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On the Bounded Oscillation of Certain Fourth Order Functional Differential Equations

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Abstract: Some new criteria for the bounded oscillation of a fourth order functional differential equation are established. Comparison results with first/second order equations as well as necessary and sufficient conditions are developed.

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1 Introduction

In this paper we are concerned with the oscillatory behavior of the fourth order functional differential equations of the type

$$\frac{d}{dt}\left(\frac{1}{a_3(t)}\left(\frac{d}{dt}\left(\frac{1}{a_2(t)}\left(\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\alpha_3}\right)\right) + q(t)f(x[g(t)]) = 0,$$

or, written more compactly as

$$L_4x(t) + q(t)f(x[g(t)]) = 0, \tag{1.1}$$

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where

$$\begin{gathered} L_0x(t) = x(t), \qquad L_4x(t) = \frac{d}{dt}L_3x(t), \\ L_kx(t) = \frac{1}{a_k(t)}\left(\frac{d}{dt}L_{k-1}x(t)\right)^{\alpha_k}, \quad k = 1, 2, 3. \end{gathered} \tag{1.2}$$

In what follows, we shall assume that

(i) $a_i(t), q(t) \in C([t_0, \infty), R^+)$, where $R^+ = (0, \infty)$, $t_0 \geq 0$ and

$$\int^{\infty} a_i^{1/\alpha_i}(s)\, ds = \infty, \quad i = 1, 2, 3; \tag{1.3}$$

(ii) $g(t) \in C([t_0, \infty), R)$, where $R = (-\infty, \infty)$, $g'(t) \geq 0$ for $t \geq t_0$ and $\lim_{t\to\infty} g(t) = \infty$;

(iii) $f \in C(R, R)$, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$;

(iv) α_i, $i = 1, 2, 3$, are the ratios of positive odd integers.

The domain $\mathcal{D}(L_4)$ of L_4 is defined to be the set of all functions $x\colon [t_x, \infty) \to R$, $t_x \geq t_0$ such that $L_jx(t)$, $0 \leq j \leq 4$ exist and are continuous on $[t_x, \infty)$. Our attention is restricted to those solutions $x \in \mathcal{D}(L_4)$ of (1.1) which satisfy $\sup\{|x(t)|\colon t \geq T\} > 0$ for $T \geq t_x$. We make the standing hypothesis that equation (1.1) does possess such solutions.

A solution of equation (1.1) is called *oscillatory* if it has arbitrarily large zeros, otherwise, it is called *nonoscillatory*. Equation (1.1) is called *B-oscillatory* if all its bounded solutions are oscillatory and is called *oscillatory* if all its solutions are oscillatory.

In the last three decades there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of functional differential equations. Most of the work on the subject, however, has been restricted to first and second order equations, as well as, higher order equations of the type

$$L_kx(t) + q(t)f(x[g(t)]) = 0,$$

where

$$L_0x(t) = x(t), \quad L_kx(t) = \frac{1}{a_k(t)}\frac{d}{dt}L_{k-1}x(t), \quad k = 1, 2, \dots, n-1, \quad L_nx(t) = \frac{d}{dt}L_{n-1}x(t).$$

For recent contributions, we refer to [1–13] and the references cited therein.

It appears that little is known regarding the oscillation of equation (1.1). Therefore, our main goal here is to present a systematic study of the oscillation of all bounded solutions of equation (1.1). We shall establish some necessary and sufficient conditions for the bounded oscillation and nonoscillation of equation (1.1). Moreover, our equation is quite general and therefore the results of this paper even in some special cases complement and generalize some known results appeared recently in the literature (see [4–8, 10–13]).

2 Main Results

Consider the inequalities

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right) + q(t)f(x[g(t)]) \le 0, \tag{2.1}$$

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right) + q(t)f(x[g(t)]) \ge 0 \tag{2.2}$$

and the equation

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right) + q(t)f(x[g(t)]) = 0, \tag{2.3}$$

where (ii) and (iii) hold, $a_1(t)$ and α_1 are as in (i) and (iv) respectively.

Now we shall prove the following lemma.

Lemma 2.1 *If inequality (2.1) (inequality (2.2)) has an eventually positive (negative) solution, then equation (2.3) also has an eventually positive (negative) solution.*

Proof Let $x(t)$ be an eventually positive solution of inequality (2.1). It is easy to see that $x'(t) > 0$ eventually. Let

$$y(t) = \frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}.$$

Then,

$$x'(t) = (a_1(t)y(t))^{1/\alpha_1} \ge 0 \quad \text{for} \quad t \ge t_0 \ge 0. \tag{2.4}$$

Integrating (2.4) from t_0 to t, we have

$$x(t) = x(t_0) + \int_{t_0}^{t} (a_1(s)y(s))^{1/\alpha_1}\, ds.$$

Thus, (2.1) becomes

$$\frac{dy}{dt} + q(t)f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)y(s))^{1/\alpha_1} ds\right) \le 0. \tag{2.5}$$

Integrating (2.5) from t to $T \ge t \ge t_0$ and letting $T \to \infty$, we have

$$y(t) \ge \int_t^\infty q(u)f\left(x(t_0) + \int_{t_0}^{g(u)} (a_1(s)y(s))^{1/\alpha_1}\, ds\right) du.$$

Next, we define a sequence of successive approximations $\{z_j(t)\}$ as follows:

$$z_0(t) = y(t),$$

$$z_{j+1}(t) = \int_t^\infty q(u)f\left(x(t_0) + \int_{t_0}^{g(u)} (a_1(s)z_j(s))^{1/\alpha_1} ds\right) du, \quad j = 0, 1, \dots .$$

Obviously, we can prove that

$$0 < z_j(t) \le y(t) \quad \text{and} \quad z_{j+1}(t) \le z_j(t), \quad j = 0, 1, \dots .$$

Thus the sequence $\{z_j(t)\}$ is positive nonincreasing in j for each $t \ge t_0$. This means we may define $z(t) = \lim_{j\to\infty} z_j(t) > 0$. Since $0 < z(t) \le z_j(t) \le y(t)$ for all $j \ge 0$, we see that

$$f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)z_j(s))^{1/\alpha_1}\, ds\right) \le f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)y(s))^{1/\alpha_1}\, ds\right).$$

Now, by the Lebesgue dominated convergence theorem, one can easily obtain

$$z(t) = \int_t^\infty q(u) f\left(x(t_0) + \int_{t_0}^{g(u)} (a_1(s)z(s))^{1/\alpha_1}\, ds\right) du.$$

Therefore,

$$\frac{dz}{dt} = -q(t) f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)z(s))^{1/\alpha_1}\, ds\right). \tag{2.6}$$

We denote by

$$v(t) = x(t_0) + \int_{t_0}^{t} (a_1(s)z(s))^{1/\alpha_1}\, ds.$$

Then, $v(t) > 0$ and

$$\frac{dv}{dt} = (a_1(t)z(t))^{1/\alpha_1},$$

or

$$z(t) = \frac{1}{a_1(t)}\left(\frac{dv}{dt}\right)^{\alpha_1}.$$

Equation (2.6) then gives

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{dv}{dt}\right)^{\alpha_1}\right) + q(t) f(v[g(t)]) = 0.$$

Hence, equation (2.3) has a positive solution $v(t)$. For the case (2.2) the argument is similar and hence is omitted. This completes the proof.

We set

$$Q(t) = a_2^{1/\alpha_2}(t)\left(\int_t^\infty a_3^{1/\alpha_3}(s)\left(\int_s^\infty q(u)\, du\right)^{1/\alpha_3} ds\right)^{1/\alpha_2}, \quad t \ge t_0 \ge 0,$$

and $F(x) = f^{1/(\alpha_2\alpha_3)}(x)$, $x \in R$.

Now, we present the following comparison result.

Theorem 2.1 *Let conditions (i) – (iv) hold. If the equation*

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right) + Q(t)F(x[g(t)]) = 0 \tag{2.7}$$

is oscillatory, then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. By condition (1.3), it is easy seen that $x(t)$ satisfies the inequalities

$$x'(t) > 0, \quad L_2x(t) < 0, \quad L_3x(t) > 0 \quad \text{and} \quad L_4x(t) \leq 0 \quad \text{for} \quad t \geq t_1 \geq t_0. \tag{2.8}$$

Integrating equation (1.1) from t to $T \geq t \geq t_1$ and letting $T \to \infty$, we find

$$L_3x(t) \geq \int_t^\infty q(s)f(x[g(s)])\,ds,$$

or

$$\frac{1}{a_3(t)}\left(\frac{d}{dt}L_2x(t)\right)^{\alpha_3} \geq \left(\int_t^\infty q(s)\,ds\right)f(x[g(t)]).$$

Thus,

$$\frac{d}{dt}L_2x(t) \geq a_3^{1/\alpha_3}(t)\left(\int_t^\infty q(s)\,ds\right)^{1/\alpha_3} f^{1/\alpha_3}(x[g(t)]), \quad t \geq t_1. \tag{2.9}$$

Once again, we integrate (2.9) from t to $T_1 \geq t \geq t_1$ and let $T_1 \to \infty$, to obtain

$$-L_2x(t) \geq \left(\int_t^\infty a_3^{1/\alpha_3}(u)\left(\int_u^\infty q(s)\,ds\right)^{1/\alpha_3} du\right) f^{1/\alpha_3}(x[g(t)]), \quad t \geq t_1,$$

or

$$\begin{aligned} -\frac{d}{dt}L_1x(t) &\geq a_2^{1/\alpha_2}(t)\left(\int_t^\infty a_3^{1/\alpha_3}(u)\left(\int_u^\infty q(s)\,ds\right)^{1/\alpha_3} du\right)^{1/\alpha_2} f^{1/(\alpha_2\alpha_3)}(x[g(t)]) \\ &= Q(t)F(x[g(t)]), \end{aligned} \tag{2.10}$$

for $t \geq t_1$. By applying Lemma 2.1, we see that equation (2.7) has a positive solution, a contradiction. This completes the proof.

Now we assume that the function $F(x) = f^{1/(\alpha_2\alpha_3)}(x)$, $x \in R$, satisfies

$$-F(-xy) \geq F(xy) \geq F(x)F(y) \quad \text{for} \quad xy > 0 \tag{2.11}$$

and

$$g(t) \leq t. \tag{2.12}$$

Also, we let

$$\eta[t, t_0] = \int_{t_0}^t a_1^{1/\alpha_1}(s)\,ds$$

and for $g(t) \geq T$ for some $T \geq t_0$,

$$\overline{Q}(t) = Q(t)F(\eta[g(t), T]).$$

Now, we present the following result.

Theorem 2.2 *Let conditions (i) – (iv), (2.11) and (2.12) hold. If the first order equation*

$$\frac{d}{dt}y(t) + \overline{Q}(t)F\Big(y^{1/\alpha_1}[g(t)]\Big) = 0 \tag{2.13}$$

is oscillatory, then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.1, we obtain (2.8) and (2.10) for $t \geq t_1$. Now

$$x(t) - x(t_1) = \int_{t_1}^{t} x'(s)\,ds = \int_{t_1}^{t} \Big(a_1^{-1/\alpha_1}(s)x'(s)\Big)\, a_1^{1/\alpha_1}(s)\,ds.$$

Using the fact that $a_1^{-1/\alpha_1}(t)x'(t)$ is nonincreasing on $[t_1, \infty)$, we find

$$x(t) \geq \Big(a_1^{-1/\alpha_1}(t)x'(t)\Big) \int_{t_1}^{t} a_1^{1/\alpha_1}(s)\,ds,$$

or

$$x(t) \geq \eta[t, t_1]\Big(a_1^{-1/\alpha_1}(t)x'(t)\Big) \quad \text{for} \quad t \geq t_1.$$

Thus, there exists a $t_2 \geq t_1$ such that

$$x[g(t)] \geq \eta[g(t), t_1]\big(Z^{1/\alpha_1}[g(t)]\big) \quad \text{for} \quad t \geq t_2, \tag{2.14}$$

where $Z(t) = (x'(t))^{\alpha_1}/a_1(t)$, $t \geq t_2$. Using (2.11) and (2.14) in (2.10) we get

$$\frac{d}{dt}Z(t) + \overline{Q}(t)F\big(Z^{1/\alpha_1}[g(t)]\big) \leq 0 \quad \text{for} \quad t \geq t_2. \tag{2.15}$$

Integrating (2.15) from t to $T \geq t \geq t_2$ and letting $T \to \infty$, we obtain

$$Z(t) \geq \int_{t}^{\infty} \overline{Q}(s)F\big(Z^{1/\alpha_1}[g(s)]\big)\,ds.$$

As in [9, 12], it is now easy to conclude that there exists a positive solution $y(t)$ of the equation (2.13) with $\lim\limits_{t\to\infty} y(t) = 0$. This contradicts the hypothesis and completes the proof.

By using a well known oscillation result in [9, Corollary 7.6.1], the following corollary is immediate.

Corollary 2.1 *Let conditions (i) – (iv), (2.11) and (2.12) hold. Then, equation (1.1) is B-oscillatory if one of the following conditions holds:*

(I$_1$) $F(y^{1/\alpha_1})/y \geq k > 0$, $y \neq 0$, *where k is a constant,* (2.16)

and

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \overline{Q}(s)\,ds > \frac{1}{ek}. \tag{2.17}$$

(I$_2$) $\displaystyle\int\limits_{\pm 0} \frac{du}{F(u^{1/\alpha_1})} < \infty,$ (2.18)

and

$$\int^{\infty} \overline{Q}(s)\,ds = \infty. \tag{2.19}$$

Next, we let $\overline{F}(x) = f^{1/(\alpha_1\alpha_2\alpha_3)}(x)$, $x \in R$ and assume that

$$\int^{\pm\infty} \frac{du}{\overline{F}(u)} < \infty. \tag{2.20}$$

Now, we prove the following oscillation result.

Theorem 2.3 *Let conditions (i) – (iv), (2.12) and (2.20) hold. If*

$$\int^{\infty} g'(u) a_1^{1/\alpha_1}[g(u)] \left(\int_u^{\infty} Q(s)\,ds \right)^{1/\alpha_1} du = \infty, \tag{2.21}$$

then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.1, we obtain (2.10) for $t \geq t_1 \geq t_0$. Now, one can easily see that

$$L_1 x(t) \geq \left(\int_t^{\infty} Q(s)\,ds \right) F(x[g(t)]), \tag{2.22}$$

or

$$a_1^{-1/\alpha_1}[g(t)] x'[g(t)] \geq a_1^{-1/\alpha_1}(t) x'(t) \geq \left(\int_t^{\infty} Q(s)ds \right)^{1/\alpha_1} \overline{F}(x[g(t)])$$

for $t \geq t_2 \geq t_1$. Hence, it follows that

$$\frac{x'[g(t)]g'(t)}{\overline{F}(x[g(t)])} \geq g'(t) a_1^{1/\alpha_1}[g(t)] \left(\int_t^{\infty} Q(s)\,ds \right)^{1/\alpha_1} \quad \text{for} \quad t \geq t_2. \tag{2.23}$$

Integrating both sides of (2.23) from t_2 to t, we get

$$\int_{t_2}^{t} g'(u) a_1^{1/\alpha_1}[g(u)] \left(\int_u^{\infty} Q(s)\,ds \right)^{1/\alpha_1} du \leq \int_{x[g(t_2)]}^{x[g(t)]} \frac{dv}{\overline{F}(v)} \leq \int_{x[g(t_2)]}^{\infty} \frac{dv}{\overline{F}(v)} < \infty,$$

which contradicts condition (2.21). This completes the proof.

In [5], we have compared the oscillation of nonlinear equations of type (2.7) with those of second order linear equations. In fact, we obtained the following results.

Lemma 2.2 *Let* $0 < \alpha_1 \le 1$, $g'(t) > 0$ *for* $t \ge t_0$, $0 < \overline{q}(t) = \int_t^\infty Q(s)\,ds < \infty$ *and* $F(x) = x^\beta$, *where* β *is the ratio of positive odd integers. Then, equation (2.7) is oscillatory if for all large* t, *the linear second order equation*

$$\left(\frac{C(t)}{g'(t)}\left(\frac{(\overline{q}(t))^{\alpha_1-1}}{a_1[g(t)]}\right)^{1/\alpha_1} y'(t)\right)' + \beta Q(t)y(t) = 0 \tag{2.24}$$

is oscillatory, where

$$C(t) = \begin{cases} c_1, \ \ c_1 > 0 \quad \text{is any constant,} & \text{when} \quad \beta > \alpha_1, \\ 1, & \text{when} \quad \beta = \alpha_1, \\ c_2\eta^{(\alpha_1-\beta)/\alpha_1}[g(t), t_0], \ \ c_2 > 0 \quad \text{is any constant,} & \text{when} \quad \beta < \alpha_1. \end{cases}$$

Lemma 2.3 *Let* $\alpha_1 \ge 1$, $g'(t) > 0$ *for* $t \ge t_0$ *and* $F(x) = x^\beta$, *where* β *is the ratio of positive odd integers. Then, equation (2.7) is oscillatory if for all large* t, *the linear second order equation*

$$\left(\frac{\overline{C}(t)}{a_1^{1/\alpha_1}[g(t)]g'(t)\eta^{\alpha_1-1}[g(t), t_0]} Z'(t)\right)' + \beta Q(t)Z(t) = 0 \tag{2.25}$$

is oscillatory, where

$$\overline{C}(t) = \begin{cases} c_1, \ \ c_1 > 0 \quad \text{is any constant,} & \text{when} \quad \beta > \alpha_1, \\ 1, & \text{when} \quad \beta = \alpha_1, \\ c_2\eta^{\alpha_1-\beta}[g(t), t_0], \ \ c_2 > 0 \quad \text{is any constant,} & \text{when} \quad \beta < \alpha_1. \end{cases}$$

By Lemmas 2.2 and 2.3 we can replace equation (2.7) in Theorem 2.1 by equation (2.24), or equation (2.25). The statements and formulations of the results are left to the reader.

Next, we present the following result.

Theorem 2.4 *Let conditions (i) – (iv) hold. If*

$$\int^\infty a_1^{1/\alpha_1}(u)\left(\int_u^\infty Q(s)\,ds\right)^{1/\alpha_1} du = \infty, \tag{2.26}$$

then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 \ge 0$. As in the proof of Theorem 2.3, we obtain (2.22) for $t \ge t_1$. Since $x(t)$ is an increasing function on $[t_1, \infty)$, there exist a $t_2 \ge t_1$ and a constant $C > 0$ such that

$$x[g(t)] \ge C \quad \text{for} \quad t \ge t_2. \tag{2.27}$$

Using (2.27) in (2.22), one can easily see that

$$x'(t) \ge a_1^{1/\alpha_1}(t)\left(\int_t^\infty Q(s)\,ds\right)^{1/\alpha_1} \overline{F}(c), \quad t \ge t_2.$$

Integrating the above inequality from t_2 to t and using (2.26) we arrive at the desired contradiction.

Next, we will give some necessary and sufficient conditions for all bounded solutions of equation (1.1) to be oscillatory or nonoscillatory.

Theorem 2.5 *Let conditions (i) – (iv) hold. Then, equation (1.1) is B-oscillatory if and only if condition (2.26) is satisfied.*

Proof Suppose that (2.26) holds and assume that equation (1.1) has a bounded nonoscillatory solution $x(t)$. The proof is similar to that of Theorem 2.4 and hence omitted.

Assume that (2.26) does not hold. We may suppose that

$$\int_{t_0}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1<\infty,\quad t_0\ge 0. \tag{2.28}$$

Then, we can choose $T\ge t_0$ sufficiently large such that for $t\ge T$,

$$\int_{T}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}f(\gamma)q(s)\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1<\frac{\gamma}{2} \tag{2.29}$$

for some constant $\gamma>0$. Let $x(t)$ be a solution of the following equation

$$x(t)=\gamma-\int_{t}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1. \tag{2.30}$$

Then we easily see that $x(t)$ is a solution of equation (1.1). Next, we shall show that equation (2.30) has a bounded nonoscillatory solution $x(t)$ by using the fixed point theorem of Schauder.

We introduce the Banach space X of all continuous and bounded real-valued functions on the interval $[t_0,\infty)$ endowed with the usual sup norm $\|\cdot\|$. We define a bounded, convex and closed subset $\mathcal{B}$ of X as

$$\mathcal{B}=\left\{x\in X\colon\ \frac{\gamma}{2}\le x(t)\le\gamma,\quad t\ge t_0\right\}.$$

Next, let S be a mapping defined on $\mathcal{B}$ as follows: For $x\in\mathcal{B}$,

$$(Sx)(t)=\begin{cases}\gamma-\displaystyle\int_{t}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1, & t\ge T,\\ (Sx)(T), & t_0\le t\le T.\end{cases} \tag{2.31}$$

Then the mapping S satisfies the following:

(I_1) S maps $\mathcal{B}$ into $\mathcal{B}$. In fact, for any $x\in\mathcal{B}$, from (2.29) and (2.31) we have

$$\gamma\ge(Sx)(t)\ge\gamma-\frac{\gamma}{2}=\frac{\gamma}{2},\quad t\ge t_0.$$

So $Sx \in \mathcal{B}$.

(I_2) The mapping S is continuous on $\mathcal{B}$. Let $x \in \mathcal{B}$ and $\{x_j\}$ be a sequence in $\mathcal{B}$ converging to x. We shall show that Sx_j converges to Sx. By (2.29), for any $\epsilon > 0$, we can choose $T_0 \geq T$ such that

$$\int_{T_0}^{\infty} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 < \frac{\epsilon}{3}. \quad (2.32)$$

Furthermore, we can see that the series $f(x_j)$ converges to $f(x)$ uniformly with respect to j. So, we can choose m such that for all $j \geq m$,

$$\Bigg| \int_{t_0}^{T_0} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x_j[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1$$
$$- \int_{t_0}^{T_0} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \Bigg| < \frac{\epsilon}{3}. \quad (2.33)$$

In the following, we shall show that $|(Sx_j)(t) - (Sx)(t)| < \epsilon$ for any t and $j \geq m$.

(i) If $t \geq T_0$, then from (2.31) and (2.32), we can easily find

$$|(Sx_j)(t) - (Sx)(t)|$$
$$\leq 2 \left| \int_{t}^{\infty} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|$$
$$< \frac{2\epsilon}{3} < \epsilon \quad \text{for} \quad j \geq m.$$

(ii) If $t \leq T_0$, from (2.31), (2.32) and (2.33), we have

$$|(Sx_j)(t) - (Sx)(t)|$$
$$\leq \Bigg| \int_{t}^{T_0} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x_j[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1$$
$$- \int_{t}^{T_0} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \Bigg|$$
$$+ \left| \int_{T_0}^{\infty} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x_j[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|$$
$$+ \left| \int_{T_0}^{\infty} \left(a_1(s_1) \int_{s_1}^{\infty} \left(a_2(s_2) \int_{s_2}^{\infty} \left(a_3(s_3) \int_{s_3}^{\infty} q(s) f(x[g(s)])\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{for} \quad j \ge m.$$

Clearly, (i) and (ii) together yield that $|(Sx_j)(t) - (Sx)(t)| < \epsilon$ for any t and $j \ge m$ which completes the proof that the mapping S is continuous on $\mathcal{B}$.

(I_3) The set $S(\mathcal{B})$ is relatively compact. For any $x \in \mathcal{B}$ and every $t \ge t_0$, we have $|(Sx)(t)| \le \gamma$. Therefore, $S\mathcal{B}$ is uniformly bounded. Furthermore, we find

$$|(Sx)(t) - \gamma| \le \left| \int_t^\infty \left(a_1(s_1) \int_{s_1}^\infty \left(a_2(s_2) \int_{s_2}^\infty \left(a_3(s_3) \int_{s_3}^\infty q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|. \tag{2.34}$$

Thus, from (2.28) and (2.34), we conclude that $S\mathcal{B}$ is equiconvergent at ∞. Now, for any $x \in \mathcal{B}$ and every t_1, t_2 with $T \le t_1 \le t_2$, we get

$$\begin{aligned} &|(Sx)(t_2) - (Sx)(t_1)| \\ &\quad \le \left| \int_{t_1}^{t_2} \left(a_1(s_1) \int_{s_1}^\infty \left(a_2(s_2) \int_{s_2}^\infty \left(a_3(s_3) \int_{s_3}^\infty q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|. \end{aligned}$$

From this it follows that $S\mathcal{B}$ is equicontinuous. Finally, by the given compactness criterion (see [13]), we conclude that $S\mathcal{B}$ is relatively compact.

Thus, by the Schauder fixed point theorem [13], it follows that (2.30) has a positive solution $x(t)$. This proves the necessity.

The following theorem provides a necessary and sufficient condition for the existence of a bounded solution of equation (1.1).

Theorem 2.6 *Assume that (i) – (iv) except condition (1.3) hold, and*

$$\int^\infty q(s)\, ds = \infty. \tag{2.35}$$

Then a necessary and sufficient condition for equation (1.1) to have a positive solution $x(t)$ which satisfies $\beta_2 \ge x(t) \ge \beta_1 > 0$ (β_1 and β_2 are constants) for $t \ge t_0$ is that

$$\int_{t_0}^\infty \left(a_1(s_1) \int_{t_0}^{s_1} \left(a_2(s_2) \int_{t_0}^{s_2} \left(a_3(s_3) \int_{t_0}^{s_3} q(s)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 < \infty. \tag{2.36}$$

Proof **Necessity** If $x(t)$ is a positive solution of equation (1.1) and the condition $\beta_2 \ge x(t) \ge \beta_1 > 0$ is satisfied, then we have in view of equation (1.1),

$$L_3 x(t) = L_3 x(t_0) - \int_{t_0}^t q(s) f(x[g(s)])\, ds \le L_3 x(t_0) - f(\beta_1) \int_{t_0}^t q(s)\, ds.$$

If t is large enough, in view of (2.35), we have $L_3x(t) < 0$. Then, for all large t_0,

$$L_3x(t) < -f(\beta_1)\int_{t_0}^{t} q(s)\,ds,$$

or

$$\frac{d}{dt}\,L_2x(t) < -f^{1/\alpha_3}(\beta_1)\left(a_3(t)\int_{t_0}^{t} q(s)\,ds\right)^{1/\alpha_3}.$$

The rest of the proof is similar to the proof of the sufficiency part of Theorem 2.5 and hence omitted.

The proof of sufficiency is similar to the proof of necessity part of Theorem 2.5. This completes the proof.

Remark 2.1 From the above study of B–oscillation of equation (1.1), we are concerned with the nonexistence of solutions of equation (1.1) satisfying (2.8). This class of solutions of (1.1) may include some unbounded solutions. Therefore, some modification in the definition of B-oscillation of equation (1.1) is required to include bounded as well as some unbounded solutions of equation (1.1). The details are left to the reader.

Remark 2.2 The results of this paper can be extended to neutral equations of the form

$$L_4(x(t) + p(t)x[\tau(t)]) + q(t)f(x[g(t)]) = 0, \tag{2.37}$$

where $p(t) \in C([t_0,\infty),[0,\infty))$ and $\tau(t) \in C([t_0,\infty),R)$, $\tau'(t) > 0$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = 0$. Here, we refer to our papers [4–6] and omit the details.

The following example illustrates some of the results obtained.

Example 2.1 Consider the differential equation

$$\frac{d}{dt}\left(\frac{1}{t^2}\left(\frac{d}{dt}\left(t\left(\frac{d}{dt}\left(t\left(\frac{d}{dt}x(t)\right)^3\right)\right)\right)^3\right)\right) + \frac{2}{t^4}\,x(t) = 0. \tag{2.38}$$

This is actually (1.1) with

$$\alpha_1 = 3, \quad \alpha_2 = 1, \quad \alpha_3 = 3, \quad a_1(t) = \frac{1}{t}, \quad a_2(t) = \frac{1}{t}, \quad a_3(t) = t^2,$$
$$q(t) = \frac{2}{t^4}, \quad g(t) = t, \quad f(x) = x.$$

By direct computation we obtain

$$Q(t) = \frac{1}{2}\,t^{-7/3}, \quad \eta[g(t),T] \leq \frac{3}{2}\,t^{2/3}, \quad \overline{Q}(t) = Q(t)F(\eta[g(t),T]) \leq t^{-19/9}.$$

Clearly, conditions (i)$-$(iv), (2.11) and (2.12) are fulfilled. Further, it can be easily checked that (2.17) is not satisfied, and also

$$\int^{\infty} \overline{Q}(s)\,ds \leq \int^{\infty} s^{-19/9}\,ds < \infty$$

which implies (2.19) is not met. Thus, we see that both conditions (I_1) and (I_2) of Corollary 2.1 are not fulfilled.

Moreover, we can verify easily that condition (2.20) is not satisfied but (2.21) and (2.26) are met. Thus, the conditions of Theorem 2.3 are not all satisfied, whereas those of Theorems 2.4 and 2.5 are fulfilled.

Hence, on one hand we *cannot* conclude from Corollary 2.1 and Theorem 2.3 that (2.38) is B-oscillatory, while on the other hand Theorems 2.4 and 2.5 give that (2.38) is B-oscillatory. In fact, we observe that (2.38) has a solution given by $x(t) = t$, which is unbounded and nonoscillatory.

References

[1] Agarwal, R.P., Grace, S.R. and O'Regan, D. *Oscillation Theory for Difference and Functional Differential Equations.* Kluwer, Dordrecht, 2000.

[2] Agarwal, R.P., Grace, S.R. and O'Regan, D. *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations.* Kluwer, Dordrecht, 2002.

[3] Agarwal, R.P., Grace, S.R. and O'Regan, D. *Oscillation Theory for Second Order Dynamic Equations.* Taylor & Francis, U.K., 2003.

[4] Agarwal, R.P., Grace, S.R. and O'Regan, D. On the oscillation of certain functional differential equations via comparison methods. *J. Math. Anal. Appl.* **286** (2003) 577–600.

[5] Agarwal, R.P., Grace, S.R. and O'Regan, D. Linearization of second order sublinear oscillation theorems. *Communications in Appl. Anal.* **8** (2004) 219–235.

[6] Agarwal, R.P., Grace, S.R. and O'Regan, D. The oscillation of certain higher order functional differential equations. *Mathl. Comput. Modelling* **37** (2003) 705–728.

[7] Agarwal, R.P., Grace, S.R. and O'Regan, D. On the oscillation of second order functional differential equations. *Adv. Math. Sci. Appl.* **12** (2002) 257–272.

[8] Agarwal, R.P., Grace, S.R. and O'Regan, D. Nonoscillatory solutions for higher order dynamic equations. *J. London Math. Soc.* **67** (2003) 165–179.

[9] Györi, I. and Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications.* Clarendon Press, Oxford, 1991.

[10] Kitamura, Y. Oscillation of functional differential equations with general deviating arguments. *Hiroshima Math. J.* **15** (1985) 445–491.

[11] Kusano, T. and Naito, M. On fourth order nonlinear oscillations. *J. London Math. Soc.* **14**(2) (1976) 91–105.

[12] Philos, Ch.G. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math.* **36** (1981) 168–178.

[13] Philos, Ch.G. Oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments. *Hiroshima Math. J.* **8** (1978) 31–48.