tau - t)} L(x, \theta, u_1)\, d\tau \right\} \\ &- \int\limits_t^{t_f} e^{-\beta(\tau - t)} |R^{1/2}(x, i) u_1 + R^{-1/2} B^{\mathrm{T}}(x, i) p(\tau, x, i)|^2\, d\tau. \end{aligned}$$

If we set

$$\begin{aligned} u_1 &= \mu_1^*(t, x, i) = -\frac{1}{2} R^{-1}(x, i) B^{\mathrm{T}}(x, i) p(t, x, i) \\ u_2 &= \mu_2^*(t, x, i) = f(t, x, i) = \operatorname*{argmin}_{a \in U_2} \left\{ \sum_{j \in \mathcal{S}} \lambda_{iaj} V(t; x, i; t_f) \right\} \end{aligned}$$

then we have

$$\begin{aligned} V(t; x, i; t_f) &= E^{(\mu_1^*, \mu_2^*)}_{x,i} \left\{ g(x(t_f), \theta(t_f)) e^{-\beta t_f} + \int\limits_t^{t_f} e^{-\beta\tau} L(x(\tau), \theta(\tau), \mu_1^*)\, dt \right\} \\ &= \inf_{\mu \in \mathcal{U}} E^{\mu}_{x,i} \left\{ g(x(t_f), \theta(t_f)) e^{-\beta(t_f - t)} + \int\limits_t^{t_f} e^{-\beta(\tau - t)} L(x(\tau), \theta(\tau), u_1(\tau))\, d\tau \right\} \end{aligned}$$

since $V(t; x, i; t_f)$ is the optimal value function according to Theorem 2.1, and this completes the proof of the theorem.

For the infinite horizon case we have a similar result:

Theorem 3.2 *Assume that (2.12) has viscosity solution* $\{V(\cdot,i)\}_{i=1}^{s}$. *Suppose that for each* i *there exists a* p *with* $(q,p)\in D^{-}V(x,i)$ *such that*

$$\mu_1^*(x,i) = -\frac{1}{2}\,R^{-1}(x,i)B^{\mathrm{T}}(x,i)p(x,i) \tag{3.26}$$

is an admissible feedback controller for system (1.1). Let

$$\mu_2^*(x,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(x,i)\Big\}. \tag{3.27}$$

Then (μ_1^*,μ_2^*) *is a pair of optimal feedback controls for system (1.1) under the cost function*

$$J_\beta(t;x,i,\mu) = E_{x,i}^{\mu}\Big\{\int_t^\infty e^{-\beta(\tau-t)}L(x(\tau),\theta(\tau),u_1(\tau))\,d\tau\Big\}, \tag{3.28}$$

Proof According to Theorem 2.1, the value function $V(x,i)$ given by (1.10) is the viscosity solution of (2.12). For any $t_f\in(t,\infty)$, $V(x,i)$ is also the (steady-state) solution of the Cauchy problem

$$\beta V(t;x,i;t_f) + \sup_{(u_1,u_2)}\Big[\mathcal{A}^{(u_1,u_2)}V(t;x,i;t_f) - L(x,i,u_1)\Big] = 0 \quad \text{in} \quad (t,t_f)\times\Omega,$$
$$V(t_f;x,i;t_f) = V(x,i) \quad \text{on} \quad R^n.$$

Now by applying the previous theorem it yields that

$$\mu_1^*(x,i) = -\frac{1}{2}\,R^{-1}(x,i)B^{\mathrm{T}}(x,i)p(x,i), \tag{3.29}$$

$$\mu_2^*(x,i) = \operatorname*{argmin}_{a\in U_2}\Big\{\sum_{j\in\mathcal{S}}\lambda_{iaj}V(x,i)\Big\} \tag{3.30}$$

is a pair of optimal feedback controllers in the time duration $(0,t_f]$ for any $t_f>t$, this leads to

$$V(x,i) \ge E_{x,i}\Big\{\int_t^\infty e^{-\beta(\tau-t)}L(x(\tau),\theta(\tau),\mu_1^*(\tau))\,d\tau\Big\}, \tag{3.31}$$

while Theorem 2.1 implies that the above inequality should be an equality. This yields the conclusion of the theorem.

Remark 3.2 Theorem 3.2 implies that if (2.12) admits a viscosity solution and μ_1^* given in the theorem is an admissible feedback control, then (μ_1^*,μ_2^*) is β-stochastically stabilizable for (1.1).

4 Linear Quadratic Case

In order to make the outcome of the paper more transparent, let us consider the scalar linear-quadratic problem. Let $n=1$ and

$$f(x,i) = A(i)x, \quad B(x,i) = B(i), \quad Q(x,i) = Q(i)x^2, \quad R(x,i) = R(i),$$

and $g(x,i) = Q_{t_f}(i)x^2$ for $i = 1,2,\dots,s$. In this case (2.11) admits a unique viscosity solution V. Moreover, V is convex with respect to x and Lipschitz with respect to t (c.f. [5]). One can show that $x \to p(t,x,i)$ in this case is linear, thus self-adjoint on R, and therefore

$$V(t;x,i;t_f) = P(t,i)x^2 \tag{4.32}$$

for any $x \in R$. Substituting (4.32) into (2.11), we obtain a system of coupled ordinary differential equations

$$\begin{gathered} -\frac{\partial}{\partial t}P(t,i) + \beta P(t,i) - 2A(i)P(t,i) - Q(i) \\ + P^2(t,i)B^2(i)R^{-1}(i) - \inf_{u_2}\sum_{j\in\mathcal{S}}\lambda_{iaj}P(t,j) = 0, \\ P(t_f,i) = Q_{t_f}, \end{gathered} \tag{4.33}$$

for $i = 1,2,\dots,s$. The solution of (4.33) now is in the sense that $P(t,i)$ is absolutely continuous and satisfies (4.33) almost everywhere on $[0,t_f]$. According to Theorem 3.1, the pair of the optimal feedback control $\mu = (\mu_1,\mu_2)$ is given by

$$\mu_1^*(t,x,i) = -R^{-1}(i)B(i)P(t,i)x, \tag{4.34}$$

$$\mu_2^*(t,i) = \operatorname*{argmin}_{a\in U_2}\left\{\sum_{j\in\mathcal{S}}\lambda_{iaj}P(t,j)\right\}. \tag{4.35}$$

Similarly, in the infinite horizon case, for each $i \in \mathcal{S}$

$$V(x,i) = P(i)x^2 \tag{4.36}$$

where $(P(1),P(2),\dots,P(s))$ satisfies a system of algebraic coupled equations

$$\beta P(i) - 2A(i)P(i) - Q(i) + P^2(i)B^2(i)R^{-1}(i) - \inf_{u_2}\sum_{j\in\mathcal{S}}\lambda_{iaj}P(j) = 0 \tag{4.37}$$

for $i = 1,2,\dots,s$. In this case the optimal feedback control $\mu = (\mu_1,\mu_2)$ is given by

$$\mu_1^*(x,i) = -R^{-1}(i)B(i)P(i)x, \tag{4.38}$$

$$\mu_2^*(i) = \operatorname*{argmin}_{a\in U_2}\left\{\sum_{j\in\mathcal{S}}\lambda_{iaj}P(j)\right\} \tag{4.39}$$

from Theorem 3.2.

5 Concluding Remarks

In this paper, we study the optimum stochastic control problem, in which the controls enter through the system dynamics as well as the transitions for the underlying Markov chain process, and are allowed to depend on both the continuous state and the current state of the Markov chain. The structure of the optimal controller is obtained in this paper which therefore makes possible to construct the optimal control by the approach of numerical method.

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