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On the Bounded Oscillation of Certain Fourth Order Functional Differential Equations

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Abstract: Some new criteria for the bounded oscillation of a fourth order functional differential equation are established. Comparison results with first/second order equations as well as necessary and sufficient conditions are developed.

Keywords: *Oscillation; nonoscillation; half-linear; comparison; necessary conditions.*

Mathematics Subject Classification (2000): 34C10, 34C15.

1 Introduction

In this paper we are concerned with the oscillatory behavior of the fourth order functional differential equations of the type

$$\frac{d}{dt}\left(\frac{1}{a_3(t)}\left(\frac{d}{dt}\left(\frac{1}{a_2(t)}\left(\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right)^{\alpha_3}\right)\right) + q(t)f(x[g(t)]) = 0,$$

or, written more compactly as

$$L_4x(t) + q(t)f(x[g(t)]) = 0, \tag{1.1}$$

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where

$$
\begin{aligned}
L_0x(t) = x(t), \qquad L_4x(t) = \frac{d}{dt}L_3x(t),\\
L_kx(t) = \frac{1}{a_k(t)}\left(\frac{d}{dt}L_{k-1}x(t)\right)^{\alpha_k}, \quad k = 1,2,3.
\end{aligned}
\tag{1.2}
$$

In what follows, we shall assume that

(i) $a_i(t), q(t) \in C([t_0,\infty),\, R^+)$, where $R^+ = (0,\infty)$, $t_0 \geq 0$ and

$$\int^{\infty} a_i^{1/\alpha_i}(s)\, ds = \infty, \quad i = 1,2,3; \tag{1.3}$$

(ii) $g(t) \in C([t_0,\infty),\, R)$, where $R = (-\infty,\infty)$, $g'(t) \geq 0$ for $t \geq t_0$ and $\lim_{t\to\infty} g(t) = \infty$;

(iii) $f \in C(R,R)$, $xf(x) > 0$ and $f'(x) \geq 0$ for $x \neq 0$;

(iv) α_i, $i = 1,2,3$, are the ratios of positive odd integers.

The domain $\mathcal{D}(L_4)$ of L_4 is defined to be the set of all functions $x\colon [t_x,\infty) \to R$, $t_x \geq t_0$ such that $L_jx(t)$, $0 \leq j \leq 4$ exist and are continuous on $[t_x,\infty)$. Our attention is restricted to those solutions $x \in \mathcal{D}(L_4)$ of (1.1) which satisfy $\sup\{|x(t)|\colon\ t \geq T\} > 0$ for $T \geq t_x$. We make the standing hypothesis that equation (1.1) does possess such solutions.

A solution of equation (1.1) is called *oscillatory* if it has arbitrarily large zeros, otherwise, it is called *nonoscillatory*. Equation (1.1) is called *B-oscillatory* if all its bounded solutions are oscillatory and is called *oscillatory* if all its solutions are oscillatory.

In the last three decades there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of solutions of functional differential equations. Most of the work on the subject, however, has been restricted to first and second order equations, as well as, higher order equations of the type

$$L_kx(t) + q(t)f(x[g(t)]) = 0,$$

where

$$L_0x(t) = x(t), \quad L_kx(t) = \frac{1}{a_k(t)}\frac{d}{dt}L_{k-1}x(t), \quad k = 1,2,\ldots,n-1, \quad L_nx(t) = \frac{d}{dt}L_{n-1}x(t).$$

For recent contributions, we refer to [1–13] and the references cited therein.

It appears that little is known regarding the oscillation of equation (1.1). Therefore, our main goal here is to present a systematic study of the oscillation of all bounded solutions of equation (1.1). We shall establish some necessary and sufficient conditions for the bounded oscillation and nonoscillation of equation (1.1). Moreover, our equation is quite general and therefore the results of this paper even in some special cases complement and generalize some known results appeared recently in the literature (see [4–8, 10–13]).

2 Main Results

Consider the inequalities

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)+q(t)f(x[g(t)])\le 0, \tag{2.1}$$

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)+q(t)f(x[g(t)])\ge 0 \tag{2.2}$$

and the equation

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)+q(t)f(x[g(t)])= 0, \tag{2.3}$$

where (ii) and (iii) hold, $a_1(t)$ and α_1 are as in (i) and (iv) respectively.

Now we shall prove the following lemma.

Lemma 2.1 *If inequality (2.1) (inequality (2.2)) has an eventually positive (negative) solution, then equation (2.3) also has an eventually positive (negative) solution.*

Proof Let $x(t)$ be an eventually positive solution of inequality (2.1). It is easy to see that $x'(t)>0$ eventually. Let

$$y(t)=\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}.$$

Then,

$$x'(t)=(a_1(t)y(t))^{1/\alpha_1}\ge 0 \quad \text{for} \quad t\ge t_0\ge 0. \tag{2.4}$$

Integrating (2.4) from t_0 to t, we have

$$x(t)=x(t_0)+\int_{t_0}^{t}(a_1(s)y(s))^{1/\alpha_1}\,ds.$$

Thus, (2.1) becomes

$$\frac{dy}{dt}+q(t)f\left(x(t_0)+\int_{t_0}^{g(t)}(a_1(s)y(s))^{1/\alpha_1}ds\right)\le 0. \tag{2.5}$$

Integrating (2.5) from t to $T\ge t\ge t_0$ and letting $T\to\infty$, we have

$$y(t)\ge\int_t^{\infty}q(u)f\left(x(t_0)+\int_{t_0}^{g(u)}(a_1(s)y(s))^{1/\alpha_1}\,ds\right)du.$$

Next, we define a sequence of successive approximations $\{z_j(t)\}$ as follows:

$$z_0(t)=y(t),$$

$$z_{j+1}(t)=\int_t^{\infty}q(u)f\left(x(t_0)+\int_{t_0}^{g(u)}(a_1(s)z_j(s))^{1/\alpha_1}ds\right)du, \quad j=0,1,\dots.$$

Obviously, we can prove that

$$0 < z_j(t) \le y(t) \quad \text{and} \quad z_{j+1}(t) \le z_j(t), \quad j = 0, 1, \dots .$$

Thus the sequence $\{z_j(t)\}$ is positive nonincreasing in j for each $t \ge t_0$. This means we may define $z(t) = \lim_{j\to\infty} z_j(t) > 0$. Since $0 < z(t) \le z_j(t) \le y(t)$ for all $j \ge 0$, we see that

$$f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)z_j(s))^{1/\alpha_1}\, ds\right) \le f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)y(s))^{1/\alpha_1}\, ds\right).$$

Now, by the Lebesgue dominated convergence theorem, one can easily obtain

$$z(t) = \int_t^\infty q(u) f\left(x(t_0) + \int_{t_0}^{g(u)} (a_1(s)z(s))^{1/\alpha_1}\, ds\right) du.$$

Therefore,

$$\frac{dz}{dt} = -q(t) f\left(x(t_0) + \int_{t_0}^{g(t)} (a_1(s)z(s))^{1/\alpha_1}\, ds\right). \tag{2.6}$$

We denote by

$$v(t) = x(t_0) + \int_{t_0}^{t} (a_1(s)z(s))^{1/\alpha_1}\, ds.$$

Then, $v(t) > 0$ and

$$\frac{dv}{dt} = (a_1(t)z(t))^{1/\alpha_1},$$

or

$$z(t) = \frac{1}{a_1(t)}\left(\frac{dv}{dt}\right)^{\alpha_1}.$$

Equation (2.6) then gives

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{dv}{dt}\right)^{\alpha_1}\right) + q(t) f(v[g(t)]) = 0.$$

Hence, equation (2.3) has a positive solution $v(t)$. For the case (2.2) the argument is similar and hence is omitted. This completes the proof.

We set

$$Q(t) = a_2^{1/\alpha_2}(t)\left(\int_t^\infty a_3^{1/\alpha_3}(s)\left(\int_s^\infty q(u)\, du\right)^{1/\alpha_3} ds\right)^{1/\alpha_2}, \quad t \ge t_0 \ge 0,$$

and $F(x) = f^{1/(\alpha_2\alpha_3)}(x)$, $x \in R$.

Now, we present the following comparison result.

Theorem 2.1 *Let conditions (i) – (iv) hold. If the equation*

$$\frac{d}{dt}\left(\frac{1}{a_1(t)}\left(\frac{d}{dt}x(t)\right)^{\alpha_1}\right)+Q(t)F(x[g(t)])=0 \tag{2.7}$$

is oscillatory, then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t)>0$ for $t\ge t_0\ge 0$. By condition (1.3), it is easy seen that $x(t)$ satisfies the inequalities

$$x'(t)>0,\quad L_2x(t)<0,\quad L_3x(t)>0 \quad\text{and}\quad L_4x(t)\le 0 \quad\text{for}\quad t\ge t_1\ge t_0. \tag{2.8}$$

Integrating equation (1.1) from t to $T\ge t\ge t_1$ and letting $T\to\infty$, we find

$$L_3x(t)\ge\int_t^\infty q(s)f(x[g(s)])\,ds,$$

or

$$\frac{1}{a_3(t)}\left(\frac{d}{dt}L_2x(t)\right)^{\alpha_3}\ge\left(\int_t^\infty q(s)\,ds\right)f(x[g(t)]).$$

Thus,

$$\frac{d}{dt}L_2x(t)\ge a_3^{1/\alpha_3}(t)\left(\int_t^\infty q(s)\,ds\right)^{1/\alpha_3}f^{1/\alpha_3}(x[g(t)]),\quad t\ge t_1. \tag{2.9}$$

Once again, we integrate (2.9) from t to $T_1\ge t\ge t_1$ and let $T_1\to\infty$, to obtain

$$-L_2x(t)\ge\left(\int_t^\infty a_3^{1/\alpha_3}(u)\left(\int_u^\infty q(s)\,ds\right)^{1/\alpha_3}du\right)f^{1/\alpha_3}(x[g(t)]),\quad t\ge t_1,$$

or

$$\begin{aligned}-\frac{d}{dt}L_1x(t)&\ge a_2^{1/\alpha_2}(t)\left(\int_t^\infty a_3^{1/\alpha_3}(u)\left(\int_u^\infty q(s)\,ds\right)^{1/\alpha_3}du\right)^{1/\alpha_2}f^{1/(\alpha_2\alpha_3)}(x[g(t)])\\&=Q(t)F(x[g(t)]),\end{aligned} \tag{2.10}$$

for $t\ge t_1$. By applying Lemma 2.1, we see that equation (2.7) has a positive solution, a contradiction. This completes the proof.

Now we assume that the function $F(x)=f^{1/(\alpha_2\alpha_3)}(x)$, $x\in R$, satisfies

$$-F(-xy)\ge F(xy)\ge F(x)F(y)\quad\text{for}\quad xy>0 \tag{2.11}$$

and

$$g(t)\le t. \tag{2.12}$$

Also, we let

$$\eta[t,t_0]=\int_{t_0}^t a_1^{1/\alpha_1}(s)\,ds$$

and for $g(t)\ge T$ for some $T\ge t_0$,

$$\overline{Q}(t)=Q(t)F(\eta[g(t),T]).$$

Now, we present the following result.

Theorem 2.2 *Let conditions (i) – (iv), (2.11) and (2.12) hold. If the first order equation*

$$\frac{d}{dt}y(t) + \overline{Q}(t)F\Big(y^{1/\alpha_1}[g(t)]\Big) = 0 \tag{2.13}$$

is oscillatory, then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.1, we obtain (2.8) and (2.10) for $t \geq t_1$. Now

$$x(t) - x(t_1) = \int_{t_1}^{t} x'(s)\,ds = \int_{t_1}^{t} \Big(a_1^{-1/\alpha_1}(s)x'(s)\Big)\, a_1^{1/\alpha_1}(s)\,ds.$$

Using the fact that $a_1^{-1/\alpha_1}(t)x'(t)$ is nonincreasing on $[t_1, \infty)$, we find

$$x(t) \geq \Big(a_1^{-1/\alpha_1}(t)x'(t)\Big) \int_{t_1}^{t} a_1^{1/\alpha_1}(s)\,ds,$$

or

$$x(t) \geq \eta[t, t_1]\Big(a_1^{-1/\alpha_1}(t)x'(t)\Big) \quad \text{for} \quad t \geq t_1.$$

Thus, there exists a $t_2 \geq t_1$ such that

$$x[g(t)] \geq \eta[g(t), t_1]\big(Z^{1/\alpha_1}[g(t)]\big) \quad \text{for} \quad t \geq t_2, \tag{2.14}$$

where $Z(t) = (x'(t))^{\alpha_1}/a_1(t),\ t \geq t_2$. Using (2.11) and (2.14) in (2.10) we get

$$\frac{d}{dt}Z(t) + \overline{Q}(t)F\big(Z^{1/\alpha_1}[g(t)]\big) \leq 0 \quad \text{for} \quad t \geq t_2. \tag{2.15}$$

Integrating (2.15) from t to $T \geq t \geq t_2$ and letting $T \to \infty$, we obtain

$$Z(t) \geq \int_{t}^{\infty} \overline{Q}(s)F\big(Z^{1/\alpha_1}[g(s)]\big)\,ds.$$

As in [9, 12], it is now easy to conclude that there exists a positive solution $y(t)$ of the equation (2.13) with $\lim\limits_{t\to\infty} y(t) = 0$. This contradicts the hypothesis and completes the proof.

By using a well known oscillation result in [9, Corollary 7.6.1], the following corollary is immediate.

Corollary 2.1 *Let conditions (i) – (iv), (2.11) and (2.12) hold. Then, equation (1.1) is B-oscillatory if one of the following conditions holds:*

(I_1) $F(y^{1/\alpha_1})/y \geq k > 0,\ y \neq 0$, *where k is a constant,* (2.16)

and

$$\liminf_{t\to\infty} \int_{g(t)}^{t} \overline{Q}(s)\,ds > \frac{1}{ek}. \tag{2.17}$$

$$(\mathrm{I}_2) \int\limits_{\pm 0} \frac{du}{F(u^{1/\alpha_1})} < \infty, \tag{2.18}$$

and

$$\int^{\infty} \overline{Q}(s)\, ds = \infty. \tag{2.19}$$

Next, we let $\overline{F}(x) = f^{1/(\alpha_1\alpha_2\alpha_3)}(x)$, $x \in R$ and assume that

$$\int^{\pm\infty} \frac{du}{\overline{F}(u)} < \infty. \tag{2.20}$$

Now, we prove the following oscillation result.

Theorem 2.3 *Let conditions (i) – (iv), (2.12) and (2.20) hold. If*

$$\int^{\infty} g'(u) a_1^{1/\alpha_1}[g(u)] \left(\int\limits_u^{\infty} Q(s)\, ds \right)^{1/\alpha_1} du = \infty, \tag{2.21}$$

then equation (1.1) is B-oscillatory.

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.1, we obtain (2.10) for $t \geq t_1 \geq t_0$. Now, one can easily see that

$$L_1 x(t) \geq \left(\int\limits_t^{\infty} Q(s)\, ds \right) F(x[g(t)]), \tag{2.22}$$

or

$$a_1^{-1/\alpha_1}[g(t)] x'[g(t)] \geq a_1^{-1/\alpha_1}(t) x'(t) \geq \left(\int\limits_t^{\infty} Q(s) ds \right)^{1/\alpha_1} \overline{F}(x[g(t)])$$

for $t \geq t_2 \geq t_1$. Hence, it follows that

$$\frac{x'[g(t)]g'(t)}{\overline{F}(x[g(t)])} \geq g'(t) a_1^{1/\alpha_1}[g(t)] \left(\int\limits_t^{\infty} Q(s)\, ds \right)^{1/\alpha_1} \quad \text{for} \quad t \geq t_2. \tag{2.23}$$

Integrating both sides of (2.23) from t_2 to t, we get

$$\int\limits_{t_2}^{t} g'(u) a_1^{1/\alpha_1}[g(u)] \left(\int\limits_u^{\infty} Q(s)\, ds \right)^{1/\alpha_1} du \leq \int\limits_{x[g(t_2)]}^{x[g(t)]} \frac{dv}{\overline{F}(v)} \leq \int\limits_{x[g(t_2)]}^{\infty} \frac{dv}{\overline{F}(v)} < \infty,$$

which contradicts condition (2.21). This completes the proof.

In [5], we have compared the oscillation of nonlinear equations of type (2.7) with those of second order linear equations. In fact, we obtained the following results.

Lemma 2.2 *Let* $0 < \alpha_1 \leq 1$, $g'(t) > 0$ *for* $t \geq t_0$, $0 < \overline{q}(t) = \int\limits_t^\infty Q(s)\,ds < \infty$ *and* $F(x) = x^\beta$, *where* β *is the ratio of positive odd integers. Then, equation (2.7) is oscillatory if for all large* t, *the linear second order equation*

$$\left(\frac{C(t)}{g'(t)}\left(\frac{(\overline{q}(t))^{\alpha_1-1}}{a_1[g(t)]}\right)^{1/\alpha_1} y'(t)\right)' + \beta Q(t)y(t) = 0 \tag{2.24}$$

is oscillatory, where

$$C(t) = \begin{cases} c_1, \ \ c_1 > 0 \quad \textit{is any constant}, & \textit{when} \quad \beta > \alpha_1, \\ 1, & \textit{when} \quad \beta = \alpha_1, \\ c_2\eta^{(\alpha_1-\beta)/\alpha_1}[g(t), t_0], \ \ c_2 > 0 \quad \textit{is any constant}, & \textit{when} \quad \beta < \alpha_1. \end{cases}$$

Lemma 2.3 *Let* $\alpha_1 \geq 1$, $g'(t) > 0$ *for* $t \geq t_0$ *and* $F(x) = x^\beta$, *where* β *is the ratio of positive odd integers. Then, equation (2.7) is oscillatory if for all large* t, *the linear second order equation*

$$\left(\frac{\overline{C}(t)}{a_1^{1/\alpha_1}[g(t)]g'(t)\eta^{\alpha_1-1}[g(t), t_0]} Z'(t)\right)' + \beta Q(t)Z(t) = 0 \tag{2.25}$$

is oscillatory, where

$$\overline{C}(t) = \begin{cases} c_1, \ \ c_1 > 0 \quad \textit{is any constant}, & \textit{when} \quad \beta > \alpha_1, \\ 1, & \textit{when} \quad \beta = \alpha_1, \\ c_2\eta^{\alpha_1-\beta}[g(t), t_0], \ \ c_2 > 0 \quad \textit{is any constant}, & \textit{when} \quad \beta < \alpha_1. \end{cases}$$

By Lemmas 2.2 and 2.3 we can replace equation (2.7) in Theorem 2.1 by equation (2.24), or equation (2.25). The statements and formulations of the results are left to the reader.

Next, we present the following result.

Theorem 2.4 *Let conditions (i) – (iv) hold. If*

$$\int^\infty a_1^{1/\alpha_1}(u)\left(\int\limits_u^\infty Q(s)\,ds\right)^{1/\alpha_1} du = \infty, \tag{2.26}$$

then equation (1.1) is B*-oscillatory.*

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 2.3, we obtain (2.22) for $t \geq t_1$. Since $x(t)$ is an increasing function on $[t_1, \infty)$, there exist a $t_2 \geq t_1$ and a constant $C > 0$ such that

$$x[g(t)] \geq C \quad \text{for} \quad t \geq t_2. \tag{2.27}$$

Using (2.27) in (2.22), one can easily see that

$$x'(t) \geq a_1^{1/\alpha_1}(t)\left(\int\limits_t^\infty Q(s)\,ds\right)^{1/\alpha_1} \overline{F}(c), \quad t \geq t_2.$$

Integrating the above inequality from t_2 to t and using (2.26) we arrive at the desired contradiction.

Next, we will give some necessary and sufficient conditions for all bounded solutions of equation (1.1) to be oscillatory or nonoscillatory.

Theorem 2.5 *Let conditions (i) – (iv) hold. Then, equation (1.1) is B-oscillatory if and only if condition (2.26) is satisfied.*

Proof Suppose that (2.26) holds and assume that equation (1.1) has a bounded nonoscillatory solution $x(t)$. The proof is similar to that of Theorem 2.4 and hence omitted.

Assume that (2.26) does not hold. We may suppose that

$$\int_{t_0}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1<\infty,\quad t_0\ge 0. \tag{2.28}$$

Then, we can choose $T \ge t_0$ sufficiently large such that for $t \ge T$,

$$\int_{T}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}f(\gamma)q(s)\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1<\frac{\gamma}{2} \tag{2.29}$$

for some constant $\gamma > 0$. Let $x(t)$ be a solution of the following equation

$$x(t)=\gamma-\int_{t}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1. \tag{2.30}$$

Then we easily see that $x(t)$ is a solution of equation (1.1). Next, we shall show that equation (2.30) has a bounded nonoscillatory solution $x(t)$ by using the fixed point theorem of Schauder.

We introduce the Banach space X of all continuous and bounded real-valued functions on the interval $[t_0,\infty)$ endowed with the usual sup norm $\|\cdot\|$. We define a bounded, convex and closed subset $\mathcal{B}$ of X as

$$\mathcal{B}=\left\{x\in X:\ \ \frac{\gamma}{2}\le x(t)\le\gamma,\quad t\ge t_0\right\}.$$

Next, let S be a mapping defined on $\mathcal{B}$ as follows: For $x \in \mathcal{B}$,

$$(Sx)(t)=\begin{cases}\gamma-\displaystyle\int_{t}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty}q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3}ds_3\right)^{1/\alpha_2}ds_2\right)^{1/\alpha_1}ds_1, & t\ge T,\\ (Sx)(T), & t_0\le t\le T.\end{cases} \tag{2.31}$$

Then the mapping S satisfies the following:

(I_1) S maps $\mathcal{B}$ into $\mathcal{B}$. In fact, for any $x \in \mathcal{B}$, from (2.29) and (2.31) we have

$$\gamma\ge(Sx)(t)\ge\gamma-\frac{\gamma}{2}=\frac{\gamma}{2},\quad t\ge t_0.$$

So $Sx \in \mathcal{B}$.

(I_2) The mapping S is continuous on $\mathcal{B}$. Let $x \in \mathcal{B}$ and $\{x_j\}$ be a sequence in $\mathcal{B}$ converging to x. We shall show that Sx_j converges to Sx. By (2.29), for any $\epsilon > 0$, we can choose $T_0 \geq T$ such that

$$\int_{T_0}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(\gamma)\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1 < \frac{\epsilon}{3}. \tag{2.32}$$

Furthermore, we can see that the series $f(x_j)$ converges to $f(x)$ uniformly with respect to j. So, we can choose m such that for all $j \geq m$,

$$\begin{aligned}\Bigg|&\int_{t_0}^{T_0}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x_j[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\\ &-\int_{t_0}^{T_0}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\Bigg| < \frac{\epsilon}{3}.\end{aligned} \tag{2.33}$$

In the following, we shall show that $|(Sx_j)(t) - (Sx)(t)| < \epsilon$ for any t and $j \geq m$.

(i) If $t \geq T_0$, then from (2.31) and (2.32), we can easily find

$$\begin{aligned}&|(Sx_j)(t) - (Sx)(t)|\\ &\qquad \leq 2\left|\int_{t}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(\gamma)\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\right|\\ &\qquad < \frac{2\epsilon}{3} < \epsilon \quad \text{for} \quad j \geq m.\end{aligned}$$

(ii) If $t \leq T_0$, from (2.31), (2.32) and (2.33), we have

$$\begin{aligned}&|(Sx_j)(t) - (Sx)(t)|\\ &\quad \leq \Bigg|\int_{t}^{T_0}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x_j[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\\ &\qquad -\int_{t}^{T_0}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\Bigg|\\ &\qquad +\left|\int_{T_0}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x_j[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\right|\\ &\qquad +\left|\int_{T_0}^{\infty}\left(a_1(s_1)\int_{s_1}^{\infty}\left(a_2(s_2)\int_{s_2}^{\infty}\left(a_3(s_3)\int_{s_3}^{\infty} q(s)f(x[g(s)])\,ds\right)^{1/\alpha_3} ds_3\right)^{1/\alpha_2} ds_2\right)^{1/\alpha_1} ds_1\right|\end{aligned}$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{for} \quad j \geq m.$$

Clearly, (i) and (ii) together yield that $|(Sx_j)(t) - (Sx)(t)| < \epsilon$ for any t and $j \geq m$ which completes the proof that the mapping S is continuous on $\mathcal{B}$.

(I_3) The set $S(\mathcal{B})$ is relatively compact. For any $x \in \mathcal{B}$ and every $t \geq t_0$, we have $|(Sx)(t)| \leq \gamma$. Therefore, $S\mathcal{B}$ is uniformly bounded. Furthermore, we find

$$|(Sx)(t)-\gamma| \leq \left| \int_t^\infty \left(a_1(s_1) \int_{s_1}^\infty \left(a_2(s_2) \int_{s_2}^\infty \left(a_3(s_3) \int_{s_3}^\infty q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|. \tag{2.34}$$

Thus, from (2.28) and (2.34), we conclude that $S\mathcal{B}$ is equiconvergent at ∞. Now, for any $x \in \mathcal{B}$ and every t_1, t_2 with $T \leq t_1 \leq t_2$, we get

$$|(Sx)(t_2) - (Sx)(t_1)|$$
$$\leq \left| \int_{t_1}^{t_2} \left(a_1(s_1) \int_{s_1}^\infty \left(a_2(s_2) \int_{s_2}^\infty \left(a_3(s_3) \int_{s_3}^\infty q(s) f(\gamma)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 \right|.$$

From this it follows that $S\mathcal{B}$ is equicontinuous. Finally, by the given compactness criterion (see [13]), we conclude that $S\mathcal{B}$ is relatively compact.

Thus, by the Schauder fixed point theorem [13], it follows that (2.30) has a positive solution $x(t)$. This proves the necessity.

The following theorem provides a necessary and sufficient condition for the existence of a bounded solution of equation (1.1).

Theorem 2.6 *Assume that (i) – (iv) except condition (1.3) hold, and*

$$\int^\infty q(s)\, ds = \infty. \tag{2.35}$$

Then a necessary and sufficient condition for equation (1.1) to have a positive solution $x(t)$ which satisfies $\beta_2 \geq x(t) \geq \beta_1 > 0$ (β_1 and β_2 are constants) for $t \geq t_0$ is that

$$\int_{t_0}^\infty \left(a_1(s_1) \int_{t_0}^{s_1} \left(a_2(s_2) \int_{t_0}^{s_2} \left(a_3(s_3) \int_{t_0}^{s_3} q(s)\, ds \right)^{1/\alpha_3} ds_3 \right)^{1/\alpha_2} ds_2 \right)^{1/\alpha_1} ds_1 < \infty. \tag{2.36}$$

Proof **Necessity** If $x(t)$ is a positive solution of equation (1.1) and the condition $\beta_2 \geq x(t) \geq \beta_1 > 0$ is satisfied, then we have in view of equation (1.1),

$$L_3 x(t) = L_3 x(t_0) - \int_{t_0}^t q(s) f(x[g(s)])\, ds \leq L_3 x(t_0) - f(\beta_1) \int_{t_0}^t q(s)\, ds.$$

If t is large enough, in view of (2.35), we have $L_3x(t) < 0$. Then, for all large t_0,

$$L_3x(t) < -f(\beta_1)\int_{t_0}^{t} q(s)\,ds,$$

or

$$\frac{d}{dt}\,L_2x(t) < -f^{1/\alpha_3}(\beta_1)\left(a_3(t)\int_{t_0}^{t} q(s)\,ds\right)^{1/\alpha_3}.$$

The rest of the proof is similar to the proof of the sufficiency part of Theorem 2.5 and hence omitted.

The proof of sufficiency is similar to the proof of necessity part of Theorem 2.5. This completes the proof.

Remark 2.1 From the above study of B–oscillation of equation (1.1), we are concerned with the nonexistence of solutions of equation (1.1) satisfying (2.8). This class of solutions of (1.1) may include some unbounded solutions. Therefore, some modification in the definition of B-oscillation of equation (1.1) is required to include bounded as well as some unbounded solutions of equation (1.1). The details are left to the reader.

Remark 2.2 The results of this paper can be extended to neutral equations of the form

$$L_4(x(t) + p(t)x[\tau(t)]) + q(t)f(x[g(t)]) = 0, \tag{2.37}$$

where $p(t) \in C([t_0,\infty),[0,\infty))$ and $\tau(t) \in C([t_0,\infty),R)$, $\tau'(t) > 0$ for $t \ge t_0$ and $\lim_{t\to\infty} \tau(t) = 0$. Here, we refer to our papers [4–6] and omit the details.

The following example illustrates some of the results obtained.

Example 2.1 Consider the differential equation

$$\frac{d}{dt}\left(\frac{1}{t^2}\left(\frac{d}{dt}\left(t\left(\frac{d}{dt}\left(t\left(\frac{d}{dt}x(t)\right)^3\right)\right)\right)^3\right)\right) + \frac{2}{t^4}\,x(t) = 0. \tag{2.38}$$

This is actually (1.1) with

$$\alpha_1 = 3, \quad \alpha_2 = 1, \quad \alpha_3 = 3, \quad a_1(t) = \frac{1}{t}, \quad a_2(t) = \frac{1}{t}, \quad a_3(t) = t^2,$$
$$q(t) = \frac{2}{t^4}, \quad g(t) = t, \quad f(x) = x.$$

By direct computation we obtain

$$Q(t) = \frac{1}{2}\,t^{-7/3}, \quad \eta[g(t),\,T] \le \frac{3}{2}\,t^{2/3}, \quad \overline{Q}(t) = Q(t)F(\eta[g(t),\,T]) \le t^{-19/9}.$$

Clearly, conditions (i) – (iv), (2.11) and (2.12) are fulfilled. Further, it can be easily checked that (2.17) is not satisfied, and also

$$\int^{\infty} \overline{Q}(s)\,ds \le \int^{\infty} s^{-19/9}\,ds < \infty$$

which implies (2.19) is not met. Thus, we see that both conditions (I_1) and (I_2) of Corollary 2.1 are not fulfilled.

Moreover, we can verify easily that condition (2.20) is not satisfied but (2.21) and (2.26) are met. Thus, the conditions of Theorem 2.3 are not all satisfied, whereas those of Theorems 2.4 and 2.5 are fulfilled.

Hence, on one hand we *cannot* conclude from Corollary 2.1 and Theorem 2.3 that (2.38) is B-oscillatory, while on the other hand Theorems 2.4 and 2.5 give that (2.38) is B-oscillatory. In fact, we observe that (2.38) has a solution given by $x(t) = t$, which is unbounded and nonoscillatory.

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A Fredholm Operator and Solution Sets to Evolution Systems

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Abstract: In this paper we deal with the Peano phenomenon for general initial boundary-value problems of quasilinear evolution systems with arbitrary even order space derivatives. The nonlinearity is a continuous or continuously Frechét differentiable function. Qualitative and quantitative structure of solution sets is studied by the theory of proper, Fredholm and Nemitskiĭ operators. These results can be applied to the different technical and natural science models.

Keywords: *Evolution systems; an initial boundary-value problem; a linear Fredholm operator; a proper and coercive operator; a bifurcation point; a surjectivity.*

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0 Introduction

The Peano phenomenon of the existence of a solution continuum of the initial value problem for ordinary differential systems is well-known. This phenomenon has been studied by many authors in [3 − 5, 8, 17, 27]. The structure of solution sets for second order partial differential problems was observed in the authors papers [12, 13].

In this paper we shall study generic properties of quasilinear initial boundary-value problems for evolution systems of an even order with the continuous or continuous differentiable nonlinearities and the general boundary value conditions. In special Hölder spaces we use the Nikoľskiĭ decomposition theorem from [29, P. 233] for linear Fredholm operators, the global inversion theorem of [9, 6] and [7, PP. 42 − 43] and the Ambrosetti

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solution quantitive results from [2, P. 216]. In the consideration on surjectivity the generalized Leray–Schauder condition is employed which is similar to that one in [20]. In the case of nonlinear Fredholm operators we use the main Quinn and Smale theorem from [22] and [24].

The present results allow us to observe different problems describing dynamics of mechanical processes (bendding, vibration), phisycal-heating processes, reaction-diffusion processes in chemical and biological technologies or in the ecology.

1 The Formulation of Problem, Assumptions and Spaces

The set $\Omega \subset R^n$ for $n \in N$ means a bounded domain with the boundary $\partial\Omega$. The real number T will be positive and $Q = (0,T] \times \Omega$, $\Gamma = (0,T] \times \partial\Omega$. If the multiindex $k = (k_1, \dots, k_n)$ with $|k| = \sum\limits_{i=1}^{n} k_i$, then we use the notation D_x^k for the differential operator $\dfrac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$ and D_t for $\dfrac{\partial}{\partial t}$. If the module $|k| = 0$ then D_x^k means an identity mapping. The symbol $cl\, M$ means the closure of the set M in R^n.

In this paper we consider the general system of $p \geq 1$ nonlinear differential equations (parabolic or non-parabolic type) of an arbitrary even order $2b$ (b is a positive integer) with p unknown functions in the column vector form $(u_1, \dots, u_p)^{\mathrm{T}} = u \colon cl\, Q \to R^p$. Its matrix form is given as follows:

$$A(t,x,D_t,D_x)u + f(t,x,\overline{D}_x^{\gamma}u) = g(t,x) \quad \text{for} \quad (t,x) \in Q, \tag{1.1}$$

where

$$A(t,x,D_t,D_x)u = D_t u - \sum_{|k|=2b} a_k(t,x) D_x^k u - \sum_{0 \leq |k| \leq 2b-1} a_k(t,x) D_x^k u,$$

and $\overline{D}_x^{\gamma} u$ is a vector function whose components are derivatives $D_x^{\gamma} u_l$ with the different multiindices $0 \leq |\gamma| \leq 2b-1$ for $l = 1, \dots, p$.

The system of boundary conditions is given by the vector equation with the bp components

$$B(t,x,D_x)u\Big|_{cl\,\Gamma} = (B_1(t,x,D_x)u, \dots, B_{b\,p}(t,x,D_x)u)^{\mathrm{T}}\big|_{cl\,\Gamma} = 0 \tag{1.2}$$

in which

$$B_j(t,x,D_x)u = \sum_{0 \leq |k| \leq r_j} b_{jk}(t,x) D_x^k u$$

for an integer $0 \leq r_j \leq 2b-1$ and $j = 1, \dots, bp$.

Further the initial value homogeneous condition

$$u(0,x) = 0 \quad \text{for} \quad x \in \bar{\Omega} \tag{1.3}$$

is considered.

Here the given functions are the following mappings: $a_k = (a_k^{hl})_{h,l=1}^{p} \colon cl\, Q \to R^{p^2}$ for $0 \leq |k| \leq 2b$ are $(p \times p)$-matrix functions; $b_{jk} = (b_{jk}^1, \dots, b_{jk}^p) \colon cl\, \Gamma \to R^p$ for $0 \leq |k| \leq r_j$, $j = 1, \dots, b_p$ are row vector functions; $f = (f_1, \dots, f_p)^{\mathrm{T}} \colon cl\, Q \times R^{\kappa} \to R^p$

and $g = (g_1, \dots, g_p)^{\mathrm{T}}: cl\, Q \to R^p$ are column vector functions, where κ is a positive integer given by the inequality

$$\kappa \le \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \dots + \binom{n+|\gamma|-2}{|\gamma|-1} + \binom{n+|\gamma|-1}{|\gamma|}\right] p.$$

Under several supplementary assumptions, problem (1.1) – (1.3) defines homeomorphism between some Hölder spaces. Now, we formulate these suppositions.

(P) A δ-uniform parabolic condition holds for system (1.1) in the sense of J.G. Petrovskiiĭ, $\delta > 0$.

The system (1.1) and boundary condition (1.2) are connected by

(C) a δ^+-uniform complementary condition with $\delta^+ > 0$ and

(Q) a compatibility condition.

The coefficients of the operator $A(t, x, D_t, D_x)$ from (1.1) and of $B(t, x, D_x)$ from (1.2) and the boundary $\partial\Omega$ satisfy

($\mathrm{S}^{l+\alpha}$) a smoothness condition for a nonnegative integer l and a number $\alpha \in (0, 1)$.

We shall be employed with the Banach spaces of continuously differentiable functions $C^l_x(cl\, Q, R^p)$ and $C^{l/2b,l}_{t,x}(cl\, Q, R^p)$ and the Hölder spaces $C^{l+\alpha}_x(cl\, Q, R^p)$, $C^{(l+\alpha)/2b,l+\alpha}_{t,x}(cl\, Q, R^p)$ for a nonnegative integer l and $\alpha \in (0, 1)$.

For the exact definition of conditions (P), (C), (Q), ($S^{l+\alpha}$) see [19, PP. 12–21] and for the definition of spaces see [19, PP. 8–12] or [11].

The homeomorphism result for (1.1) – (1.3) can be formulated as follows:

Proposition 1.1 (see [19, P. 21] and [15, PP. 182–183]) Let the conditions (P), (C) and (S^α) be satisfied for $\alpha \in (0, 1)$. Necessary and sufficient conditions for the existence and uniqueness of the solution

$$u \in C^{(2b+\alpha)/2b,2b+\alpha}_{t,x}(cl\, Q, R^p)$$

of linear problem (1.1)–(1.3) for $f = 0$ is

$$g \in C^{\alpha/2b,\alpha}_{t,x}(cl\, Q, R^p)$$

and the compatibility condition (Q).

Moreover, there exists a constant $c > 0$ independent of g such that

$$c^{-1}\|g\|_{\alpha/2b,\alpha,Q,p} \le \|u\|_{(2b+\alpha)/2b,2b+\alpha,Q,p} \le c\|g\|_{\alpha/2b,\alpha,Q,p}$$

2 General Results

In this part we remind some notions and assertions from the nonlinear functional analysis applied in the fundamental lemmas and theorems.

Throughout this paper we shall assume that X and Y are Banach spaces either both over the real or complex field.

In the Zeidler books [31, PP. 365–366] and [32, PP. 667–668] we find definitions of the linear and nonlinear Fredholm operator.

The following proposition gives the necessary and sufficient condition for a linear operator to be Fredholm.

Proposition 2.1 (S.M. Nikoľskiĭ [29, P. 233]) A linear bounded operator $A\colon X \to Y$ is Fredholm of the zero index iff $A = C+T$, where $C\colon X \to Y$ is a linear homeomorphism and $T\colon X \to Y$ is a linear completely continuous operator.

In the theory and applications of nonlinear operators, the notions as a proper, *σ-proper*, closed, coercive operator (for definitions see books [31] and [32]) are very frequent. Their significant application gives the following statements.

Proposition 2.2 (the Ambrosetti theorem [2, P. 216]) Let $F \in C(X,Y)$ be a proper mapping. Then the cardinal number $\operatorname{card} F^{-1}(q)$ of the set $F^{-1}(q)$ is constant and finite (it may be zero) for every q taken from the same component (nonempty and connected subset) of the set $Y \setminus F(\Sigma)$. Here Σ means a closed set of all points $u \in X$ at which F is not locally invertible.

A relation between the local invertibility and homeomorphism of X onto Y gives the global inverse mapping theorem.

Proposition 2.3 (R. Caccioppoli [9], E. Zeidler [31, P. 174]) Let $F \in C(X,Y)$ be a locally invertible mapping in X. Then F is a homeomorphism of X onto Y iff F is proper.

The following propositions give necessary and sufficient conditions for the proper mapping.

Proposition 2.4 (see [31, P. 176], [23, P. 49] and [27, P. 20]) Let $F \in C(X,Y)$.

(i) If F is proper, then F is a nonconstant closed mapping.
(ii) If $\dim X = +\infty$ and F is a nonconstant closed mapping, then F is proper.

Proposition 2.5 (see [23, PP. 58–59], [31, P. 498] and [27, P. 20]) Suppose that $F\colon X \to Y$ and $F = F_1 + F_2$, where

(i) $F_1 : X \to Y$ is a continuous proper mapping on X and
(ii) $F_2 : X \to Y$ is complete continuous.

Then

(i) the restriction of the mapping F to an arbitrary bounded closed set in X is a proper mapping;
(ii) if moreover, F is coercive, then F is a proper mapping.

Now we can formulate some sufficient conditions for the surjectivity of an operator.

Proposition 2.6 (see [27, PP. 24 and 27]) Let X be a real Banach space. Suppose

(i) $P = I - f\colon X \to X$ is a condensing field, where $I\colon X \to X$ is the identity,
(ii) P is coercive,
(iii) there exists a strictly solvable field $G = I - g\colon X \to X$ and $R > 0$ such that for all solutions $u \in X$ of the equation

$$P(u) = kG(u)$$

and for all $k < 0$ the estimation $\|u\|_X < R$ holds.

Then the following statements are true:

(i) P is a proper mapping,
(ii) P is strictly surjective,
(iii) $\operatorname{card} F^{-1}(q)$ is constant, finite and nonzero for every q from the same connected component of the set $Y \setminus F(\Sigma)$. For Σ see Proposition 2.2.

The definition of a condensing field is understood in the sense given in [10, P. 69]. For the definition of a strict solvable field and strict surjective field see in [29].

Remark 2.1 It is clear that an operator F is strictly surjective, then it is surjective and if F is strictly solvable, then it is also solvable. Moreover, if F is strictly surjective, then it is strictly solvable, too.

Proposition 2.7 (the Schauder invariance of domain theorem [31, P. 705]) Let $F\colon (M \subseteq X) \to X$ be continuous and locally compact perturbation of identity on the open nonempty set M in the Banach space X. Then

(i) if F is locally injective on M so F is an open mapping;
(ii) if F is injective on M so F is a homeomorphism from M onto the open set $F(M)$.

For the compact perturbation of C^1–Fredholm operator we shall use the following proposition.

Proposition 2.8 (E. Zeidler [32, P. 672]) Let $A\colon D(A) \subset X \to Y$ be a C^1–Fredholm operator on the open set $D(A)$ and $B\colon D(A) \to Y$ be a compact mapping from the class C^1. Then $A+B\colon D(A) \to Y$ is a Fredholm (possible nonlinear) operator with the same index as A at each point of $D(A)$.

In the following propositions we use the notion of *a regular, singular, critical point* of an operator and *a regular, singular values* of operators. The reader finds these definitions in [32, P. 668] or [31, P. 184].

Also, we need a residual set. A subset of a topological space Z is called *residual* iff it is a countable intersection of dense and open subsets of Z.

By the Baire theorem in any complete metric space or locally compact Hausdorff topological space, a residual set is dense in this space.

The most important theorem for nonlinear Fredholm mappings is due to S. Smale [24, P. 862] and Quinn [22]. It is also in [7, PP. 11–12].

Proposition 2.9 (a Smale–Quinn Theorem) If $F\colon X \to Y$ is a Fredholm mapping (possible nonlinear) of the class $C^k(X,Y)$ in the Frechét sense and either

(i) X has a countable basis (S. Smale) or
(ii) F is σ-proper (Quinn),

then the set R_F of all regular values of F is residual in Y. Moreover, if F is proper, then R_F is open and dense set in Y.

A necessary and sufficient condition for a local diffeomorphism (see [31, p. 171]) is given in the following proposition.

Proposition 2.10 (a Local Inverse Mapping Theorem, [31, p. 172]) Let $F\colon U(u_0) \subset X \to Y$ be a C^1-mapping in the Frechét sense. Then F is a local C^1-diffeomorphism at u_0 iff u_0 is a regular point of F.

Proposition 2.11 ([23, P. 89]) Let $\dim Y \geq 3$ and $F\colon X \to Y$ be a Fredholm mapping of the zero index. If $u_0 \in X$ is an isolated singular point of F, then F is locally invertible at u_0.

To illustrate the following results we shall need estimations of a Green $p \times p$-matrix for linear problem (1.1)–(1.3).

Lemma 2.1 *Let the assumptions* (P), (C), (S$^\alpha$) *be satisfied for* $\alpha \in (0,1)$. *Then we have for the Green matrix* G *of linear problem (1.1) – (1.3) with* $f = 0$

$$|D_t^{k_0} D_x^k G(t,x;\tau,\xi)| \le c(t-\tau)^{-\mu} \|x-\xi\|_{R^n}^{2b\mu-(n+2bk_0+|k|)} E \tag{2.1}$$

for $0 \le 2bk_0 + |k| \le 2b$ *and* $\mu \le (n+2bk_0+|k|)/2b$, *thereby* $0 \le \tau < t \le T$ *and* $x, \xi \in cl\,\Omega$, $x \ne \xi$. *The positive constant* c *does not depend on* t, x, τ, ξ *and* E *means the* $p \times p$*-matrix consisting only of units,* $r = 2b/(2b-1)$.

Proof Since $n+2bk_0+|k|-2b\mu \ge 0$ and $\|x-\xi\|_{R^n} < \operatorname{diam}\Omega$ so for $0 < \delta \le t-\tau \le T$ we obtain (2.1) by the estimation (see [15, PP. 182–183])

$$\begin{aligned}
&|D_t^{k_0} D_x^k G(t,x;\tau,\xi)| \le c_1(t-\tau)^{-\frac{n+2bk_0+|k|}{2b}} \exp\{-c_2 \frac{\|x-\xi\|_{R^n}^r}{(t-\tau)^{1/(2b-1)}}\} \\
&\quad \le c_1(t-\tau)^{-\mu}\|x-\xi\|_{R^n}^{2b\mu-(n+2bk_0+|k|)} \\
&\quad \times [\,\|x-\xi\|_{R^n}^{2b}/(t-\tau)\,]^{(n+2bk_0+|k|-2b\mu)/2b} \exp\{-c_2[\,\|x-\xi\|_{R^n}^{2b}/(t-\tau)\,]^{1/(2b-1)}\}E.
\end{aligned}$$

If $0 < t-\tau < \delta$ with respect to

$$\lim_{y\to+\infty} y^u \exp\{-cy^v\} = 0$$

for every u, $v \in R$ and $c > 0$, we get estimation (2.1).

Remark 2.2 For any $x = (x_1, \dots, x_n) \in R^n$ the inequalities

$$c_n \sum_{i=1}^n |x_i| \le \|x\|_{R^n} \le \sum_{i=1}^n |x_i| \tag{2.2}$$

hold, if $c_n \in (0, 1/(\sqrt{2})^{n-1})$, $n \in N$, does not depend of x.

Remark 2.3 Also, we see that the mild solution $u \in C_x^{|\gamma|}(cl\,Q, R)$ of problem (1.1)–(1.3) satisfies the column vector integro-differential equation

$$\begin{aligned}
u(t,x) &= \int_0^t d\tau \int_\Omega G(t,x;\tau,\xi)\,[\,g(\tau,\xi) - f(\tau,\xi,\overline{D}^{\,\gamma}u(\tau,\xi))\,]\,d\xi =: \\
&= (Su)(t,x) \quad \text{for} \quad (t,x) \in cl\,Q
\end{aligned} \tag{2.3}$$

for $0 \le |\gamma| \le 2b-1$ and on the contrary the solution $v \in C_x^{|\gamma|}(cl\,Q, R^p)$ satisfying (2.3) is a mild solution of (1.1)–(1.3).

3 Operator Formulation and Fundamental Lemmas

Consider the following operators:

(i)

$$A\colon X \to Y, \tag{3.1}$$

where

$$(Au)(t,x) = A(t,x,D_t,D_x)u(t,x) = D_t u(t,x) - \sum_{0\le|k|\le 2b} a_k(t,x)D_x^k u(t,x)$$

for $(t,x)\in cl\,Q,\ u\in X$,

$$X = \{u\in X_\rho; B_j(t,x,D_x)u|_\Gamma = 0, \quad j=1,2,\dots,bp,$$
$$u(0,x)=0 \quad \text{for} \quad x\in cl\,Q\} \subset C(cl\,Q,\,R^p).$$

Here

$$X_\rho \subset C_{t,x}^{(2b+\alpha)/2b,2b+\alpha}(cl\,Q,R^p)$$

is the Banach space of continuous functions $u\colon cl\,Q\to R^p$ with the continuous derivatives $D_x^k u$ for $|k|=1,\dots,2b$ and $D_x^{k_0}D_x^k u$ for $1\le 2bk_0+|k|\le 2b$ on $cl\,Q$ and with the finite norm

$$\|u\|_{X_\rho} = \max_{l=1,\dots,p}\Big[\sum_{0\le 2bk_0+|k|\le 2b}\ \sup_{(t,x)\in cl\,Q}\Big|D_t^{k_0}D_x^k u_l(t,x)\Big| + \langle D_t u_l\rangle^y_{x,\alpha,Q}$$
$$+\sum_{|k|=2b}\langle D_x^k u_l\rangle^y_{x,\alpha+\rho,Q} + \langle D_t u_l\rangle^s_{t,\alpha/2b,Q}$$
$$+\sum_{|k|=1}^{2b-1}\langle D_x^k u_l\rangle^s_{t,(2b+\alpha-|k|)/2b,Q} + \sum_{|k|=2b}\langle D_x^k u_l\rangle^s_{t,(\alpha+\rho)/2b,Q}\Big],$$

where $\rho>0$ and $\alpha+\rho<1$. Further

$$Y = TX \subset C_{t,\,x}^{\alpha/2b,\,\alpha}(cl\,Q,R^p)$$

for $\alpha\in(0,1)$ with the norm

$$\|u\|_Y = \max_{l=1,\dots,p}\Big[\sup_{(t,x)\in cl\,Q}|u_l(t,x)| + \langle u_l\rangle^y_{x,\alpha,Q} + \langle u_l\rangle^s_{t,\alpha/2b,Q}\Big].$$

We understand

$$\langle v\rangle^s_{t,\mu,Q} = \sup_{\substack{(t,x),(s,x)\in cl\,Q\\ t\ne s}} \frac{|v(t,x)-v(s,x)|}{|t-s|^\mu},$$

$$\langle v\rangle^y_{x,\mu,Q} = \sup_{\substack{(t,x),(t,y)\in cl\,Q\\ x\ne y}} \frac{|v(t,x)-v(t,y)|}{\|x-y\|^\mu_{R^n}}.$$

for $v\colon cl\,Q\to R$.

(ii) The *Nemitskiĭ operator*

$$N\colon X\to Y, \tag{3.2}$$

where

$$(Nu)(t,x) = (f\circ u)(t,x) = f(t,x,\overline{D}{}_x^\gamma u(t,x))$$

for $(t,x)\in cl\,Q,\ u\in X$.

(iii) The operator

$$F\colon X\to Y, \tag{3.3}$$

where

$$(Fu)(t,x) = (Au)(t,x) + (Nu)(t,x) \quad \text{for} \quad (t,x)\in cl\,Q, \quad u\in X.$$

Together with the solution sets of given problem (1.1)−(1.3) we shall search for the bifurcation points sets.

Definition 3.1 (i) A couple $(u,g) \in X \times Y$ will be called the *bifurcation point* of $(1.1)-(1.3)$ iff u is a solution of this problem and there exists a sequence $\{g_k\}_{k\in N} \subset Y$ such that $\lim\limits_{k\to\infty} g_k = g$ in Y and initial boundary value problem $(1.1)-(1.3)$ with $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $\lim\limits_{k\to\infty} u_k = \lim\limits_{k\to\infty} v_k = u$ in X.

(ii) The set of all solutions $u \in X$ of $(1.1)-(1.3)$ (or the set of all functions $g \in Y$) such that (u,g) is a bifurcation point of $(1.1)-(1.3)$ will be called the *domain of bifurcation* (the *bifurcation range*) of $(1.1)-(1.3)$.

Example 3.1 The point $(u_r, 0) \in X \times Y$ for $r \in \langle 0, T\rangle$ is a bifurcation point of the Neumann problem (parabolic and non-parabolic)

$$\frac{\partial u}{\partial t} = \pm\frac{\partial^2 u}{\partial x^2} + f(t,x,u), \quad (t,x) \in (0,T\rangle \times \Omega = Q \subset R^2, \tag{3.1*}$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \qquad t \in \langle 0,T\rangle, \tag{3.2*}$$

$$u(0,x) = 0, \qquad x \in \overline{\Omega} \tag{3.3*}$$

for $f(t,x,u) = |u|^{1/2} - au$, $a > 0$. Here for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0, & \text{if } (t,x) \in \langle 0,r\rangle \times \overline{\Omega}, \\ \dfrac{1}{\alpha^2}\Big(1 - \exp\Big\{-\dfrac{a}{2}(t-r)\Big\}\Big)^2, & \text{if } (t,x) \in (r,T\rangle \times \overline{\Omega}. \end{cases}$$

The functions $u_0(t,x) = \frac{1}{\alpha^2}(1-\exp\{-at/2\})^2$, $u_T(t,x) = 0$ are solutions of the given problem, too.

Really, there is the zero sequence $\{g_k\}_{k\in N}$ of the right hand side of (1.1) for which there exist two different sequences of solutions

$$\{u_k\}_{k\in N} = \Big\{u_{\frac{r(k+1)}{k+2}}\Big\}_{k\in N} \quad \text{and} \quad \{v_k\}_{k\in N} = \Big\{v_{\frac{rk}{k+1}}v\Big\}_{k\in N}$$

with the same limit $u_r \in X$.

The following equivalence result is true.

Lemma 3.1 (i) *The function $u \in X$ is a solution of initial boundary-value problem (1.1) – (1.3) for $g \in Y$ iff $Fu = g$.*

(ii) *The couple $(u,g) \in X \times Y$ is a bifurcation point of (1.1) – (1.3) iff $Fu = g$ and u is a point at which F is not locally invertible, i.e. $u \in \Sigma$.*

Proof The first assertion is clear.

If (u,g) is a bifurcation point of $(1.1)-(1.3)$, then with respect to Definition 3.1 we get $F(u) = g$, $F(u_k) = g_k = F(v_k)$, $u_k \neq v_k$. Thus F is not locally injective at u. Hence, F is not locally invertible at u, i.e. $u \in \Sigma$. On the contrary, if F is not locally invertible at u and $F(u) = g$, then F is not locally injective at u. Hence, it follows that the couple $(u,g) \in X \times Y$ is a bifurcation point of $(1.1)-(1.3)$. The second assertion is proved.

The following lemma gives sufficient conditions under which the operator A is a Fredholm type.

Assumption A.1 There exists a linear homeomorphism $H\colon\ X \to Y$ with

$$Hu = D_t u - H(t,x,D_x)u, \quad u \in X,$$

where

$$H(t,x,D_x)u = \sum_{|k|=2b} h_k(t,x)D_x^k u + \sum_{0\le|k|\le 2b-1} h_k(t,x)D_x^k u$$

satisfies $(\mathrm{S}^{\alpha+\rho})$ for $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$.

Lemma 3.2 *Let the operator A from (3.1) satisfy the smoothness hypothesis* $(\mathrm{S}^{\alpha+\rho})$*, $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$ (it has not to satisfy the conditions* (P), (C), (Q)*). Further let Assumption A.1 hold.*

Then

(i) $\dim X = +\infty$*;*

(ii) *the operator $A\colon X \to Y$ is a linear bounded Fredholm operator of the zero index.*

Proof (i) The equation

$$\dim C_0^\infty(Q,R) = +\infty$$

and the inclusion

$$C_0^\infty(Q,R) \subset X$$

imply $\dim X = +\infty$.

(ii) Since the coefficients a_k for $0 \le |k| \le 2b$ are continuous on the compact set $cl\,Q$, there is a positive constant $K > 0$ such that

$$\|Au\|_Y \le K(\|D_t u\|_Y + \sum_{0\le|k|\le 2b} \|D_x^k u\|_Y) = K\|u\|_X$$

for all $u \in X$, whence the operator A is bounded on X.

If the operator A is a homeomorphism, then statement (ii) is clear.

If A is not the homeomorphism, then by the Nikol̆skiĭ decomposition theorem from Proposition 2.1, it is sufficient to show that

$$Au = Hu + (H(t,x,D_x) - A(t,x,D_x))u = Hu + Tu,$$

thereby the mapping $T\colon X \to Y$ is the linear completely continuous operator. It will be proved by generalized Ascoli - Arzelà theorem from [21, P. 31].

From the hypothesis $(\mathrm{S}^{\alpha+\rho})$, the equi-boundedness of

$$Tu = \sum_{|k|=2b} (h_k(t,x) - a_k(t,x))D_x^k u + \sum_{0\le|k|\le 2b-1} (h_k(t,x) - a_k(t,x))D_x^k u$$

holds at the bounded set $S \subset X$, i.e. there is a constant $K_1(n,\alpha,T,\Omega) > 0$ such that $\|Tu\|_Y \le K_1\|u\|_X$ for all $u \in S$.

Now for the equi-continuity of the set $TS \subset Y$ we have to prove the inequality (for every element u_l, $l = 1,\dots,p$, of $u = (u_1,\dots,u_p)$)

$$|(Tu)_l(t,x) - (Tu)_l(s,y)| + \frac{|(Tu)_l(t,x) - (Tu)_l(t,y)|}{\|x-y\|_{R^n}^\alpha} + \frac{|(Tu)_l(t,x) - (Tu)_l(s,x)|}{|t-s|^{\alpha/2b}} < \varepsilon$$

for all $u \in S$ and (t,x), (s,y), (t,y), $(s,x) \in cl\,Q$, $x \neq y$, $t \neq s$ for which the norms $\|x-y\|_{R^n}$ and $|t-s|$ are sufficiently small, $\varepsilon > 0$.

With respect to ($\mathrm{S}^{\alpha+\rho}$) we obtain for the first member of the previous inequality

$$
\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(s,y)| \\
&\qquad \leq \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)|\,|D_x^k u_l(t,x)| \\
&\qquad\quad + \sum_{|k|=2b} |h_k(s,y) - a_k(s,y)|\,|D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\qquad\quad + \sum_{0\leq|k|\leq 2b-1} |h_k(s,y) - a_k(s,y)|\,|D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\qquad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)| \\
&\qquad\quad + K_3 \sum_{|k|=2b} |D_x^k u_l(t,x) - D_x^k u_l(s,y)| \\
&\qquad\quad + K_3 \sum_{0\leq|k|\leq 2b-1} |D_x^k u_l(t,x) - D_x^k u_l(s,y|,
\end{aligned}
$$

where K_2, K_3 are positive constants dependent only on n, α, T, Ω. For $|t-s| < \delta$, $\|x-y\|_{R^n} < \delta$ with a sufficiently small $\delta > 0$ the every member of the last inequality is smaller than fixed arbitrary $\varepsilon > 0$. (Since $u \in S \subset X$, the number δ does not depend on u.)

For the second member we get by the condition ($\mathrm{S}^{\alpha+\rho}$) and using the mean value theorem

$$
\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(t,y)|\,\|x-y\|_{R^n}^{-\alpha} e \\
&\qquad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(t,y)|\,\|x-y\|_{R^n}^{-\alpha} \\
&\qquad\quad + K_3 \sum_{|k|=2b} \left|D_x^k u_l(t,x) - D_x^k u_l(t,y)\right|\,\|x-y\|_{R^n}^{-\alpha} \\
&\qquad\quad + K_3 \sum_{0\leq|k|\leq 2b-1} \left|D_x^k u_l(t,x) - D_x^k u_l(t,y)\right|\,\|x-y\|_{R^n}^{-\alpha} \\
&\qquad \leq K(2\|x-y\|_{R^n}^{\rho} + \|x-y\|^{1-\alpha})
\end{aligned}
$$

By the similar way we have for the third member

$$
\begin{aligned}
&|(Tu)_l(t,x) - (Tu)_l(s,x)| \cdot |t-s|^{-\alpha/2b} \\
&\qquad \leq K_2 \sum_{0\leq|k|\leq 2b} |(h_k - a_k)(t,x) - (h_k - a_k)(s,x)|\,|t-s|^{-\alpha/2b} \\
&\qquad\quad + K_3 \sum_{|k|=2b} \left|D_x^k u_l(t,x) - D_x^k u_l(s,x)\right|\,|t-s|^{-\alpha/2b} \\
&\qquad\quad + K_3 \sum_{0\leq|k|\leq 2b-1} \left|D_x^k u_l(t,x) - D_x^k u_l(s,x)\right|\,|t-s|^{-\alpha/2b}
\end{aligned}
$$

$$\leq K\bigg(2|t-s|^{\rho/2b} + |t-s|^{1-\alpha/2b} + \sum_{|k|=1}^{2b-1} |t-s|^{1-|k|/2b}\bigg).$$

By these three estimations the assertion (ii) is proved.

Remark 3.1 Necessary and sufficient conditions for the existence of a linear homeomorphism $H\colon X \to Y$ from the assumption (A.1) are given in Proposition 1.1. Concretely, for example, $Hu = \dfrac{\partial u}{\partial t} - \Delta u$, $u \in X$.

Corollary 3.1 *Let $\mathcal{L}$ mean the set of all linear differential operators $A = D_t - A(t,x,D_x)\colon X \to Y$ satisfying the hypothesis $(S^{\alpha+\rho})$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$. Then for each $A \in \mathcal{L}$ the initial boundary-value homogeneous problem $Au = 0$, (1.2), (1.3) has a nontrivial solution or any $A \in \mathcal{L}$ is a linear bounded Fredholm operator of the zero index.*

Proof Really, if there exists an operator $A \in \mathcal{L}$ such that the problem $Au = 0$, (1.2), (1.3) has only trivial solution, then A is homeomorphism X onto Y (see Proposition 1.1). Then by Lemma 3.2 all operators of $\mathcal{L}$ are Fredholm of the zero index.

Assumption N.1 The vector function $f \in C(cl\, Q \times R^{\kappa}, R^p)$ satisfies the following local grown vector condition

$$|\, f(t,x,u^{\gamma}) - f(s,y,v^{\gamma})\,| \leq L\,\Big[\,|t-s|^{\beta_1} + \|x-y\|_{R^n}^{\beta_2} + \sum_{l=1}^{p} \sum_{0\leq|\gamma|\leq 2b-1} |u_l^{\gamma} - v_l^{\gamma}|^{\beta_{\gamma,l}}\,\Big]\, J$$

for (t,x,u^{γ}), (s,y,v^{γ}) from a compact subset of R^{κ} and $\beta_1 > \alpha/2b$, $\beta_2 > \alpha$, $\beta_{\gamma,l} > \alpha/(\alpha+\rho)$, $0 \leq |\gamma| \leq 2b-1$, $l = 1,\dots,p$, where $L > 0$.

Lemma 3.3 *Suppose Assumption N.1 holds. Then the Nemitskiĭ operator $N\colon X \to Y$ from (3.2) is completely continuous on X.*

Proof For any bounded set $S \subset X$ the N is equi-bounded in Y. Indeed, for all $u \in S$ using (N.1) the norm

$$\begin{aligned}
\|Nu\|_Y \leq \max_{l=1,\dots,p} \Bigg[& \sup_{(t,x)\in cl\, Q} |f_l(t,x,\overline{D}_x^{\gamma} u(t,x))| \\
& + L \sup_{\substack{(t,x),(t,y)\in cl\, Q \\ x\neq y}} \frac{\|x-y\|_{R^n}^{\beta_2} + \sum\limits_{l=1}^{p} \sum\limits_{0\leq|\gamma|\leq 2b-1} |D_x^{\gamma} u_l(t,x) - D_x^{\gamma} u_l(t,y)|^{\beta_{\gamma,l}}}{\|x-y\|_{R^n}^{\alpha}} \\
& + \sup_{\substack{(t,x),(s,x)\in cl\, Q \\ t\neq s}} \frac{|t-s|^{\beta_1} + \sum\limits_{l=1}^{p} \sum\limits_{0\leq|\gamma|\leq 2b-1} |D_x^{\gamma} u_l(t,x) - D_x^{\gamma} u_l(s,x)|^{\beta_{\gamma,l}}}{|t-s|^{\alpha/2b}} \Bigg]
\end{aligned}$$

Hence, it is bounded by a positive constant $K(\Omega, T, L, \alpha, \beta_1, \beta_2, \beta_{\gamma,l})$.

Also, for $|t-s|^2 + \|x-y\|^2_{R^n} < \delta^2$ with a sufficiently small $\delta > 0$ we get the equicontinuity of N. It is sufficient to prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$|(Nu)_l(t,x) - (Nu)_l(s,y)| \\ + \frac{|(Nu)_l(t,x) - (Nu)_l(t,y)|}{\|x-y\|^\alpha_{R^n}} + \frac{|(Nu)_l(t,x) - (Nu)_l(s,x)|}{|t-s|^{\alpha/2b}} < \varepsilon$$

is true for all $u \in S$, if both t, s and x, y to be sufficiently near and $l = 1, \dots, p$.

Assumption F.1 For each bounded set $S \subset Y$ there is a constant $K^a > 0$ such that for all solutions $u \in X$ of (1.1) – (1.3) with $g \in S$ the inequality

$$\|u\|_{a,\,Q} = \max_{l=1,\dots,p} \sum_{0 \le |k| \le a} \sup_{(t,x) \in cl\, Q} |D^k_x u_l(t,x)| \le K^a \tag{3.4}$$

holds for $a = \max\{|\gamma|, r\}$. Here r is an integer $0 \le r \le 2b-1$ for which the coefficients of operators A and H from (3.1) and (A.1), respectively satisfy the relations $a_k = h_k$ for $|k| = r+1, \dots, 2b$ and $a_k \neq h_k$ for at least one multiindex k with $|k| = r$ on $cl\, Q$.

Lemma 3.4 *Let* ($S^{\alpha+\rho}$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho > 1$), (A.1), (N.1) *and an almost coercivity condition of Assumption F.1 be satisfied. Then*

(i) *F from (3.3) is coercive at X.*
(ii) *F is proper and continuous at X.*

Proof (i) We need to prove that if the set $S \subset Y$ is bounded in Y, then the set of arguments $F^{-1}(S) \subset X$ is bounded in X.

By (3.4) and the Assumption F.1 it follows that the set $F^{-1}(S)$ is bounded in the norm $\|\cdot\|_{a,\,Q}$. Hence and by Assumption N.1 one obtains the estimation $\|Nu\|_Y \le K_4$ for all $u \in F^{-1}(S)$. From Lemma 3.2 (ii) also $\|Au\|_Y \le \|Fu\|_Y + \|Nu\|_Y \le K_5$ for any $u \in F^{-1}(S)$, where K_4, K_5 are positive constants.

On the other hand, Assumption A.1 ensures the existence and uniqueness of the solution $u \in X$ of the linear equation $Hu = y$ for any $y \in Y$ and (see the Green representation of solution from (2.3) and [15, PP. 182–183] and estimation (2.1)) the estimation

$$\|u\|_X \le K_6 \|y\|_Y, \quad K_6 > 0, : u \in F^{-1}(S) \tag{3.5}$$

is true.

Then for $u \in F^{-1}(S)$ we have

$$Hu = Au + \sum_{0 \le |k| \le 2b} (a_k(t,x) - h_k(t,x)) D^k_x u.$$

With respect to (S^α) and Assumption F.1

$$\|y\|_Y = \|Hu\|_Y \le \|Au\|_Y + \sum_{0 \le |k| \le r} \|a_k - h_k\|_Y \, \|D^k_x u\|_Y \le$$

$$K_5 + K_7 \|u\|_{r,\,Q} \le K_5 + K_7 \|u\|_{a,\,Q} \le K_5 + K_7 K^a, \quad K_7 > 0.$$

Hence and by (3.5)

$$\|u\|_X \le K_6(K_5 + K_7 K^a), \quad u \in F^{-1}(S).$$

(ii) Since $\dim X = +\infty$ and A is a nonconstant and closed mapping on X, then by Proposition 2.4 (ii) it is proper on X. From Lemma 3.3 the operator N is completely continuous on X. From (i) of this lemma F is coercive on X. The Proposition 2.5 (ii) concludes the proof of (ii) and the proof of Lemma 3.4.

In the following lemmas we shall consider the continuous nonlinearity f. Conditions for the continuous F-differentiability of the Nemitskiĭ operator N give the following lemma.

Assumption N.2 For $l = 1,\dots,p$ and the multiindices β with the modulus $0 \le |\beta| \le 2b-1$,

$$\frac{\partial f}{\partial v_{\beta,\,l}} \in C(cl\,Q \times R^{\kappa}, R^p)$$

where κ represents the number of all components in the vector function $\overline{D}{}_x^{\beta} u$ from (1.1).

Lemma 3.5 *Let the Nemitskiĭ operator $N\colon X \to Y$ satisfy Assumptions N.1 and N.2. Then*

(i) *the operator N is continuously Frechét differentiable on X, i.e. $N \in C^1(X,Y)$;*
(ii) *if moreover* $(\mathrm{S}^{\alpha+\rho})$ *for $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$ holds, then $F \in C^1(X,Y)$.*

Proof (i) We need to prove that Frechét derivative $N'\colon X \to L(X,Y)$ defined by the vector equation

$$N'(u)h(t,x) = \sum_{\substack{0\le|\beta|\le 2b-1\\ \mathrm{card}\{\beta,l\}=\kappa\\ l=1,\dots,p}} \frac{\partial f}{\partial v_{\beta}}[t,x,\overline{D}{}_x^{\gamma}u(t,x)]D_x^{\beta}h_l(t,x) \tag{3.6}$$

is continuous on X for every $u, h \in X$. Here $\beta = (\beta_1,\dots,\beta_n)$ represents every multiindex $\gamma = (\gamma_1,\dots,\gamma_n)$ appearing in the nonlinearity f. It is sufficient to show for every fixed $v \in X$ the implication:

$$\forall\,\varepsilon > 0 \quad \exists\,\delta(\varepsilon,v) > 0 \quad \forall\,u \in X, \quad \|u-v\|_X < \delta \Rightarrow \|N'u - N'v\|_{L(X,Y)} < \varepsilon,$$

i.e.

$$\sup_{h\in X,\,\|h\|_X\le 1} \|N'(u)h - N'(v)h\|_Y < \varepsilon \tag{3.7}$$

Let us take an arbitrary $\varepsilon > 0$ and $u \in X$ such that $\|u-v\|_X < \delta$, i.e. $|D_t u_l(t,x) - D_t v_l(t,x)| < \delta$ and $|D_x^k u_l(t,x) - D_x^k v_l(t,x)| < \delta$ for all multiindices $0 \le |k| \le 2b$ on $cl\,Q$. Hence with respect to the uniform continuity of $\frac{\partial f}{\partial v_{\beta,l}}$ for $0 \le |\beta| \le 2b-1$, $l = 1,\dots,p$, on every compact of $cl\,Q \times R^{\kappa}$ we get the vector inequality

$$\begin{aligned} &|N'(u)h(t,x) - N'(v)h(t,x)| \\ &\quad \le \sum_{\substack{0\le|\beta|\le 2b-1\\ \mathrm{card}\{\beta\}=\kappa\\ l=1,\dots,p}} \left|\frac{\partial f}{\partial v_{\beta,l}}[t,x,\overline{D}{}_x^{\gamma}u(t,x)] - \frac{\partial f}{\partial v_{\beta,l}}[t,x,\overline{D}{}_x^{\gamma}v(t,x)]\right| \left|D_x^{\beta}h_l(t,x)\right| < \varepsilon J \end{aligned}$$

for $\|h\|_X \leq 1$ and all $(t,x) \in cl\, Q$. It finishes the proof of (3.7).

(ii) We easily see that Fréchet derivative $F'\colon X \to L(X,Y)$ is defined by the vector equation

$$F'(u)h(t,x) = D_t h(t,x) - \sum_{0\leq |k| \leq 2b} a_k(t,x) D_x^k h(t,x) + N'(u)h(t,x)$$

for $u, h \in X$. Hence and by (i) we get $F \in C^1(X,Y)$.

Lemma 3.6 *Let the hypotheses* $(\mathrm{S}^{\alpha+\rho})$, $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$, *(A.1), (N.1) and (N.2) be satisfied. Then* $F = A + N\colon X \to Y$ *is a nonlinear Fredholm operator of the zero index on* X*.*

Proof According to Lemma 3.2 the operator $A\colon X \to Y$ is a linear continuous and C^1–Fredholm mapping of the zero index. By the statement of Lemma 3.3 the operator $N\colon X \to Y$ is compact. By Lemma 3.5 it belongs to the class C^1. Then Proposition 2.8 implies that F is a nonlinear Fredholm operator with the zero index.

4 The Solution Set for Continuous Nonlinearities

The first results for that proper mapping F give the following theorem.

Theorem 4.1 *Let hypotheses* $(\mathrm{S}^{\alpha+\rho})$ *for* $\alpha \in (0,1)$, $\rho > 0$, $\alpha+\rho < 1$, *and Assumptions A.1, N.1 hold. Then*

(a) *for any compact set of the right hand sides* $g \in Y$ *of (1.1) the corresponding set of all solutions of (1.1) – (1.3) is a countable union of compact sets;*

(b) *for* $u_0 \in X$ *there exists a neighborhood* $U(u_0)$ *of* u_0 *and* $U(F(u_0))$ *of* $F(u_0) \in Y$ *such that for each* $g \in U(F(u_0))$ *there is an unique solution of (1.1) – (1.3) iff the operator* F *is locally injective at* u_0*;*

(c) *let moreover (F.1) hold. Then for any compact set of the right hand sides* $g \in Y$ *from (1.1), the set of all solutions of (1.1) – (1.3) is compact (possible empty).*

Proof (a) Since $F = A + N$ (see (3.3)) by the decomposition of $A = C + T$ (Proposition 2.1) we have $F = C + (T + N)$, where C is a continuous and proper mapping X onto Y (see Proposition 2.4), A is a Fredholm operator of the zero index, T and N are completely continuous mappings. Since X is a countable union of closed balls in X, so with respect to Proposition 2.5 (i) the operator F is σ-proper (continuous). Lemma 3.1 (i) implies assertion (a).

(b) Suppose that F is injective in a neighborhood $U(u_0)$ of $u_0 \in X$. From the decomposition (for H see Lemma 3.2)

$$F = H + (T + N)$$

we obtain $H^{-1}F = I + H^{-1}(T+N)$ which is a completely continuous and injective perturbation of the identity $I\colon X \to Y$ in $U(u_0)$. According to Proposition 2.7 (i) the set $H^{-1}F(U(u_0))$ is open in X and the restriction $H^{-1}F|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $H^{-1}F(U(u_0))$. Therefore F is locally invertible at u_0. Again by Lemma 3.1 (i) we obtain (b).

(c) By Lemma 3.4 (ii) the operator $F\colon X \to Y$ is proper which implies the given assertion and includes the proof of Theorem 4.1.

We have the following theorem on further qualitative and quantitative properties of the set solutions of (1.1) – (1.3).

Proof First of all we see that conditions (k) and (l) are mutually equivalent to the conditions

(k′) $F(D_b) \subset F(X \setminus D_b)$

and

(l′) $Y \setminus R_b$ is a connected set and $F(X \setminus D_b) \setminus R_b \neq \varnothing$,

respectively.

From the proof of Theorem 4.2 (f) we have $D_b = \Sigma$.

(k) From (k′) we have $F(X) = F(D_b) \cup F(X \setminus D_b) = F(X \setminus D_b)$. So $R(F) = F(X)$ is closed and connected in Y (Theorem 4.2 (e)) as well as open set in Y (see Theorem 4.2 (f)). Thus $R(F) = Y$ which implies the surjectivity of F.

(l) By (h) of Theorem 4.2, $\operatorname{card} F^{-1}(\{g\})$ is a constant $k \geq 0$ for every g from the same component of $Y \setminus R_b$.

If $k = 0$ for all $g \in Y \setminus R_b$ such that $F(X) = R_b$, whence $F(X \setminus D_b) \subset R_b$. It is a contradiction with (l′).

Assumptions S.1 There exists a constant $K^a > 0$ such that all solutions $u \in X$ of the initial boundary-value problem for the equation

$$Hu + \mu(Au - Hu + Nu) = 0, \quad \mu \in (0,1)$$

with data (1.2), (1.3) fulfil inequality (3.4) from Lemma 3.4. H is the linear homeomorphism from Assumption A.1.

Theorem 4.4 *Let* ($\mathrm{S}^{\alpha+\rho}$) *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 hold together with the hypothesis S.1.*

Then

(m) *problem (1.1) – (1.3) has at least one solution for each* $g \in Y$*;*

(n) *the number* n_g *of solutions (1.1) – (1.3) is finite, constant and different from zero on each component of the set* $Y \setminus R_b$ *(for all* g *belonging to the same component of* $Y \setminus R_b$*).*

Proof (m) It is sufficient to prove the surjectivity of $F\colon X \to Y$. By Lemma 3.2 (see the proof of (ii)) we can write

$$F = A + N = H + (T + N)$$

The mapping

$$H^{-1}F = I + H^{-1}(T + N)\colon\ X \to X$$

is a completely continuous and condensing field (see [31, P. 496]).

Let $S \subset X$ be a bounded set. Then $H(S)$ is a bounded set in Y. From the coercivity of F (see Lemma 3.4 (i)) the set $F^{-1}[H(S)] = (H^{-1}F)^{-1}(S)$ is bounded at X. Hence $H^{-1}F$ is coercive.

Now we show that condition (iii) from Proposition 2.6 is satisfied for the condensing and coercive field $P = H^{-1}F$. Take the strictly solvable field $G(u) = u$. Then the equation $P(u) = kG(u)$ implies

$$(H^{-1}F)(u) = ku.$$

Hence we get for $u \in X$ and $k < 0$

$$Hu + (1-k)^{-1}[Au - Hu + Nu] = 0$$

Theorem 4.2 *Let hypotheses* ($S^{\alpha+\rho}$) *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 be satisfied. For solutions of (1.1) – (1.3) the following statements are true:*

(d) *the set of solution for each* $g \in Y$ *is compact (possible empty);*
(e) *the set* $R(F) = g \in Y$ *such that there exists at least one solution* $u \in X$ *of (1.1) – (1.3) is closed and connected in* Y*;*
(f) *the domain of bifurcation* D_b *is closed in* X *and the bifurcation range* R_b *is closed in* Y*. The set* $F(X \setminus D_b)$ *is open in* Y*;*
(g) *if* $Y \setminus R_b \neq \varnothing$, *then each component of* $Y \setminus R_b$ *is a nonempty open set (i.e. domain);*
(h) *if* $Y \setminus R_b \neq \varnothing$, *the number* n_g *of solutions is finite and constant (it may be zero) on each component of* $Y \setminus R_b$, *i.e.* n_g *is the same nonnegative integer for each* g *belonging to the same component of* $Y \setminus R_b$*;*
(i) *if* $R_b = \varnothing$, *then the given problem has a unique solution* $u \in X$ *for each* $g \in Y$ *and this solution continuously depends on* g *as a mapping from* Y *onto* X*;*
(j) *if* $R_b \neq \varnothing$, *then the boundary* $\partial F(X \setminus D_b)$ *is a subset of* $F(D_b) = R_b$ ($\partial F(X \setminus D_b) \subset F(D_b)$)*.*

Proof The assertion (d) follows directly from Theorem 4.1 (c).

(e) Take the sequence $\{g_n\}_{n\in N} \subset R(F) \subset Y$ converging to $g \in Y$ as $n \to \infty$. By (d) there is a compact set of all solutions $\{u_\gamma\}_{\gamma\in I} \subset X$ (here I means an index set) of the equations $F(u) = g_n$ for $n = 1, 2, \dots$. Thus there exists a subsequence $\{u_{n_k}\}_{k\in N} \subset \{u_\gamma\}_{\gamma\in I}$ converging to $u \in X$ and $F(u_{n_k}) = g_{n_k} \to g$ in Y as $n \to \infty$. Since the mapping F is proper (Lemma 3.4 (ii)) by Proposition 2.4 (i) it is closed, whence $F(u) = g$, i.e. $g \in R(F)$. The set $R(F)$ is closed. $R(F) = F(X)$ is connected as a continuous image of the connected set X.

(f) According to Lemma 3.1 (ii) $D_b = \Sigma$ and $R_b = F(D_b) = F(\Sigma)$. Since $X \setminus \Sigma$ is an open set then D_b is closed in X and its continuous image R_b is a closed set in Y.

Since, $X \setminus D_b = X \setminus \Sigma$ is the set of all points at which the mapping F is locally invertible, to each $u_0 \in X \setminus D_b$ there is a neighborhood $U_1(F(u_0)) \subset F(X \setminus D_b)$. It means, the set $F(X \setminus D_b)$ is open.

(g) The set $Y \setminus R_b = Y \setminus F(D_b) \neq \varnothing$ is open in Y. Then each its component is nonempty and open, too.

(h) This directly follows from Proposition 2.2.

(i) By $R_b = \varnothing$ is $D_b = \varnothing$ and the mapping F is locally invertible in X. Proposition 2.5 (ii) asserts that F is a proper mapping. Then from the global inverse mapping theorem (Proposition 2.3) implies F is homeomorphism X onto Y.

(j) From Lemma 3.1 (ii) $D_b = \Sigma$ and by (f) D_b and $F(D_b)$ are closed. Then $\partial F(X \setminus D_b) = \partial F(D_b) \subset F(D_b)$.

This finishes the proof of the theorem.

The following two theorems are on the surjectivity of (1.1) – (1.3).

Theorem 4.3 *Under the assumptions* ($S^{\alpha+\rho}$), $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1 and F.1 each of the following conditions is sufficient for the solvability of problem (1.1) – (1.3) for each* $g \in Y$*:*

(k) *for each* $g \in R_b$ *there is a solution* $u \in X \setminus D_b$ *of (1.1) – (1.3);*
(l) *the set* $Y \setminus R_b$ *is connected and there is* $g \in R(F) \setminus R_b$ *(for* $R(F)$ *see Theorem 4.2 (e)).*

where $(1-k)^{-1} \in (0,1)$. With respect to Assumption S.1

$$\|u\|_{a,Q} \le K^a$$

for $a = \max\{|\gamma|, r\}$, where $|\gamma| = 0,1,\dots,2b-1$ and $0 \le r \le 2b-1$ are fixed. Using the same method as in Lemma 3.4 (i) we obtain for all solutions of

$$(H^{-1}F)u = ku$$

the estimation $\|u\|_X \le K_8$, $K_8 > 0$. By Proposition 2.6 we have the strict surjectivity of $H^{-1}F$ and so F. This proves (m).

(n) From the surjectivity of F on X it follows $n_g \neq 0$. The other assertions of (n) follow from Theorem 4.2 (h).

Example 4.1 The simple example illustrating results of this part can be the initial boundary-value problem for the system of p equations.

$$\frac{\partial u_l}{\partial t} - K_l \frac{\partial^2 u_l}{\partial x^2} + f_l(u) = 0, \quad (t,x) \in \langle 0, T\rangle \times \Omega \subset R \times R,$$

where $l = 1,\dots,p$ with the conditions

$$\frac{\partial u_l}{\partial x}(t,0) = \frac{\partial u_l}{\partial x}(t,1) = 0, \quad t \in \langle 0, T\rangle,$$
$$u_l(0,x) = 0, \ x \in cl\,\Omega.$$

We take $K_l > 0$ and

$$f_l(u) = \begin{cases} u_l^{1/2}, & \text{if } u_l \in \langle 0, a\rangle, \\ a^{1/2}, & \text{if } u_l \in \langle a, \infty), \\ 0, & \text{if } u_l \le 0, \end{cases}$$

for $l = 1,\dots,p$. Assumption A.1 is satisfied by Proposition 1.1. The condition N.1 can be verified by elementary calculus. The supposition F.1 follows from equation (2.3) and Green matrix estimations (2.1). The condition $(\mathrm{S}^{\alpha+\rho})$ holds for $0 < \alpha < 1/2$, $1/2 < \rho < 1$ and $\alpha + \rho < 1$ (for example $\alpha = 1/5$, $\rho = 3/5$).

5 The Solution Set for C^1-nonlinearities

With respect to the C^1-,differentiability of the operator N from (3.2) we prove here several stronger results than in Chapter 4 for the solutions of (1.1) – (1.3).

Theorem 5.1 *Suppose that* $(\mathrm{S}^{\alpha+\rho})$ *for* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$ *and Assumptions A.1, N.1, N.2 and F.1 are satisfied and* R_b *means the bifurcation range of (1.1) – (1.3) from Definition 3.1. Then the set* $Y \setminus R_b$ *is open and dense in* Y *and thus the bifurcation range* R_b *of initial boundary-value problem (1.1) – (1.3) is nowhere dense in* Y.

Proof The openness of $Y \setminus R_b$ follows from the statement (f) of Theorem 4.2.

From previous lemmas the operator $A\colon X \to Y$ is a linear continuous Fredholm mapping of the zero index and the Nemitskiĭ operator $N\colon X \to Y$ is compact and $N \in C^1(X,Y)$.

For every $u \in X$ the linear operator $N'\colon X \to Y$ from (3.6) is completely continuous on X. By the Nikoľskiĭ decomposition theorem (see Proposition 2.1) the operator $F'(u) = A + N'(u)\colon X \to Y$ is a linear Fredholm mapping of the zero index for each $u \in X$. By Lemma 3.5 (ii) there is $F \in C^1(X,Y)$ and by Lemma 3.6 the F is a nonlinear Fredholm operator of the zero index.

According to the Banach open mapping theorem (see [30, P. 77]) the mutual equivalence is true: $F'(u)$ is a linear homeomorphism iff it is a bijective mapping. Since $F'(u)$ for every $u \in X$ is a linear Fredholm mapping of the zero index so $F'(u)$ is bijective iff it is injective (in this case the injectivity implies surjectivity, see Proposition 8.14 (1) from [31, P. 366]). We see that $u \in X$ is a singular point of the Fredholm operator F iff u is a critical point of F.

From Proposition 2.10 we obtain that set Σ (of all points $u \in X$ for which F is not locally invertible) is a subset of all critical point F. Then, evidently Σ is a subset of all singular points S of F, i.e. $\Sigma \subset S$. Hence we get for the set of regular values R_F of the operator F the relations

$$R_F = Y \setminus F(S) \subset Y \setminus F(\Sigma) \subset Y \setminus R_b \subset Y,$$

where $R_b \subset F(\Sigma)$ is a bifurcation range of F.

Since $F\colon X \to Y$ is nonconstant closed mapping with $\dim X = \infty$, by Proposition 2.4 we obtain that F is a proper mapping. By Proposition 2.9 (the Quinn version) the set R_F is residual, open and dense in Y. Hence $Y \setminus R_b$ is dense in Y, too. With respect to Lemma 3.1 (ii) we can conclude the proof.

In the following results we shall deal with the linear problem in $h \in X$

$$Ah(t,x) + \sum_{\substack{0 \le |\beta| \le 2b-1 \\ \mathrm{card}\{|\beta|\} = \kappa}} \frac{\partial f}{\partial v_\beta}[t,x,D_x^\gamma u(t,x)] D_x^\beta h(t,x) = g(t,x) \tag{5.1}$$

for $(t,x) \in Q$ and some fixed $u \in X$ with condition (1.2), (1.3). The left side of equation (5.1) represents the Frechét derivative $F'(u)h$ of the operator $F = A + N\colon X \to Y$.

Theorem 5.2 *Let the hypotheses* $\mathrm{S}^{\alpha+\rho}$ *with* $\alpha \in (0,1)$, $\rho > 0$, $\alpha + \rho < 1$, *and Assumptions A.1, N.1, N.2 and F.1 are satisfied. Then*

(o) *the number solutions of (1.1) – (1.3) is constant and finite (it may be zero) on each connected component of the open set* $Y \setminus F(S)$, *i.e. for any* g *belonging to the same connected component of* $Y \setminus F(S)$. *Here* S *means the set of all critical points of the operator* $F = A + N\colon X \to Y$;

(p) *let* $u_0 \in X$ *be a regular solution of (1.1) – (1.3) with the right hand side* $g_0 \in Y$. *Then there exists a neighborhood* $U(g_0) \subset Y$ *of* g_0 *such that for any* $g \in U(g_0)$ *initial-boundary value problem (1.1) – (1.3) has one and only one solution* $u \in X$. *This solution continuously depends on* g.
The associated linear problem (5.1), (1.2), (1.3) for $u = u_0$ *has a unique solution* $h \in X$ *for any* g *from a neighborhood* $U(g_0)$ *of* $g_0 = F(u_0)$. *This solution continuously depends on* g;

(q) *denote by G the set of all right hand side $g \in Y$ of equation (1.1) for which the corresponding solutions $u \in X$ of (1.1) – (1.3) are its critical points. Then G is closed nowhere dense in Y;*

(r) *if the singular points set of (1.1) – (1.3) is empty, then this problem has unique solution $u \in X$ for each $g \in Y$. It continuously depends on the right hand side g.*

Proof (o) In the proof of Theorem 5.1 we have shown that the set of all singular points of F is equal to the set of all critical points of F. Then the Ambrosetti theorem (see Proposition 2.2) implies the statement (o).

(p) Since $u_0 \in X \setminus S$, where S is a set of all singular (in our case all critical) points then by Proposition 2.10 the mapping F is a local C^1-diffeomorphism at u_0. This proves first part of (p) for (1.1) – (1.3).

From F as the C^1-diffeomorphism follows that $F' \in C(X,Y)$, $(F^{-1})' \in C(X,Y)$, where $F'(u)h$ is the left hand side of (5.1) and $(F^{-1})'(Fu) = (F'(u))^{-1}$ for every $u \in X$. Hence linear problem (5.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X$ for any $g \in U(g_0)$ with $g_0 = F(u_0)$. This solution continuously depends on all right hand side g. The proof of (p) is completed.

(q) In our case the equality $G = F(S)$ holds, where S is the set of all critical (all singular) points of F. By the Smale–Quinn theorem (Proposition 2.9) we obtain the expected results.

(r) By Proposition 2.10, the operator $F\colon X \to Y$ is a local C^1-diffeomorphism at any point $u \in X$. Hence follows the last assertion.

Assumption H.1 Linear homogeneous problem (5.1), (1.2), (1.3) (for $g = 0$) has only zero solution $h = 0 \in X$ for any $u \in X$.

By the point (p) of Theorem 5.2 we obtain the following corollary.

Corollary 5.1 *Let the hypotheses of Theorem 5.2 and Assumption H.1 hold. Then initial boundary-value problem (1.1) – (1.3) has a unique solution $u \in X$ for any $g \in Y$. Moreover, linear problem (5.1), (1.2), (1.3) has a unique solution $h \in X$ for any $u \in X$ and the right hand side $g \in Y$ of (5.1). This solution continuously depends on g.*

Corollary 5.2 *Let the assumptions of Theorem 5.2 be satisfied. Then we have:*

(s) *if the set S of all singular (in our case all critical) points of F is nonempty, then $\partial F(X \setminus S) \subset F(S)$;*

(t) *if $F(S) \subset F(X \setminus S)$, then problem (1.1) – (1.3) has the solution $u \in X$ for any $g \in Y$, i.e. $R(F) = Y$ (F is a surjectivity of X onto Y);*

(u) *if $Y \setminus F(S)$ is connected and $X \setminus S \neq \varnothing$, then $R(F) = Y$ (the solvability of (1.1) – (1.3) for any $g \in Y$).*

Proof By Theorem 5.2 (q) the set $F(S)$ is closed in Y and by Proposition 2.9 $F(X \setminus S)$ is open in Y. Hence we have the equations

$$F(X) = F(S) \cup F(X \setminus S) = F(S) \cup \overline{F(X \setminus S)} = \overline{F(X)} \tag{5.2}$$

which implies that $F(X)$ is a closed set.

(s) Since $F \in C^1(X,Y)$ we get $\Sigma \subset S$, as in Theorem 5.1. Hence and by Theorem 4.2 (i)

$$\partial F(X \setminus S) \subset \partial F(X \setminus \Sigma) \subset F(\Sigma) \subset F(S).$$

(t) From the first equation of (5.2) we have $F(X) = F(X \setminus S)$ and so $R(F)$ is an open as well as a closed subset of the connected space Y. Thus $R(F) = Y$.

(u) Since $Y \setminus F(S)$ is connected, and by Proposition 2.2 we obtain the card $F^{-1}(\{g\}) =$ const $= k \geq 0$ for each $g \in Y \setminus F(S)$.

If $k = 0$, then $F(X) = F(S)$ and $F(X \setminus S) \subset F(S)$ and this is a contradiction with $X \setminus S \neq \varnothing$. Thus $k > 0$.

Assumption H.2 Each point $u \in X$ is either a regular point or an isolated critical point of problem (1.1) – (1.3).

Theorem 5.3 *Suppose that hypotheses* $(\mathrm{S}^{\alpha+\rho})$ *with* $\alpha \in (0,1)$*,* $\rho > 0$*,* $\alpha + \rho < 1$*, and Assumptions A.1, N.1, N.2, F.1 and H.2 hold. Then for every* $g \in Y$ *there exists one solution* $u \in X$ *of (1.1) – (1.3). It continuously depends on* g*.*

Proof The associated operator $F\colon X \to Y$ is a proper C^1-Fredholm mapping of the zero index. By Proposition 2.10 F is a local C^1-diffeomorphism at a regular point of F. In the isolated singular point, by Proposition 2.11 F is locally invertible. Since F is proper, the global inverse mapping theorem (see Proposition 2.3) implies the statement of this problem.

Example 5.1 Example 4.1 illustrates the results of Chapter 5 for $f_l(u) = \sin\left(\sum_{i=1}^{l} u_i^2\right)$.

6 Conclusion

The studied models describe different natural science phenomena (a reaction-diffusion and environment models, a diffusive waves in fluid dynamics — the Burges equation, the wave propagation in a large number of biological and chemical systems — the Fisher equation, a nerve pulse propagation in nerve fibers and wall motion in liquid crystals).

We can apply the Fredholm theory to hyperbolic equations modeling different nonlinear vibration problems, to a nonlinear dispersion (the nonlinear Klein–Gordan equation), a propagation of magnetic flux and the stability of fluid notions (the nonlinear Sine–Gordan equation) and so on.

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Influence of Propellant Burn Pattern on the Attitude Dynamics of a Spinning Rocket

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Abstract: This study examines the effect of various propellant burn geometries on the attitude dynamics of a rocket-type variable mass system. The three burn scenarios studied are the end burn, the centripetal burn, and the radial burn. Results of this study indicate that a change in burn scenario changes the predicted attitude motion. The differences are more pronounced for spin motion than for transverse attitude motion. The end burn is recommended whenever it is practically feasible; it is found to be the least disruptive from the point of view of attitude dynamics.

Keywords: *Rockets; variable mass systems; attitude dynamics.*

Mathematics Subject Classification (2000): 70P05, 70M20, 34C60.

1 Introduction

In the study of the dynamics of rockets, the fact that the system undergoes substantial mass variation is generally captured in one of two ways. One method is to view the system as a solid whose mass and inertia vary as functions of time [4, 5]. The exact time functions used for both the mass and inertia scalars are based on reasonable guesses of what is likely to occur in real systems. Another approach is to show the propellant as a subsystem of the rocket, and then specify the physical and geometric manner in which the propellant mass is depleted. These facts are then used for the precise calculation of the mass and inertia functions for the system. Naturally, the second approach is preferable, since it eliminates the need for guessing the time histories of the mass/inertia properties. However, authors that have utilized this second approach have generally used very simple

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models for both the rocket system and the propellant [2, 3]. When a more complex model has been used [6], only one propellant burn pattern – the radial burn – is examined. The radial burn assumes that a cylindrically shaped solid propellant is ignited on its axis, and burns radially outwards towards its periphery. Yet, there are situations where it makes sense to assume an end burn for example; that is, a burn in which a cylindrical propellant is ignited at one of its ends, and burns towards the opposite end.

The goal of this paper is to examine if and how a change in burn pattern influences predictions of the attitude behavior of a rocket system. Specifically, three different burn patterns will be compared: the end burn, the centripetal burn, and the radial burn. This is important for two reasons. First, this study will lead to reasonably accurate predictions for a case that is in fact best captured by one of the burn scenarios studied, and for which results were previously unavailable. The second reason is that the results can be used as design tool in determining the type of propellant burn that should be implemented in order to produce certain desired dynamic effects.

2 Equations of Attitude Motion

The system studied here is a solid rocket motor and its payload, shown schematically in Figure 2.1. B represents the rocket's main body, assumed rigid, and F is the solid fuel. The products of combustion are expelled through the nozzle. Both B and F are assumed to be axisymmetric, with a common axis z, and F burns so as to remain axisymmetric at all times. The mass centers F^* of F, B^* of B, and S^* of the overall system S all lie on the axis z. Furthermore, we assume that the motion of the gas products of combustion relative to the rocket body is either axial, or symmetric with respect to the z-axis and with no transverse component. Finally, for this study, the velocity distribution of the exhaust gas particles as they traverse the nozzle exit plane is taken to be uniform as shown in Figure 2.1. The equations of attitude motion for this system can be written in the form (see, for example, [1, 6]):

$$I\dot{\omega}_1 + \left[\dot{I} - \dot{m}\left(z_e^2 + \frac{R_1^2}{4}\right)\right]\omega_1 + [(J - I)\omega_3]\omega_2 = 0, \tag{1}$$

$$I\dot{\omega}_2 + \left[\dot{I} - \dot{m}\left(z_e^2 + \frac{R_1^2}{4}\right)\right]\omega_2 - [(J - I)\omega_3]\omega_1 = 0, \tag{2}$$

$$J\dot{\omega}_3 + \left(\dot{J} - \dot{m}\,\frac{R_1^2}{2}\right)\omega_3 = 0, \tag{3}$$

where J and I are the system's overall central axial and transverse moments of inertia respectively, m is the mass, ω_i $(i = 1, 2, 3)$ are the components of the inertial angular velocity of B in the $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ directions (see Figure 2.1), R_1 is the radius of the nozzle at the exit plane, and z_e is the distance from the overall system mass center, S^*, to the nozzle exit plane.

In order to generate non-dimensional versions of equations (1) – (3), we introduce

$$m_r = -\dot{m} = \int (\boldsymbol{v} \cdot \boldsymbol{b}_3)\rho\, ds = \pi\rho U R_1^2 \tag{4}$$

and

$$m_F = m_{FO} - m_r t, \tag{5}$$

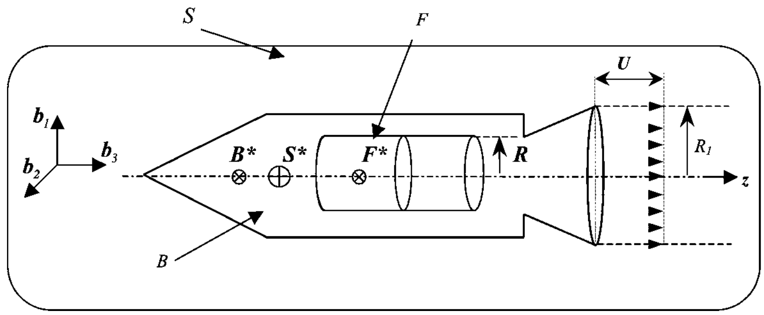

Figure 2.1. Rocket system with solid propellant.

where $\boldsymbol{v}$ is the velocity of exhaust fluid particles relative to the body B, ρ is the mass density of the exhaust gas, m_{FO} is the mass of the solid fuel at ignition, m_F is the instantaneous mass of the fuel, U is the constant magnitude of the axial velocity of the exhaust fluid particles as they cross the nozzle exit plane, and t is time. Hence, the time from ignition to burnout, t_b, is given by

$$t_b = m_{FO}/m_r. \tag{6}$$

Dimensionless time τ, is defined as

$$\tau = t/t_b = (m_r/m_{FO})t. \tag{7}$$

This means that $\tau = 0$ at fuel ignition, and $\tau = 1$ at burnout.

Other useful dimensionless quantities are

$$\bar{m} = m/m_{FO}, \quad \bar{I} = I/m_{FO}R^2, \quad \bar{J} = J/m_{FO}R^2, \quad \text{and} \quad \bar{\omega}_i = \omega_i t_b, \tag{8}$$

where R is the outer radius of the cylindrical propellant grain. Equations (1), (2), and (3) then become respectively

$$\bar{I}\bar{\omega}_1' + \left\{\bar{I}' - \bar{m}'\left[\left(\frac{z_e}{R}\right)^2 + \frac{\beta^2}{4}\right]\right\}\bar{\omega}_1 + [(\bar{J} - \bar{I})\bar{\omega}_3]\bar{\omega}_2 = 0, \tag{9}$$

$$\bar{I}\bar{\omega}_2' + \left\{\bar{I}' - \bar{m}'\left[\left(\frac{z_e}{R}\right)^2 + \frac{\beta^2}{4}\right]\right\}\bar{\omega}_2 - [(\bar{J} - \bar{I})\bar{\omega}_3]\bar{\omega}_1 = 0, \tag{10}$$

and

$$\bar{J}\bar{\omega}_3' + \left(\bar{J}' - \bar{m}'\,\frac{\beta^2}{2}\right)\bar{\omega}_3 = 0. \tag{11}$$

In the above equations, a prime indicates derivative with respect to the dimensionless time variable τ, and β is the nozzle expansion ratio (R_1/R).

From equation (11),

$$\frac{\bar{\omega}_3(\tau)}{\bar{\omega}_3(0)} = \exp\left[-\int_0^\tau \frac{\psi(\tau)}{\bar{J}}\,d\tau\right], \tag{12}$$

where

$$\psi(\tau) = \left(\bar{J}' - \bar{m}'\,\frac{\beta^2}{2}\right). \tag{13}$$

Next, we follow established tradition [3, 6], and define complex angular velocity

$$\bar{\omega}_T = \bar{\omega}_1 + i\bar{\omega}_2, \tag{14}$$

where $i = \sqrt{-1}$. Equations (9) and (10) are then combined to give

$$\frac{\bar{\omega}_T(\tau)}{\bar{\omega}_T(0)} = \left\langle \exp\left[-\int_0^\tau \frac{\varphi(\tau)}{\bar{J}}\,d\tau\right]\right\rangle \cdot \left\langle \exp\left[i\int_0^\tau \Theta\,d\tau\right]\right\rangle, \tag{15}$$

where

$$\varphi(\tau) = \bar{I}' - \bar{m}'\left[\left(\frac{z_e}{R}\right)^2 + \frac{\beta^2}{4}\right] \tag{16}$$

and

$$\Theta = [(\bar{J}/\bar{I}) - 1]\bar{\omega}_3. \tag{17}$$

It is clear from (15) that the magnitude of the transverse angular velocity vector is controlled by the function $\varphi(\tau)$, while $\Theta(\tau)$ governs the frequency. On the other hand, the sign of $\psi(\tau)$ [see (12)] is an indication of whether the spin rate increases or decreases with τ.

3 Spin Motion

To study the spin rate of the rocket body during propellant burn, it is necessary [see equations (12) and (13)] to determine expressions for instantaneous system mass and inertia. One way to determine these functions is to select a propellant depletion strategy. For this study, we choose to examine three different propellant depletion scenarios: the End Burn, the Centripetal Burn, and the Radial Burn. As the names indicate, End Burn refers to the case where the propellant burns from end to end. Centripetal Burn is the unusual case where propellant burn proceeds radially inwards from the outermost part of the fuel, and Radial Burn is the case where combustion starts from the propellant axis, and proceeds radially outwards.

3.1 End Burn

For the purpose of this study, the solid propellant F is assumed to be a solid cylinder prior to ignition. For the end burn, this cylindrical fuel burns from the end closest to the nozzle towards the opposite end. The burn proceeds uniformly, in the sense that the unburned fuel is always a cylinder of the same radius as at ignition but with diminishing length as shown in Figures 2.1 and 3.1.

Using the symbols defined in Figure 3.1, the mass of fuel F at ignition is

$$m_{FO} = \rho_{FO}\pi R^2 L \tag{18}$$

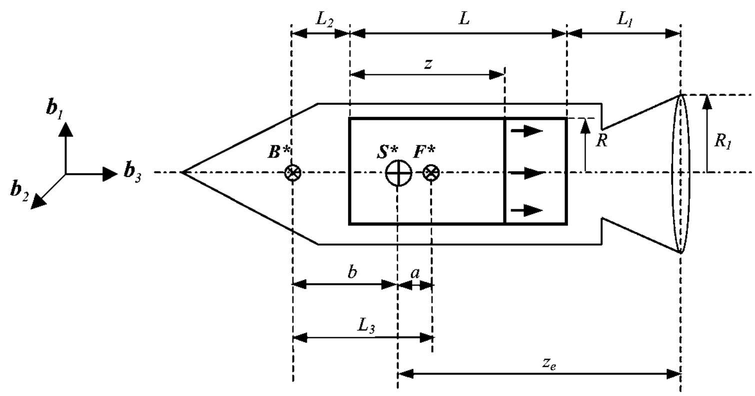

Figure 3.1. Rocket model with end burning propellant.

and the mass at some intermediate stage of the burn is

$$m_F = \rho_{FO} \pi R^2 z, \tag{19}$$

where z is the instantaneous length of the solid cylindrical propellant and ρ_{FO} is its density. From equations (6), (18), and (19), the time from ignition to burnout is

$$t_b = \frac{m_{FO}}{-\dot{m}_F} = \frac{\rho_{FO}\pi R^2 L}{-\rho_{FO}\pi R^2 \dot{z}} = \frac{L}{-\dot{z}}. \tag{20}$$

Integrating (20), we obtain

$$\frac{z}{L} = 1 - \tau. \tag{21}$$

The dimensionless mass of the propellant is

$$\bar{m}_F = \frac{m_F}{m_{FO}} = \frac{\rho_{FO}\pi R^2 z}{\rho_{FO}\pi R^2 L} = \frac{z}{L} = 1 - \tau. \tag{22}$$

Hence,

$$\bar{m}' = \bar{m}_F' = -1. \tag{23}$$

The axial moment of inertia of the propellant is

$$J_F = \frac{m_F R^2}{2} \tag{24}$$

and the dimensionless version is

$$\bar{J}_F = \frac{J_F}{m_{FO} R^2} = \frac{\bar{m}_F}{2} = \frac{1-\tau}{2}. \tag{25}$$

The combined axial moment of inertia of the system is

$$\bar{J} = \bar{J}_B + \bar{J}_F = \bar{J}_B + \frac{(1-\tau)}{2}, \tag{26}$$

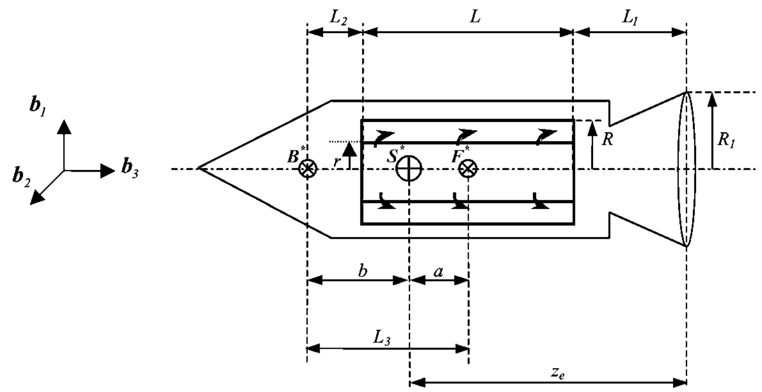

Figure 3.2. Rocket with propellant in centripetal burn.

where the subscripts B and F refer to bodies B and F respectively of Figure 2.1. Hence,

$$\bar{J}' = \bar{J}'_F = -\frac{1}{2}. \tag{27}$$

Substituting (23) and (27) into (13), we get

$$\psi(\tau) = -\frac{1}{2}(1 - \beta^2). \tag{28}$$

From (28), $\psi(\tau)$ is a constant that can be negative, zero, or positive depending on the value of the nozzle expansion ratio β. There is thus a threshold value $\beta = \beta_L = 1$ for which the spin rate remains constant throughout the burn. The spin rate increases from ignition to burnout if $\beta > \beta_L$, and decreases from ignition to burnout for $\beta < \beta_L$. From (12), (26) and (28), a closed form solution can be shown to be

$$\frac{\bar{\omega}_3(\tau)}{\bar{\omega}_3(0)} = \left[\frac{2\bar{J}_B + 1}{2\bar{J}_B + 1 - \tau}\right]^{[1-\beta^2]}. \tag{29}$$

This expression confirms the above predictions.

3.2 Centripetal Burn

In centripetal burn, the cylindrical solid fuel is ignited at its periphery but not at any of its ends. It then burns radially inwards, with the radius decreasing uniformly along its length in such a way that the intermediate shape of the propellant is always a solid cylinder that has the same length as at ignition, but of decreasing radius (see Figure 3.2).

The mass of F at ignition remains as given by (18), and the intermediate mass of F during the burn is

$$m_F = \rho_{FO}\pi L r^2, \tag{30}$$

where r is the intermediate value of the external radius of the propellant. The time from ignition to burnout in this case is

$$t_b = \frac{m_{FO}}{-\dot{m}_F} = \frac{\rho_{FO}\pi L R^2}{-\rho_{FO}\pi L \frac{d}{dt}\left(r^2\right)} = \frac{R^2}{-\frac{d}{dt}\left(r^2\right)}. \tag{31}$$

This leads to

$$\left(\frac{r}{R}\right)^2 = 1 - \tau. \tag{32}$$

The dimensionless mass of the fuel is

$$\bar{m}_F = \frac{m_F}{m_{FO}} = \frac{\rho_{FO}\pi L r^2}{\rho_{FO}\pi L R^2} = \left(\frac{r}{R}\right)^2 = 1 - \tau \tag{33}$$

and once more,

$$\bar{m}' = \bar{m}'_F = -1. \tag{34}$$

The axial moment of inertia for the propellant is

$$J_F = \frac{m_F r^2}{2}. \tag{35}$$

So,

$$\bar{J}_F = \frac{J_F}{m_{FO}R^2} = \frac{\bar{m}_F}{2}\left(\frac{r}{R}\right)^2 = \frac{(1-\tau)^2}{2}. \tag{36}$$

For the overall system, we have

$$\bar{J} = \bar{J}_B + \bar{J}_F = \bar{J}_B + \frac{(1-\tau)^2}{2}. \tag{37}$$

Thus

$$\bar{J}' = \bar{J}'_F = -(1-\tau). \tag{38}$$

We then substitute (34) and (38) into (13) to obtain

$$\psi(\tau) = \left(\frac{\beta^2}{2} - 1\right) + \tau. \tag{39}$$

Equation (39) indicates that the function $\psi(\tau)$ increases linearly with time with unit slope, and $\psi(1) = \beta^2/2$ is greater than $\psi(0) = \beta^2/2 - 1$. $\psi(1)$ is always positive; however, $\psi(0)$ can be negative, zero, or positive depending on the value of β. Figure 3.3 captures the three possibilities. If the nozzle expansion ratio is equal to or greater than $\beta_L = \sqrt{2}$, the spin rate will decrease from ignition all the way to burnout. Otherwise, the spin rate increases initially, changes sign at some point during the burn, then decreases for the remainder of the burn. The trend reversal occurs at

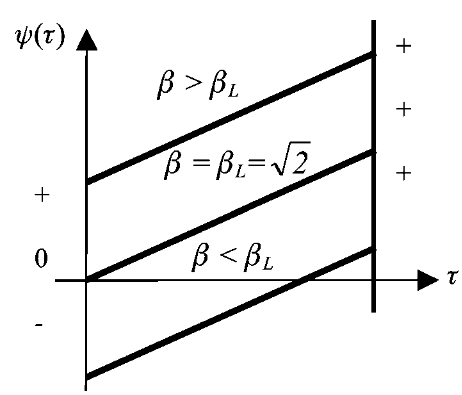

Figure 3.3. Function ψ for centripetal burn.

$$\tau = 1 - \beta^2/2. \tag{40}$$

A closed form solution is also possible for (12) in this case. Using (12), (37) and (39) we obtain

$$\frac{\bar{\omega}_3(\tau)}{\bar{\omega}_3(0)} = \left[\frac{2\bar{J}_B + 1}{2\bar{J}_B + (1-\tau)^2}\right] \cdot \exp\left\{\frac{-\beta^2}{\sqrt{2\bar{J}_B}}\left[\tan^{-1}\frac{\tau\sqrt{2\bar{J}_B}}{2\bar{J}_B + 1 - \tau}\right]\right\}. \tag{41}$$

Figure 3.4 shows plots of the normalized spin rate as a function of τ. The figure confirms the inferences given above.

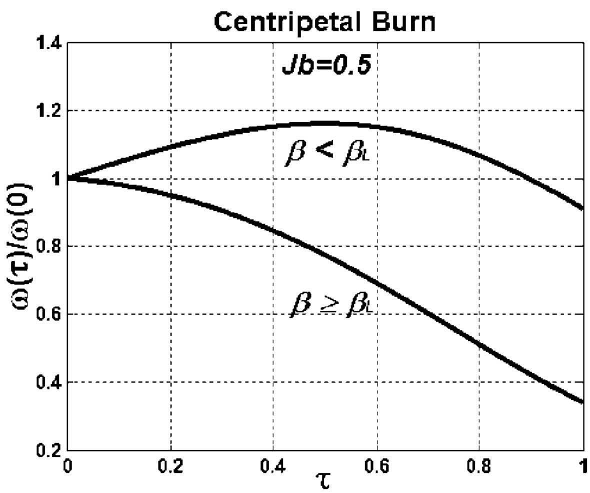

Figure 3.4. Spin behavior for centripetal burn.

3.3 Radial Burn

For radial burn, the cylindrical propellant is ignited along its axis, and burns radially outwards in such a way that the intermediate shape of the propellant is a hollow cylinder, as shown in Figure 3.5. This case was studied in detail in [6], but the highlights will be presented here for completeness.

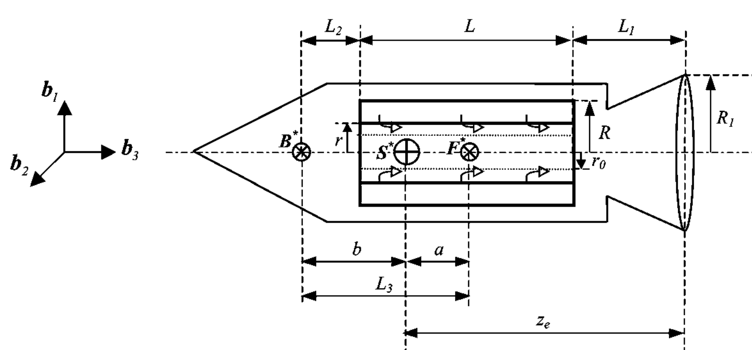

Figure 3.5. Rocket with radially burning propellant.

From Figure 3.5, the mass of propellant before ignition is

$$m_{F_O} = \rho_{FO}\pi L(R^2 - r_0^2) \tag{42}$$

and the propellant mass at some instant after ignition is

$$m_F = \rho_{FO}\pi L(R^2 - r^2). \tag{43}$$

In (42), r_0 is the internal radius of the propellant at ignition. The time from ignition to burnout is thus

$$t_b = \frac{m_{FO}}{-\dot{m}_F} = \frac{\rho_{FO}\pi L(R^2 - r_0^2)}{\rho_{FO}\pi L\,\frac{d}{dt}\,(r^2)} = \frac{R^2 - r_0^2}{\frac{d}{dt}\,(r^2)}. \tag{44}$$

Equation (44) can be integrated to give

$$\left(\frac{r}{R}\right)^2 = \left(\frac{r_0}{R}\right)^2 + \left[1 - \left(\frac{r_0}{R}\right)^2\right]\tau = \gamma^2 + (1-\gamma^2)\tau, \tag{45}$$

where γ is the ratio r_0/R. We get from (42) and (43)

$$\bar{m}_F = \frac{m_F}{m_{FO}} = \frac{\rho_F\pi L(R^2 - r^2)}{\rho_F\pi L(R^2 - r_0^2)} = \frac{1 - (r/R)^2}{1 - (r_0/R)^2} = 1 - \tau \tag{46}$$

and

$$\bar{m}' = \bar{m}_F' = -1. \tag{47}$$

The axial inertia of F is

$$\bar{J}_F = \frac{J_F}{m_{FO}R^2} = \frac{\bar{m}_F}{2}\left[1 + \left(\frac{r}{R}\right)^2\right] = \left[\frac{1-\tau}{2}\right][1 + \gamma^2 + (1-\gamma^2)\tau] \tag{48}$$

and that of the entire system is

$$\bar{J} = \bar{J}_B + \bar{J}_F = \bar{J}_B + \frac{1+\gamma^2}{2} - \gamma^2\tau - \frac{1-\gamma^2}{2}\,\tau^2. \tag{49}$$

Thus,

$$\bar{J}' = \bar{J}_F' = -[\gamma^2 + (1-\gamma^2)\tau]. \tag{50}$$

Equations (13), (47), and (50) give

$$\psi(\tau) = \left(\frac{\beta^2}{2} - \gamma^2\right) - (1-\gamma^2)\tau. \tag{51}$$

This time the function $\psi(\tau)$ varies linearly with τ, and has a slope of $(\gamma^2 - 1)$. The quantity $\gamma = r_0/R$ is strictly less than 1; hence, $\psi(\tau)$ has a negative slope. At ignition, $\psi(0) = (\beta^2/2 - \gamma^2)$, and this is likely to be positive for real rockets. At burnout, $\psi(1) = (\beta^2/2 - 1)$. Hence, when $\beta \geq \beta_L = \sqrt{2}$, the spin rate decreases all the way to burnout, and when $\beta < \beta_L$, the spin rate decreases at first, but then reaches a minimum value when $\tau = (\beta^2/2 - \gamma^2)/(1-\gamma^2)$, and starts to increase all the way to burnout.

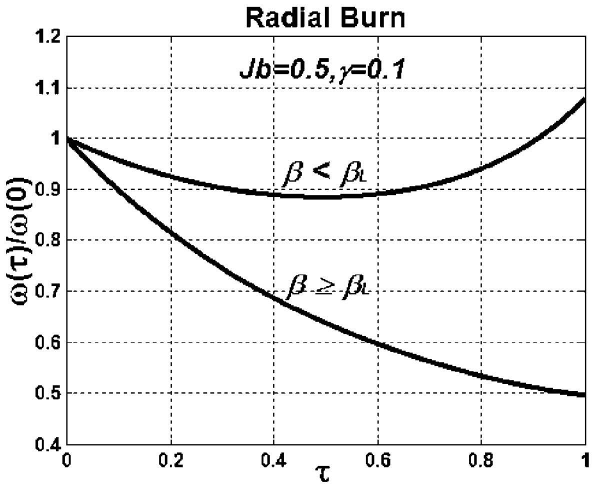

Figure 3.6. Spin rate behavior of radial burn.

Using equations (12), (49), and (51), it is again possible to solve for the spin rate in closed form:

$$\frac{\bar{\omega}_3(\tau)}{\bar{\omega}_3(0)} = \left[\frac{2\bar{J}_B(1-\gamma^2)+1-\gamma^4}{2\bar{J}_B(1-\gamma^2)+1-\left[\gamma^2+(1-\gamma^2)\tau\right]^2}\right] \tag{52}$$

$$\times\exp\left\{\frac{-\beta^2}{\sqrt{2\bar{J}_B(1-\gamma^2)+1}}\left[\tanh^{-1}\frac{\left[\gamma^2+(1-\gamma^2)\tau\right]}{\sqrt{2\bar{J}_B(1-\gamma^2)+1}}-\tanh^{-1}\frac{\gamma^2}{\sqrt{2\bar{J}_B(1-\gamma^2)+1}}\right]\right\}.$$

Figure 3.6 shows two cases that match the above predictions when $\bar{J}_B = 0.5$ and $\gamma = 0.1$ are used as an example.

4 Transverse Angular Speed

The magnitude of the transverse angular velocity is obtainable from (15), and is

$$\left|\frac{\bar{\omega}_T(\tau)}{\bar{\omega}_T(0)}\right| = \exp\left[-\int_0^\tau \frac{\varphi(\tau)}{\bar{I}}\,d\tau\right]. \tag{53}$$

The quantity $\bar{I}$ decreases with τ during a propellant burn, but is always positive. Hence, the sign of $\varphi(\tau)$ determines whether the magnitude of the transverse angular velocity increases or decreases with the burn. The central transverse moment of inertia of the rocket system can be written, in non-dimensional form as

$$\bar{I} = \bar{I}_B + \bar{I}_F + \frac{m_B b^2 + m_F a^2}{m_{FO}R^2}, \tag{54}$$

where the dimensionless transverse inertia of B is $\bar{I}_B = I_B/m_{FO}R^2$.

4.1 End Burn

For the case of End Burn [see Figure 3.1], the transverse inertia of the propellant F is

$$\bar{I}_F = \frac{I_F}{m_{FO}R^2} = \bar{m}_F\left[\frac{1}{4} + \frac{1}{12}\left(\frac{z}{R}\right)^2\right] = (1-\tau)\left[\frac{1}{4} + \frac{1}{12}\,\delta^2(1-\tau)^2\right], \tag{55}$$

where δ is the ratio L/R, which can be referred to as the shape factor of the solid propellant. In subsequent equations, we will use $\delta_i = L_i/R$, for $i = 1, 2, 3$ (see Figure 3.1 for L_1, L_2, L_3). The distances a and b can be expressed as

$$a = \frac{m_B}{m_B + m_F}\left[L_2 + \frac{L(1-\tau)}{2}\right] \tag{56}$$

and

$$b = \frac{m_F}{m_B + m_F}\left[L_2 + \frac{L(1-\tau)}{2}\right]. \tag{57}$$

Substituting equations (55), (56), and (57) into (54), and simplifying, we obtain

$$\bar{I} = \bar{I}_B + (1-\tau)\left[\frac{1}{4} + \frac{\delta^2(1-\tau)^2}{12}\right] + \frac{\bar{m}_B(1-\tau)}{\bar{m}_B + 1 - \tau}\left[\delta_2 + \frac{\delta(1-\tau)}{2}\right]^2. \tag{58}$$

Hence,

$$\bar{I}' = \left[\frac{1}{4} + \frac{\delta^2(1-\tau)^2}{4}\right] - \left[\frac{\bar{m}_B}{\bar{m}_B + 1 - \tau}\right]^2\left[\delta_2 + \frac{\delta(1-\tau)}{2}\right]^2 - \left[\frac{\bar{m}_B(1-\tau)\delta}{\bar{m}_B + 1 - \tau}\right]\left[\delta_2 + \frac{\delta(1-\tau)}{2}\right]. \tag{59}$$

Again from Figure 3.1, the distance

$$z_e = L_1 + L + a - \frac{z}{2}. \tag{60}$$

Thus

$$\frac{z_e}{R} = \frac{\bar{m}_B[\delta_2 + \delta(1-\tau)/2] + (\bar{m}_B + 1 - \tau)[\delta_1 + \delta(1+\tau)/2]}{\bar{m}_B + 1 - \tau}. \tag{61}$$

Finally, from (16), (23), (59), and (61), we get

$$\varphi(\tau) = -\frac{1}{4} + \frac{\beta^2}{4} + \delta_1^2 + \delta^2\tau + \delta\delta_1(1+\tau) + \frac{2\bar{m}_B[\delta_1 + \delta\tau]}{\bar{m}_B + 1 - \tau}\left[\delta_2 + \frac{\delta(1-\tau)}{2}\right] = -\frac{1}{4} + \varphi_e(\tau), \tag{62}$$

where

$$\varphi_e(\tau) = \frac{\beta^2}{4} + \delta_1^2 + \delta^2\tau + \delta\delta_1(1+\tau) + \frac{2\bar{m}_B[\delta_1 + \delta\tau]}{\bar{m}_B + 1 - \tau}\left[\delta_2 + \frac{\delta(1-\tau)}{2}\right]. \tag{63}$$

Since each parameter that appears in $\varphi(\tau)$ is positive, and $0 \le \tau \le 1$, it is clear that φ_e is always positive. In fact, it is most likely greater than $\frac{1}{4}$. Hence the function $\varphi(\tau)$ is likely to be always positive. We conclude then that in the case of End Burn, the magnitude of the transverse angular velocity is damped as the propellant burns. The term "jet damping" truly applies in this case.

4.2 Centripetal Burn

We now consider the case of Centripetal Burn [see Figure 3.2]. Here,

$$\bar{I}_F = \frac{I_F}{m_{FO}R^2} = \bar{m}_F\left[\frac{1}{4}\left(\frac{r}{R}\right)^2 + \frac{1}{12}\left(\frac{L}{R}\right)^2\right] = (1-\tau)\left[\frac{1-\tau}{4} + \frac{1}{12}\,\delta^2\right], \tag{64}$$

$$a = \frac{m_B L_3}{m_B + m_F} \tag{65}$$

and

$$b = \frac{m_F L_3}{m_B + m_F}. \tag{66}$$

Substituting equations (64), (65), and (66) into (54), we obtain,

$$\bar{I} = \bar{I}_B + (1-\tau)\left[\frac{1-\tau}{4} + \frac{\delta^2}{12}\right] + \frac{\bar{m}_B(1-\tau)\delta_3^2}{\bar{m}_B + 1 - \tau} \tag{67}$$

so that

$$\bar{I}' = -\left(\frac{1-\tau}{2} + \frac{\delta^2}{12}\right) - \left(\frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\right)^2. \tag{68}$$

In this case [see Figure 3.2],

$$z_e = L_1 + \frac{L}{2} + a. \tag{69}$$

So,

$$\frac{z_e}{R} = \frac{\bar{m}_B\delta_3 + (\bar{m}_B + 1 - \tau)(\delta_1 + \delta/2)}{\bar{m}_B + 1 - \tau}. \tag{70}$$

From equations (16), (34), (68), and (70)

$$\varphi(\tau) = -\frac{1}{2} + \frac{\tau}{2} + \frac{\beta^2}{4} + \frac{\delta^2}{6} + \delta_1^2 + \delta\delta_1 + \frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\,(\delta + 2\delta_1) = -\frac{1}{2} + \varphi_c(\tau), \tag{71}$$

where

$$\varphi_c(\tau) = \frac{\tau}{2} + \frac{\beta^2}{4} + \frac{\delta^2}{6} + \delta_1^2 + \delta\delta_1 + \frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\,(\delta + 2\delta_1). \tag{72}$$

We have here a situation that is similar to the End Burn case. $\varphi_c(\tau)$ is positive and increases with τ. $\varphi_c(\tau)$ is most likely greater than $\frac{1}{2}$, even at $\tau = 0$. Therefore the transverse angular speed is again a decreasing function from ignition to burnout.

4.3 Radial Burn

If the propellant undergoes a radial burn as shown in Figure 3.5,

$$\begin{aligned}\bar{I}_F = \frac{I_F}{m_{FO}R^2} &= \bar{m}_F\left[\frac{1}{4} + \frac{1}{4}\left(\frac{r}{R}\right)^2 + \frac{1}{12}\left(\frac{L}{R}\right)^2\right] \\ &= (1-\tau)\left[\frac{1+\gamma^2+(1-\gamma^2)\tau}{4} + \frac{\delta^2}{12}\right].\end{aligned} \tag{73}$$

The distances a and b become

$$a = \frac{m_B L_3}{m_B + m_F} \tag{74}$$

and

$$b = \frac{m_F L_3}{m_B + m_F}. \tag{75}$$

Substituting equations (73), (74), and (75) into (54), we obtain, after some algebra,

$$\bar{I} = \bar{I}_B + (1-\tau)\left[\frac{1+\gamma^2+(1-\gamma^2)\tau}{4} + \frac{\delta^2}{12}\right] + \frac{\bar{m}_B(1-\tau)\delta_3^2}{\bar{m}_B + 1 - \tau} \tag{76}$$

so that

$$\bar{I}' = -\left[\frac{\gamma^2+(1-\gamma^2)\tau}{2} + \frac{\delta^2}{12}\right] - \left[\frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\right]^2. \tag{77}$$

Since, the distance

$$z_e = L_1 + \frac{L}{2} + a. \tag{78}$$

We have,

$$\frac{z_e}{R} = \frac{\bar{m}_B\delta_3 + (\bar{m}_B + 1 - \tau)(\delta_1 + \delta/2)}{\bar{m}_B + 1 - \tau}. \tag{79}$$

From equations (16), (47), (77), and (79)

$$\begin{aligned} \varphi(\tau) &= -\left[\frac{\gamma^2+(1-\gamma^2)\tau}{2}\right] + \frac{\beta^2}{4} + \frac{\delta^2}{6} + \delta_1^2 + \delta\delta_1 + \frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\,(\delta + 2\delta_1) \\ &= \varphi_1(\tau) + \varphi_2(\tau), \end{aligned} \tag{80}$$

where

$$\varphi_1(\tau) = -\left[\frac{\gamma^2+(1-\gamma^2)\tau}{2}\right] \tag{81}$$

and

$$\varphi_2(\tau) = \frac{\beta^2}{4} + \frac{\delta^2}{6} + \delta_1^2 + \delta\delta_1 + \frac{\bar{m}_B\delta_3}{\bar{m}_B + 1 - \tau}\,(\delta + 2\delta_1). \tag{82}$$

Here, the minimum value that φ_1 can have is $-\frac{1}{2}$, but φ_2 is always positive and most likely greater than $\frac{1}{2}$. Hence, we have again that mass loss through radial propellant burn results in continuous damping of the transverse rate.

In summary, we find that the transverse angular velocity decreases in magnitude as propellant burn progresses for each of the three propellant-burn scenarios examined. We note, however, that this conclusion is not absolute. In other words, one cannot absolutely exclude the possibility of growth in the transverse angular speed with propellant burn. Some factors that could bring this about include small values of β, δ, δ_1, δ_2 and δ_3. We note that in [2] a variable mass cylinder model was used to show that the transverse rate can grow without bounds when the system is "short and fat," that is, for small δ. This makes sense because when a cylinder is used to model a rocket system, we automatically have that δ_i $(i = 1, 2, 3)$ are all zero and $\beta = 1$. If δ is small in addition, then there is a clear danger of having $|\varphi_2| < |\varphi_1|$ in (80). We also note that even for the extreme case of the cylinder, the authors [2] were not able to show divergence in transverse rate for End and Centripetal Burns. It is easy to see this by setting $\delta_i = 0$ and $\beta = 1$ in equations (62) and (71).

5 Conclusion

This study examines how a spinning solid rocket's propellant depletion scheme affects the rotational dynamics of the rocket. Three mass loss scenarios – end burn, centripetal burn, and radial burn – were evaluated.

Results obtained indicate that for End Burn, spin rate can remain constant, increase, or decrease throughout the propellant burn depending on the value of the nozzle expansion ratio used. For Centripetal Burn, the spin rate will either decrease through the burn or increase at first then reverse itself and decrease to the end of the burn. In the case of Radial burn, the spin rate initially decreases then it can either keep decreasing or start increasing through the end of the burn. The value of the nozzle expansion ratio plays an important role in determining the character of the spin rate curve.

The transverse angular speed normally decreases with propellant burn irrespective of the type of burn adopted. For certain extreme choices of the parameters of the system, it may be possible to have the transverse rate increase with time for the radial burn.

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A "Patched Conics" Description of the Swing-By of a Group of Particles

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Abstract: We study the close approach between a planet and a cloud of particles. It is assumed that the dynamical system is formed by two main bodies in circular orbits and a cloud of particles in planar motion. The goal is to study the change of the orbit of this cloud after the close approach with the planet. It is assumed that all the particles have semi-major axis $a \pm \Delta a$ and eccentricity $e \pm \Delta \mathrm{e}$ before the close approach with the planet. It is desired to known those values after the close approach.

Keywords: *Astrodynamics; swing-by; orbital maneuvers.*

Mathematics Subject Classification (2000): 70F15, 70F07.

1 Introduction

In astronautics, the close approach between a spacecraft and a planet is a very popular technique used to decrease fuel expenditure in space missions. This maneuver modifies the velocity, energy and angular momentum of a spacecraft. There are many important applications very well known, like the Voyager I and II that used successive close encounters with the giant planets to make a long journey to the outer Solar System; the Ulysses mission that used a close approach with Jupiter to change its orbital plane to observe the poles of the Sun, etc.

In the present paper we study the close approach between a planet and a cloud of particles. It is assumed that the dynamical system is formed by two main bodies (usually the Sun and one planet) which are in circular orbits around their center of mass and a cloud of particles that is moving under the gravitational attraction of the two primaries. The motion is assumed to be planar for all the particles and the dynamics given by the

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"patched-conic" approximation is used, which means that a series of two-body problems are used to generate analytical equations that describe the problem. The standard canonical system of units is used and it implies that the unit of distance is the distance between the two primaries and the unit of time is chosen such that the period of the orbit of the two primaries is 2π.

The goal is to study the change of the orbit of this cloud of particles after the close approach with the planet. It is assumed that all the particles that belong to the cloud have semi-major axis $a \pm \Delta a$ and eccentricity $\mathrm{e} \pm \Delta \mathrm{e}$ before the close approach with the planet. It is desired to known those values after the close approach.

Among the several sets of initial conditions that can be used to identify uniquely one swing-by trajectory, a modified version of the set used in the papers written by [18 – 20] is used here. It is composed by the following three variables: 1) V_p, the velocity of the spacecraft at periapse of the orbit around the secondary body; 2) the angle ψ, that is defined as the angle between the line $M_1 - M_2$ (the two primaries) and the direction of the periapse of the trajectory of the spacecraft around M_2; 3) r_p, the distance from the spacecraft to the center of M_2 in the moment of the closest approach to M_2 (periapse distance). The values of V_p and ψ are obtained from the initial orbit of the spacecraft around the Sun using the "patched-conics" approximation and r_p is a free parameter that is varied to obtain the results.

2 Review of the Literature for the Swing-By

The literature shows several applications of the swing-by technique. Some of them can be found in [1], that studied a mission to Neptune using swing-by to gain energy to accomplish the mission; [2], that made a similar study for a mission to Pluto; [3], that formulated a mission to study the Earth's geomagnetic tail; [4 – 6], that planned the mission ISEE-3/ICE; [7], that made the first studies for the Voyager mission; [8], that design a mission to flyby the Halley comet; [9, 10] that studied multiple flyby for interplanetary missions; [11, 12], that design missions with multiple lunar swing-bys; [13], that studied the effects of the atmosphere in a swing-by trajectory; [14], that used a swing-by in Venus to reach Mars; [15], that studied numerically a swing-by in three dimensions, including the effects in the inclination; [16], that considered the possibility of applying an impulse during the passage by the periapsis; [17], that classified trajectories making a swing-by with the Moon. The most usual approach to study this problem is to divide the problem in three phases dominated by the "two-body" celestial mechanics. Other models used to study this problem are the circular restricted three-body problem (like in [18 – 20] and the elliptic restricted three-body problem [21]).

3 Orbital Change of a Single Particle

This section will briefly describe the orbital change of a single particle subjected to a close approach with the planet under the "patched-conics" model. It is assumed that the particle is in orbit around the Sun with given semi-major axis (a) and eccentricity (e). The swing-by is assumed to occur in the planet Jupiter for the numerical calculations shown below, but the analytical equations are valid for any system of primaries. The periapse distance (r_p) is assumed to be known. As an example for the numerical calculations, the following numerical values are used: a = 1.2 canonical units, e = 0.3,

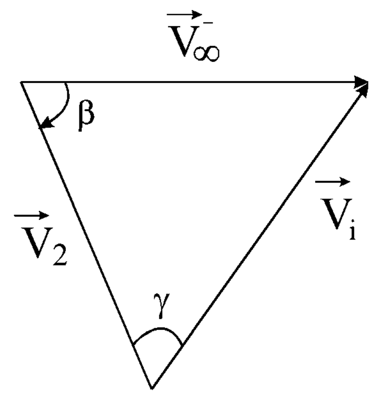

Figure 3.1. The swing-by in the three-dimensional space.

$\mu_J = 0.00094736$, $r_p = 0.0001285347$ (100000 km = 1.4 Jupiter's radius), where μ_J is the gravitational parameter of Jupiter in canonical units (total mass of the system equals to one).

The first step is to obtain the energy (EB) and angular momentum (CB) of the particle before the swing-by. They are given by

$$EB = -\frac{1-\mu_J}{2a} = -0.4162, \qquad CB = \sqrt{(1-\mu_J)\,a(1-e^2)} = 1.0445. \tag{1}$$

Then, it is possible to calculate the magnitude of the velocity of the particle with respect to the Sun in the moment of the crossing with Jupiter's orbit (V_i), as well as the true anomaly of that point (θ). They come from

$$V_i = \sqrt{(1-\mu_J)\left(\frac{2}{r_{SJ}} - \frac{1}{a}\right)} = 1.0796 \tag{2}$$

and

$$\theta = \cos^{-1}\left[\frac{1}{e}\left(\frac{a(1-e^2)}{r_{SJ}} - 1\right)\right] = 1.2591$$

using the fact that the distance between the Sun and Jupiter (r_{SJ}) is one and taking only the positive value of the true anomaly.

Next, it is calculated the angle between the inertial velocity of the particle and the velocity of Jupiter (the flight path angle γ), as well as the magnitude of the velocity of the particle with respect to Jupiter in the moment of the approach (V_∞). They are given by (assuming a counter-clock-wise orbit for the particle)

$$\gamma = \tan^{-1}\left[\frac{\mathrm{e}\sin\theta}{1+\mathrm{e}\cos\theta}\right] = 0.2558$$

and $V_\infty = \sqrt{V_i^2 + V_2^2 - 2V_iV_2\cos\gamma} = 0.2767$ using the fact that the velocity of Jupiter around the Sun (V_2) is one. Figure 3.1 shows the vector addition used to derive the equations.

The angle β shown is given by

$$\beta = \cos^{-1}\left(-\frac{V_i^2 - V_2^2 - V_\infty^{-2}}{2V_2V_\infty^-}\right) = 1.7322.$$

This information allows us to obtain the turning angle (2δ) of the particle around Jupiter, from

$$\delta = \sin^{-1}\left(1 + \frac{r_p V_\infty^{-2}}{\mu_J}\right)^{-1} = 1.4272. \quad (3)$$

The angle of approach (ψ) has two values, depending if the particle is passing in front or behind Jupiter. These two values will be called ψ_1 and ψ_2. They are obtained from $\psi_1 = \pi + \beta + \delta = 6.3011$ and $\psi_2 = 2\pi + \beta - \delta = 6.5882$.

The correspondent variations in energy and angular momentum are obtained from the equation $\Delta C = \Delta E = -2V_2V_\infty \sin\delta \sin\psi$ (since $\omega = 1$). The results are:

$$\Delta C_1 = \Delta E_1 = -0.009811, \qquad \Delta C_2 = \Delta E_2 = -0.1644. \quad (4)$$

By adding those quantities to the initial values we get the values after the swing-by. They are:

$$\begin{aligned} E_1 &= -0.4260, \qquad & C_1 &= 1.0346, \\ E_2 &= -0.5806, \qquad & C_2 &= 0.8801. \end{aligned}$$

Finally, to obtain the semi-major axis and the eccentricity after the swing-by it is possible to use the equations

$$a = -\frac{\mu}{2E} \quad \text{and} \quad \mathrm{e} = \sqrt{1 - \frac{C^2}{\mu a}}. \quad (5)$$

The results are: $a_1 = 1.1723$, $\mathrm{e}_1 = 0.2937$, $a_2 = 0.8603$, $\mathrm{e}_2 = 0.3144$.

4 Orbital Change of a Cloud of Particles

The algorithm just described can now be applied to a cloud of particles passing close to Jupiter. The idea is to simulate a cloud of particles that have orbital elements given by: $a \pm \Delta a$ and $\mathrm{e} \pm \Delta\mathrm{e}$. The goal is to map this cloud of particles to obtain the new distribution of semi-major axis and eccentricities after the swing-by. Figure 4.1 and Figure 4.2 shows some results for a cloud of particles with $r_p = 1.4R_j$, for the case $\Delta a = \Delta\mathrm{e} = 0.001$, $r_p = 1.4R_J$ and Figure 4.3 and Figure 4.4 shows the equivalent results with $r_p = 10.0R_j$ for $\Delta a = \Delta\mathrm{e} = 0.001$, $r_p = 10.0R_J$.

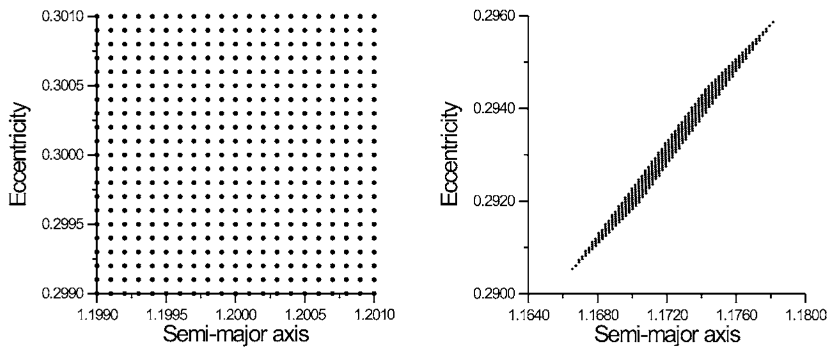

Figure 4.1. Eccentricity vs. Semi-major axis before and after the Swing-By for "Solution 1".

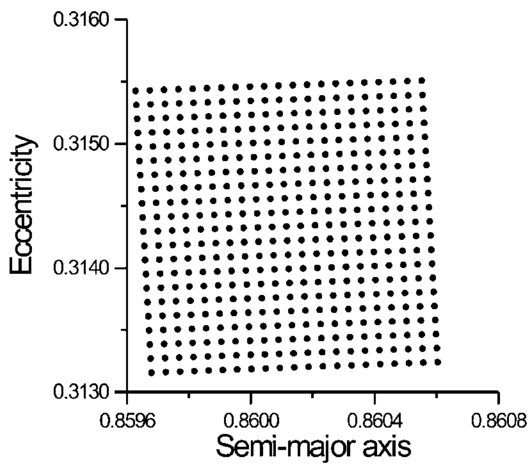

Figure 4.2. Eccentricity vs. Semi-major axis after the Swing-By for "Solution 2".

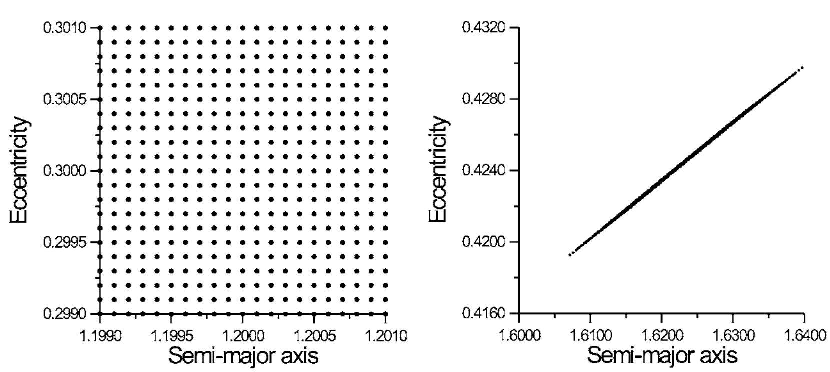

Figure 4.3. Eccentricity vs. Semi-major axis before and after the Swing-By for "Solution 1".

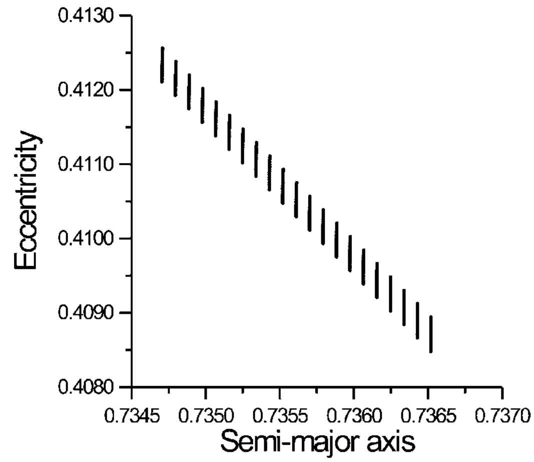

Figure 4.4. Eccentricity vs. Semi-major axis after the Swing-By for "Solution 2".

5 Conclusions

The figures above allow us to get some conclusions. The solution called "Solution 1" has a larger amplitude than the Solution 2 in both orbital elements, but it concentrates the orbital elements in a line, while the so-called "Solution 2" generates a distribution close to a square. The area occupied by the points is smaller for "Solution 1". Both vertical and horizontal lines are rotated and become diagonal lines with different inclinations. The effect of increasing the periapse distance is to generate plots with larger amplitudes, but with the points more concentrated, close to a straight line.

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Fault Detection Filter for Linear Time-Delay Systems

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Abstract: By extension of a fault detection optimization approach to linear time invariant (LTI) systems, this short paper deals with the fault detection filter (FDF) problem for linear time-delay systems with L_2-norm bounded unknown inputs. The basic idea is first to introduce a new FDF as the residual generator; and then based on an objective function to formulate the FDF design as an optimization problem. Through appropriate choice of the filter gain matrix and a post-filter, the convergence of the residual generator and satisfactory FDF performance can be achieved. A numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: *Fault detection; filter; robustness; sensitivity; time delay.*

Mathematics Subject Classification (2000): 60G35, 93E11, 94A13.

1 Introduction

Many significant approaches to the problems of robust fault detection and isolation (FDI) have been developed during the past two decades, for instance unknown input observer (UIO), parity space, H_∞ optimization, eigenstructure assignment, and H_∞ filtering [1, 5, 6, 9, 12]. However, most of these aforementioned works are about delay-free systems. Time delay is an inherent characteristic of many physical systems, such as rolling mills, chemical processes, water resources, biological, economic and traffic control systems. To the best of our knowledge, only few researches on FDI have been carried out

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for time-delay systems [4, 7, 8, 10]. Note that [7] did not consider the influence of unknown inputs; [10] formulated the fault detection filter (FDF) design problem as a two-objective nonlinear programing problem where no analytic solution can be constructed in general; [8] extended the results of [10] to the discrete-time case. The authors' earlier work in [4] developed an LMI approach to FDF design for linear time invariant (LTI) time-delay systems, but the selection of weighting transfer function matrix has strong influence on FDF performance. Research on fault detection (FD) of time-delay system is as yet an open and important issue.

The main objective of this short paper is to deal with the FDF design problem for linear systems with L_2-norm bounded unknown input and multiple time delays. An FDF will be developed such that a robustness/sensitivity based objective function is minimized. The core of this study is the introduction of a new FDF as a residual generator and an extension of the optimization FDI method for LTI systems in [2, 3] to time-delay systems. A sufficient condition to the solvability of FDF is derived and a solution can be obtained by appropriate choice of a filter gain matrix and post-filter. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

Notations. Throughout this paper, the superscript T stands for the matrix transposition, R^n denotes the n dimensional Euclidean space. $R^{n\times m}$ is the set of all $n\times m$ real matrices. I is the identity matrix with appropriate dimensions. L_2 denotes the space of square integrable vector functions over $[0,\infty)$. For $h(t)\in L_2$, $\|h\|_2$ denotes the L_2-norm of $h(t)$. For a real matrix P, $P>0$ (respectively, $P<0$), means that P is real symmetric and positive definite (respectively, negative definite). $\mathbf{RH}_\infty$ denotes the set of rational transfer functions analytic in closed right half plane. For $G(s)\in\mathbf{RH}_\infty$, $\|G(s)\|_\infty$ denotes the H_∞ norm of transfer function matrix $G(s)$.

2 Preliminaries and Problem Formulation

2.1 Brief review of related FD approach

Consider LTI systems described by

$$\dot{x}(t) = Ax(t) + Bu(t) + B_f f(t) + B_d d(t) \tag{1}$$

$$y(t) = Cx(t) + Du(t) + D_f f(t) + D_d d(t) \tag{2}$$

where $x(t)\in R^n$, $u(t)\in R^p$, $y(t)\in R^q$ are the state vector, control input and measurement output respectively. $d(t)\in R^m$ denotes the L_2-norm bounded unknown input, $f(t)\in R^l$ is the fault to be detected. A, B, B_f, B_d, C, D, D_f and D_d are known matrices with appropriate dimensions. It has been shown by Ding and Frank [3] that the dynamics of observer-based residual generator for systems (1)–(2) can be expressed as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H(y(t)-\hat{y}(t)), \tag{3}$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t), \quad r(s) = R(s)(y(s)-\hat{y}(s)) \tag{4}$$

or the frequency domain description

$$\begin{aligned} r(s) &= R(s)[(C(sI-A+HC)^{-1}(B_d-HD_d)+D_d)d(s) \\ &\quad + (C(sI-A+HC)^{-1}(B_f-HD_f)+D_f)f(s)] \\ &= R(s)G_{\varepsilon d}(s)d(s) + R(s)G_{\varepsilon f}(s)f(s) = G_{rd}(s)d(s) + G_{rf}(s)f(s), \end{aligned}$$

where $\hat{x}(t) \in R^n$ and $\hat{y}(t) \in R^q$ represent the state and output estimation vectors respectively, r is the so-called residual signal. The transfer function matrix $R(s) \in \mathbf{RH}_\infty$, also called a post-filter, and observer gain matrix H are parameters to be determined. In the case of a full decoupling of unknown input being not achievable, the main task of FDF design is to find a suitable H and $R(s)$ such that the H_∞ norm of $G_{rd}(s)$ is minimized by guaranteeing a desired sensitivity to fault. One widely accepted way is to formulate the FDF problem as the following optimal problem

$$J = \min_{R(s),H} \frac{\|R(s)G_{\varepsilon d}(s)\|_\infty}{\|R(s)G_{\varepsilon f}(s)\|_\infty}. \tag{5}$$

Under some assumptions, [2, 3] has developed an optimization method to solve the problem (5).

Lemma 1 [2, 3] *Consider system (1) – (2) and suppose the assumptions*

(A1) *system (1) – (2) is asymptotically stable when $u(t) = 0$, $d(t) = 0$ and $f(t) = 0$ for $t \geqslant 0$;*

(A2) *(C, A) is detectable;*

(A3) $\begin{bmatrix} A - j\omega I & B_d \\ C & D_d \end{bmatrix}$ *is of full row rank for $\omega \in [0, \infty)$*

hold, then

$$R^*(s) = Q^{-1/2}, \quad H^* = (B_d D_d^{\mathrm{T}} + YC^{\mathrm{T}})Q^{-1}$$

solve the optimal problem (5), where $Q = D_d D_d^{\mathrm{T}}$ and $Y \geq 0$ is a solution of the algebraic Riccati equation

$$Y(A - B_d D_d^{\mathrm{T}} Q^{-1} C)^{\mathrm{T}} + (A - B_d D_d^{\mathrm{T}} Q^{-1} C)Y - YC^{\mathrm{T}} Q^{-1} CY + B_d(I - D_d^{\mathrm{T}} Q^{-1} D_d)B_d^{\mathrm{T}} = 0$$

Moreover, $G^*_{rd}(s)$ is a co-inner matrix, where

$$G^*_{rd}(s) = R^*(s)\left[C(sI - A + H^*C)^{-1}(B_d - H^*D_d) + D_d\right].$$

Remark 1 From the view point of FDI, Assumptions A1 and A2 are trivial and do not lead to a loss of generality. The results in Lemma 1 are true only under the assumptions made, in particular, Assumption A3. Upon removing it, the lemma will lose its validity [3].

2.2 Problem formulation

In this short paper, we consider the FDF problem for a class of linear time-delay systems described by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} A_i x(t - \tau_i) + Bu(t) + \sum_{j=1}^{L} B_j u(t - \mu_j) + B_f f(t) + B_d d(t), \tag{6}$$

$$y(t) = Cx(t) + Du(t) + D_f f(t) + D_d d(t), \tag{7}$$

$$x(-t) = 0, \quad u(-t) = 0, \quad t \geqslant 0, \tag{8}$$

where $x(t) \in R^n$, $u(t) \in R^p$, $y(t) \in R^q$, $d(t) \in R^m$, $f(t) \in R^l$ and matrices A, B, B_f, B_d, C, D, D_f and D_d are defined as in system (1) − (2). A_i $(i = 1, 2, \dots, N)$ and B_j $(j = 1, 2, \dots, L)$ are known matrices with appropriate dimensions. τ_i and μ_j denote known constant time delays. Throughout this work, Assumptions A1 to A3 corresponding to system (6) − (8) are also made, that is

(A4) system (6) − (8) is asymptotically stable when $u(t) = 0$, $d(t) = 0$ and $f(t) = 0$ for $t \geqslant 0$;

(A5) (C, A) is detectable;

(A6) $\begin{bmatrix} A - j\omega I & B_d \\ C & D_d \end{bmatrix}$ is of full row rank for $\omega \in [0, \infty)$.

The type of filter considered in this paper is given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \sum_{i=1}^{N} A_i x_u(t - \tau_i) + Bu(t) + \sum_{j=1}^{L} B_j u(t - \mu_j) + H(y(t) - \hat{y}(t)), \quad (9)$$

$$\dot{x}_u(t) = Ax_u(t) + \sum_{i=1}^{N} A_i x_u(t - \tau_i) + Bu(t) + \sum_{j=1}^{L} B_j u(t - \mu_j), \quad (10)$$

$$\hat{y}(t) = C\hat{x}(t) + Du(t), \quad \varepsilon(t) = y(t) - \hat{y}(t), \quad (11)$$

$$r(s) = R(s)\varepsilon(s), \quad (12)$$

$$\hat{x}(-t) = 0, \quad x_u(-t) = 0, \quad t \geqslant 0, \quad (13)$$

where $\hat{x}(t) \in R^n$, $\hat{y}(t) \in R^q$ and $x_u(t) \in R^n$ are vectors, $R(s) \in \mathbf{RH}_\infty$ is a so-called post-filter, H is the filter gain matrix, r is the generated residual. H and $R(s)$ are parameters to be determined for achieving perfect FD performance. Especially, in the case of unknown input full decoupling being not achievable, the main task of FDF design is to determine H and $R(s)$ such that

(i) When $d(t) = 0$ and $f(t) = 0$ for all t, the generated residual r asymptotically decays to zero for any $u(t)$.

(ii) The residual r achieves best compromise between sensitivity to faults and robustness to known input.

By denoting $e(t) = x(t) - \hat{x}(t)$ and $x_{df}(t) = x(t) - x_u(t)$, the overall dynamics of the residual generator are governed by

$$\dot{e}(t) = (A - HC)e(t) + \sum_{i=1}^{N} A_i x_{df}(t - \tau_i) + (B_d - HD_d)d(t) + (B_f - HD_f)f(t), \quad (14)$$

$$\dot{x}_{df}(t) = Ax_{df}(t) + \sum_{i=1}^{N} A_i x_{df}(t - \tau_i) + B_d d(t) + B_f f(t), \quad (15)$$

$$\varepsilon(t) = Ce(t) + D_d d(t) + D_f f(t), \quad (16)$$

$$r(s) = R(s)\varepsilon(s). \quad (17)$$

It can be seen from the above that $u(t)$ has no influence on the residual r. The main problem of FDF can be formulated as to determine H and $R(s)$ such that system (14) − (17) is asymptotically stable, while an FDF designing performance index as in (5) is satisfied.

Remark 2 Compared with the residual generator used in [4, 7, 8, 10], here $x_{df}(t-\tau_i)$ $(i=1,2,\dots,N)$ in equation (14) is used instead of the time-delay state estimate error $e(t-\tau_i)$ in [4, 7, 8, 10]. Notice that $x_{df}(t)$, which describes the effect of d and f in state x, is independent of filter gain matrix H. Especially, under the assumptions on system (6)–(8) being asymptotically stable and d, f being L_2-norm bounded, $x_{df}(t)$ is also L_2-norm bounded. Finally, the FDF problem for time-delay system can be solved by an extension of the optimization FD approach in [2, 3].

3 Design of FDF

In this section, an extension of the FD approach presented in [2, 3] will be performed for the FDF problem of time-delay system (6)–(8).

3.1 Basic idea of our study

Notice that if system (14)–(17) is asymptotically stable, then residual $r(t)$ is convergent to zero when $d(t)=0$ and $f(t)=0$. To express clearly the influences of past unknown input $d(t-\tau_i)$ and fault signal $f(t-\tau_i)$ on residual $r(t)$, we first separate $x_{df}(t)$ into $x_d(t)$ and $x_f(t)$,

$$\dot{x}_d(t) = Ax_d(t) + \sum_{i=1}^{N} A_i x_d(t-\tau_i) + B_d d(t), \tag{18}$$

$$\dot{x}_f(t) = Ax_f(t) + \sum_{i=1}^{N} A_i x_f(t-\tau_i) + B_f f(t) \tag{19}$$

and denote

$$\begin{aligned}
\theta_d(t) &= \begin{bmatrix} x_d^{\mathrm{T}}(t-\tau_1) & x_d^{\mathrm{T}}(t-\tau_2) & \cdots & x_d^{\mathrm{T}}(t-\tau_N) \end{bmatrix}^{\mathrm{T}}, \\
\theta_f(t) &= \begin{bmatrix} x_f^{\mathrm{T}}(t-\tau_1) & x_f^{\mathrm{T}}(t-\tau_2) & \cdots & x_f^{\mathrm{T}}(t-\tau_N) \end{bmatrix}^{\mathrm{T}}, \\
A_\theta &= \begin{bmatrix} A_1 & A_2 & \cdots & A_N \end{bmatrix}.
\end{aligned}$$

It is obvious that $\theta_d(t)$ and $\theta_f(t)$ respectively describe the influences of past unknown input $d(t-\tau_i)$ and fault signal $f(t-\tau_i)$ $(i=1,2,\dots,N)$, while $\theta_d(t)$ and $\theta_f(t)$ are independent of H. Recall that for L_2-norm bounded d and f, the asymptotic stability of system (6)–(8) ensures that $x_d(t)$, $x_f(t)$ and, furthermore, $\theta_d(t)$ and $\theta_f(t)$ are also L_2-norm bounded. Introduce vector $w(t) = \begin{bmatrix} d^{\mathrm{T}}(t) & \theta_d^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$ to describe both the present and past unknown input, and let $B_w \triangleq \begin{bmatrix} B_d & A_\theta \end{bmatrix}$, $D_w \triangleq \begin{bmatrix} D_d & 0 \end{bmatrix}$. From the above definitions, we have

$$\dot{e}(t) = (A-HC)e(t) + (B_w - HD_w)w(t) + (B_f - HD_f)f(t) + A_\theta\theta_f(t), \tag{20}$$

$$\varepsilon(t) = Ce(t) + D_w w(t) + D_f f(t), \tag{21}$$

$$r(s) = R(s)\varepsilon(s) \tag{22}$$

and

$$r(s) = G_{rw}(s)w(s) + G_{rf}(s)f(s), \tag{23}$$

where

$$G_{rw}(s) = R(s)G_{\varepsilon w}(s), \qquad G_{\varepsilon w}(s) = [C(sI - A + HC)^{-1}(B_w - HD_w) + D_w], \quad (24)$$

$$G_{rf}(s) = R(s)[G_{\varepsilon\theta_f}(s)G_{\theta f}(s) + G_{\varepsilon f}(s)], \qquad G_{\varepsilon\theta_f}(s) = C(sI - A + HC)^{-1}A_\theta, \quad (25)$$

$$G_{\theta f}(s) = [\, e^{-s\tau_1}I \quad e^{-s\tau_2}I \quad \cdots \quad e^{-s\tau_N}I \,]^{\mathrm{T}} \left(sI - A + \sum_{i=1}^{N} A_i e^{-s\tau_i}\right)^{-1} B_f, \quad (26)$$

$$G_{\varepsilon f}(s) = C(sI - A + HC)^{-1}(B_f - HD_f) + D_f. \quad (27)$$

As in [3], we use $\|G_{rw}(s)\|_\infty$ to measure the robustness of residual against unknown inputs, while the sensitivity of residual to faults is represented by $\|G_{rf}(s)\|_\infty$. Then the FDF problem for time-delay system (6) − (8) can be further formulated as to find H and $R(s)$ such that system (14) − (17) is asymptotically stable on one hand, while on the other hand solves the following optimization problem

$$J = \min_{R(s),H} \frac{\|G_{rw}(s)\|_\infty}{\|G_{rf}(s)\|_\infty}. \quad (28)$$

The procedure to solve the FDF problem is made of two steps, namely (a) the choice of filter gain matrix H to ensure the asymptotic stability of system (14) − (17), and (b) the derivation of $R(s)$ so that $(H, R(s))$ is an optimal solution of the problem (28).

Remark 3 By solving the above formulated FDF problem, not only the convergence of the residual but also the satisfactory robustness and sensitivity criterion of FD system defined in (28) are achieved.

3.2 Main results

The following Lemmas are required to solve the FDF problem.

Lemma 2 [11] *System*

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} A_i x(t - \tau_i),$$
$$x(t) = 0 \quad \text{for} \quad t \leqslant 0,$$

is asymptotically stable, if there exist matrices $P > 0$ and $R_i > 0$, $(i = 1, 2, \ldots, N)$ such that LMI

$$\begin{bmatrix} A^{\mathrm{T}}P + PA + \sum\limits_{i=1}^{N} R_i & PA_1 & \cdots & PA_N \\ A_1^{\mathrm{T}}P & -R_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A_N^{\mathrm{T}}P & \cdots & 0 & -R_N \end{bmatrix} < 0$$

holds.

Lemma 3 [2] *Given*

$$\widehat{M}_1(s) = V_1 - V_1C(sI - A + H_1C)^{-1}H_1,$$
$$\widehat{M}_2(s) = V_2 - V_2C(sI - A + H_2C)^{-1}H_2,$$

where H_1 and H_2 are selected such that $A - H_1C$ and $A - H_2C$ are Hurwitz, V_1 and V_2 are invertible, there exists a stable solution $Q(s)$ for the equation

$$Q(s)\widehat{M}_1(s) = \widehat{M}_2(s).$$

Furthermore, the solution can be expressed by

$$Q(s) = V_2[I + C(sI - A + H_2C)^{-1}(H_1 - H_2)]V_1^{-1}.$$

Now we are ready to present the main results of this short paper, which give a sufficient condition to solve H and parameterize FDF using the obtained solutions of H. By applying Lemma 2, we first present the determination of filter gain matrix H ensuring the asymptotic stability of system (14)–(17) (with proof omitted).

Theorem 1 *If there exist matrices $P_1 > 0$, $P_2 > 0$, $R_i > 0$, $S_i > 0$ $(i = 1, 2, \dots, N)$ and Y such that LMI*

$$\begin{bmatrix} A^{\mathrm{T}}P_1 + P_1A - C^{\mathrm{T}}Y^{\mathrm{T}} - YC + \sum\limits_{i=1}^{N} R_i & 0 & P_1A_1 & \cdots & P_1A_N \\ 0 & A^{\mathrm{T}}P_2 + P_2A + \sum\limits_{i=1}^{N} S_i & P_2A_1 & \cdots & P_2A_N \\ A_1^{\mathrm{T}}P_1 & A_1^{\mathrm{T}}P_2 & -S_1 & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ A_N^{\mathrm{T}}P_1 & A_N^{\mathrm{T}}P_2 & 0 & 0 & -S_N \end{bmatrix} < 0$$

holds, then system (14)–(17) is asymptotically stable. Moreover, the observer gain matrix is determined by

$$H = P_1^{-1}Y.$$

After designing the filter gain matrix H, the remained important task for FDF design is the determination of a post-filter $R(s)$. Following studies show that under Assumptions of A4 to A6, for all H ensuring the stability of system (14)–(17), there exists an $R(s) \in \mathbf{RH}_\infty$ such that $(H, R(s))$ is an optimal solution of the problem (28).

Theorem 2 *Given system (6)–(8) with Assumptions of A4 to A6, there exists $R_h(s) \in \mathbf{RH}_\infty$ such that $(H, R_h(s))$ is an optimal solution of (28), where $R_h(s)$ is given by*

$$R_h(s) = Q^{-1/2}(I + C(sI - A + H^*C)^{-1}(H - H^*)), \tag{29}$$
$$H^* = (B_wD_w^{\mathrm{T}} + YC^{\mathrm{T}})Q^{-1}, \quad Q = D_wD_w^{\mathrm{T}}, \tag{30}$$

and $Y \geq 0$ is a solution of the following algebraic Riccati equation

$$\begin{gathered} Y(A - B_wD_w^{\mathrm{T}}Q^{-1}C)^{\mathrm{T}} + (A - B_wD_w^{\mathrm{T}}Q^{-1}C)Y - YC^{\mathrm{T}}Q^{-1}CY \\ + B_w(I - D_w^{\mathrm{T}}Q^{-1}D_w)B_w^{\mathrm{T}} = 0. \end{gathered} \tag{31}$$

Proof Considering system (6)–(8) and the residual generator (20)–(22), define $G_{rw}(s)$, $G_{\epsilon w}(s)$, $G_{rf}(s)$, $G_{\epsilon f}(s)$, $G_{\epsilon\theta_f}(s)$, $G_{\theta f}(s)$ as in (24)–(27), and

$$\begin{aligned}
G_{yw}(s) &= C(sI-A)^{-1}B_w + D_w,\\
G^*_{\epsilon w}(s) &= C(sI-A+H^*C)^{-1}(B_w - H^*D_w) + D_w,\\
G^*_{rw}(s) &= R^*(s)G^*_{\epsilon w}(s),\\
G^*_{\epsilon f}(s) &= C(sI-A+H^*C)^{-1}(B_f - H^*D_f) + D_f,\\
G^*_{\varepsilon\theta_f}(s) &= C(sI-A+H^*C)^{-1}A_\theta,\\
G^*_{rf}(s) &= R^*(s)[G^*_{\varepsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\varepsilon f}(s)],\\
\widehat{N}_w(s) &= G_{\epsilon w}(s),\\
\widehat{N}^*_w(s) &= G^*_{\epsilon w}(s),\\
\widehat{M}(s) &= I - C(sI-A+HC)^{-1}H,\\
\widehat{M}^*(s) &= I - C(sI-A+H^*C)^{-1}H^*.
\end{aligned}$$

Based on the left coprime factorization of $G_{yw}(s)$, it is easy to get

$$G_{yw}(s) = \widehat{M}^{-1}(s)\widehat{N}_w(s) = (\widehat{M}^*(s))^{-1}\widehat{N}^*_w(s).$$

For any available H ensuring the asymptotic stability of system (14)–(17), we then have

$$\begin{aligned}
G_{rw}(s) = R(s)G_{\varepsilon w}(s) = R(s)\widehat{N}_w(s) &= R(s)\widehat{M}(s)(\hat{M}^*(s))^{-1}\widehat{N}^*_w(s)\\
&= R(s)\widehat{M}(s)(\widehat{M}^*(s))^{-1}G^*_{\epsilon w}(s).
\end{aligned} \tag{32}$$

Moreover, from Lemma 3, it is easy to verify that, for $R^*(s) = Q^{-1/2}$ and the above defined $\widehat{M}(s)$ and $\widehat{M}^*(s)$, there exists a matrix $\Gamma(s)$,

$$\Gamma(s) = [I + C(sI-A+HC)^{-1}(H^* - H)]Q^{1/2} \tag{33}$$

such that

$$\widehat{M}(s) = \Gamma(s)R^*(s)\widehat{M}^*(s). \tag{34}$$

It follows from (32)–(34) that

$$G_{rw}(s) = R(s)\Gamma(s)R^*(s)G^*_{\epsilon w}(s) = R(s)\Gamma(s)G^*_{rw}(s). \tag{35}$$

Also, from Lemma 3, $R_h(s)$ in (29) and $\Gamma(s)$ in (33) satisfy

$$R_h(s)[I - C(sI-A+HC)^{-1}H] = Q^{-1/2}[I - C(sI-A+H^*C)^{-1}H^*], \tag{36}$$

$$\Gamma(s)(Q^{-1/2})(I - C(sI-A+H^*C)^{-1}H^*) = I - C(sI-A+HC)^{-1}H. \tag{37}$$

It is obtained from (36)–(37) that

$$\begin{aligned}
R_h(s)\Gamma(s)(Q^{-1/2})(I - C(sI-A+H^*C)^{-1}H^*) &= (Q^{-1/2})[I - C(sI-A+H^*C)^{-1}H^*]\\
\Rightarrow \quad R_h(s)\Gamma(s) &= I.
\end{aligned}$$

Thus, for $R(s) = R_h(s)$, we have

$$G_{rw}(s) = G^*_{rw}(s).$$

In the same way, we can get

$$\begin{aligned} G_{\epsilon\theta_f}(s) &= \Gamma(s)R^*(s)G^*_{\epsilon\theta_f}(s), \\ G_{\epsilon f}(s) &= \Gamma(s)R^*(s)G^*_{\epsilon f}(s), \\ G_{rf}(s) &= R(s)[G_{\epsilon\theta_f}(s)G_{\theta f}(s) + G_{\epsilon f}(s)] \\ &= R(s)\Gamma(s)R^*(s)[G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s)], \end{aligned} \tag{38}$$

and for $R(s) = R_h(s)$, we have

$$G_{rf}(s) = G^*_{rf}(s).$$

Under Assumptions of A4 to A6, from Lemma 1 we know that $R^*(s) = Q^{-1/2}$ and H^* given in (30) – (31) is an optimal solution of the problem (28) and, in this case, $G^*_{rw}(s)$ is a co-inner matrix. Therefore,

$$\|R^*(s)G^*_{\epsilon w}(s)\|_\infty = 1, \qquad \|R_h(s)G_{\epsilon w}(s)\|_\infty = 1,$$
$$\|R^*(s)(G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s))\|_\infty = \|R_h(s)(G_{\epsilon\theta_f}(s)G_{\theta f}(s) + G_{\epsilon f}(s))\|_\infty.$$

On the other hand, for co-inner matrix $G^*_{rw}(s) = R^*(s)G^*_{\epsilon w}(s)$ and for all $R(s) \in \mathbf{RH}_\infty$, from (35) and (38) it is easy to get

$$\|G_{rw}(s)\|_\infty = \|R(s)G_{\epsilon w}(s)\|_\infty = \|R(s)\Gamma(s)G^*_{rw}(s)\|_\infty = \|R(s)\Gamma(s)\|_\infty$$
$$\begin{aligned} \big\|R(s)(G_{\epsilon\theta_f}(s)G_{\theta f}(s) + G_{\epsilon f}(s))\big\|_\infty &= \big\|R(s)\Gamma(s)R^*(s)[G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s)]\big\|_\infty \\ &\leqslant \big\|R(s)\Gamma(s)\big\|_\infty \ \big\|R^*(s)[G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s)]\big\|_\infty, \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\|R_h(s)G_{\epsilon w}(s)\|_\infty}{\|R_h(s)(G_{\epsilon\theta_f}(s)G_{\theta f}(s) + G_{\epsilon f}(s))\|_\infty} &= \frac{\|R^*(s)G^*_{\epsilon w}(s)\|_\infty}{\|R^*(s)(G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s))\|_\infty} \\ &= \frac{1}{\|R^*(s)(G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s))\|_\infty}, \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{\|R(s)G_{\epsilon w}(s)\|_\infty}{\|R(s)(G_{\epsilon\theta_f}(s)G_{\theta f}(s) + G_{\epsilon f}(s))\|_\infty} &\geqslant \frac{\|R(s)\Gamma(s)\|_\infty}{\|R(s)\Gamma(s)t\|_\infty \|R^*(s)(G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s))\|_\infty} \\ &= \frac{1}{\|R^*(s)(G^*_{\epsilon\theta_f}(s)G_{\theta f}(s) + G^*_{\epsilon f}(s))\|_\infty}, \qquad \forall\, R(s) \in \mathbf{RH}_\infty. \end{aligned} \tag{40}$$

It concludes from (39) – (40) that both $(H^*, R^*(s))$ and $(H, R_h(s))$ are the optimal solutions of problem (28).

Remark 4 The convergence of residual r is guaranteed by a suitable selection of filter gain matrix H, while the selection of stable post-filter $R_h(s)$ in (29) delivers an optimal residual vector. Results in Theorem 2 also show that, for all H ensuring the asymptotic stability of system (14)−(17), $(H, R_h(s))$ is one of the optimal solutions of the FDF problem.

4 Numerical Example

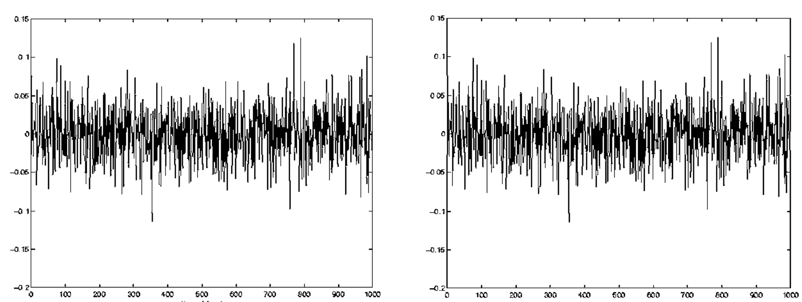
To illustrate the proposed FDF design method, a numerical example is given in this section. Consider a time-delay system of (6)−(8) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$C = [\,1 \quad 1\,], \quad D = 0, \quad D_f = 0, \quad D_d = [\,0 \quad 0.1\,], \quad N = 1, \quad L = 0, \quad \tau = 1.$$

By using the proposed approach, we obtain one solution as follows:

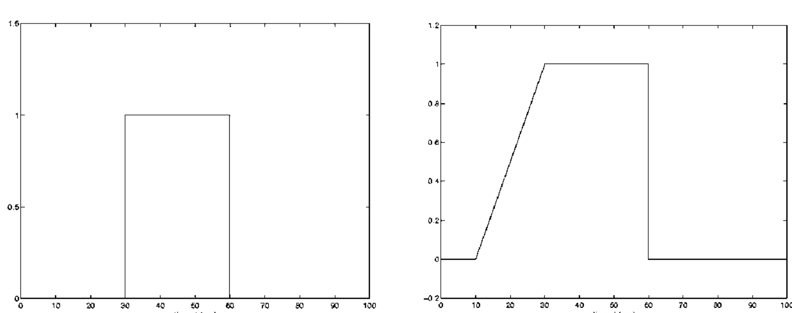
$$H^* = \begin{bmatrix} 1 \\ 1.6056 \end{bmatrix}, \quad H = \begin{bmatrix} 1.0026 \\ -0.9212 \end{bmatrix}, \quad Q = 100,$$
$$R_h(s) = Q^{-1/2}(I + C(sI - A + H^*C)^{-1}(H - H^*)).$$

Over evaluation time window $[0, 100]$ sec, suppose the unknown input is $d(t) = [\,d_1(t) \quad d_2(t)\,]^{\mathrm{T}}$, and $d_1(t)$, $d_2(t)$ are band-limited white noise as in Figure 4.1 (a) and (b). Two faulty cases are considered, where the fault signals are respectively given in Figure 4.2 (a) and (b). Figure 4.3 (a) and (b) show the two cases of residual signal whatever the control input $u(t)$.



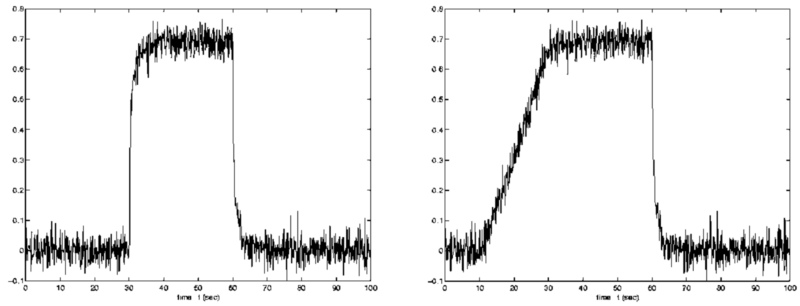
a) b)

Figure 4.1. a) Unknown input signal $d_1(t)$; b) Unknown input signal $d_2(t)$.



a) b)

Figure 4.2. a) Fault signal $f(t)$: case I; b) Fault signal $f(t)$: case II.



a) b)

Figure 4.3. a) Residual signal $r(t)$: case I; b) Residual signal $r(t)$: case II.

5 Conclusion

In this short paper, the FDF design problem for linear time-delay systems with unknown input is studied. The main contributions of this work are the introduction of a new FDF, the formulation of an optimization problem based on a performance index, and the extension of the FD optimization approach for LTI systems to the time-delay systems. The convergence of the residual generator is ensured by suitabe choice of the filter gain matrix, while the FDF performance can be guaranteed by the selection of a corresponding stable post-filter in terms of a Riccati equation. A simulation example is given to show the effectiveness of the proposed method.

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Adaptive Output Control of a Class of Time-Varying Uncertain Nonlinear Systems

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Abstract: In this paper, we present a new scheme to design adaptive controllers for single-input single-output uncertain time-varying systems in the presence of unknown bounded disturbances. No knowledge is assumed on the sign of the term multiplying the control. The control design is achieved by introducing certain well defined functions, estimating variation rates of parameters and incorporating a Nussbaum gain. To overcome the problem of overparametrization, tuning functions, which are different from the standard ones due to the use of projection operations, are employed. It is shown that the proposed controller can guarantee global uniform ultimate boundedness.

Keywords: *Adaptive control; backstepping; time-varying systems; tuning functions; Nussbaum gain.*

Mathematics Subject Classification (2000): 93C40.

1 Introduction

Adaptive control has seen significant development since the appearance of a Lyapunov-based recursive design procedure known as backstepping [7]. A great deal of attention has been paid to tackle both linear and nonlinear systems with unknown parameters and a number of results have been obtained in $[1-6]$. However, only limited number of results are available for nonlinear systems with time-varying parameters and/or without the knowledge on the sign of the term multiplying the control, i.e. high frequency gain in the case of linear systems, in the presence of external disturbances. In this paper, we shall also call this term the high frequency gain for nonlinear systems for simplicity.

In [9], output feedback control was considered for linear time-varying systems when the sign of high-frequency gain is known. In [11], the problem of adaptive control with

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unknown sign of high-frequency gain for linear time invariant systems was studied. In [2], Nussbaum gain incorporated with the backstepping technique was used to design adaptive output stabilizer for high order uncertain time invariant nonlinear systems with unknown sign of high-frequency gain in the absence of external disturbances. The nonlinearities considered should satisfy sector conditions. In [3], disturbance decoupling was addressed for nonlinear time invariant systems with known sign of the high frequency gain. The result obtained is critically depending on a function of the system output y and the reference trajectory y_r. It should be noted that such a function is undefined at the time instants when $y = y_r$. Therefore, the control signal is undefined at these time instants. In [4], a flat zone was used to handle the problem of nonlinear time invariant systems with unknown sign of high frequency gain in the presence of disturbances. The bound of the disturbance and all the unknown parameters need to be estimated at every step in the backstepping process. This results in the problem of overparametrization and makes the implementation complicated. In [6] state-feedback control was considered for a class of uncertain time-varying nonlinear systems in the presence of disturbances. Due to state feedback, no filter is required for state estimation. Thus the derivatives of the time varying parameters and the term of the disturbance need not to be considered in controller design. This also makes the stability analysis greatly simplified. Again, parameters are required to be estimated at every step, which results in overparametrization. In the case of output feedback control of nonlinear time-varying systems in the presence of disturbances, no result is available. In this case, filters similar to [7] are required to estimate system states and the state equations of the state estimation error will be used in the design and analysis. In these equations, the external disturbances and derivatives of time-varying parameters will appear and have great impact on the errors. This makes the design and analysis quite difficult, especially when the sign of high frequency gain is unknown and tuning functions are used.

In this paper, we consider such a case and propose a new control design scheme to solve the problem. The nonlinearities considered are not required to satisfy the sector type of conditions like [2]. To handle the disturbances, well defined functions are introduced to eliminate their effects in the Lyapunov functions employed in the recursive design steps. To deal with the time variation problem, an estimator is used to estimate the bound of the variation rates. Furthermore, the overparametrization problem is also solved by using the concept of tuning functions. As projection operation is used, the design of tuning functions are different from existing schemes as in [7]. With our proposed controller, global system stability is ensured.

2 System Model and Problem Formulation

Consider the following class of single-input-single-output (SISO) nonlinear time-varying systems in the feedback form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \theta_{a1}(t)\psi_{a1}(y) + d_1(t)\phi_{a1}(y) + \psi_{01}(y), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{x}_{\rho-1} &= x_\rho + \theta_{a\rho-1}(t)\psi_{a\rho-1}(y) + d_{\rho-1}(t)\phi_{a\rho-1}(y) + \psi_{0\rho-1}(y), \\
\dot{x}_\rho &= x_{\rho+1} + \theta_{a\rho}(t)\psi_{a\rho}(y) + d_\rho(t)\phi_{a\rho}(y) + \psi_{0\rho}(y) + b_m(t)u, \qquad (1) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

$$\dot{x}_n = \theta_{an}(t)\psi_{an}(y) + d_n(t)\phi_{an}(y) + \psi_{0n}(y) + b_0(t)u,$$
$$y = e_1^{\mathrm{T}} x,$$

where $x = [x_1, \cdots, x_n]^{\mathrm{T}} \in R^n$, $u \in R$ and $y \in R$ are system states, input and output respectively, $b_i(t)$, $i = m, \dots, 0$, are bounded uncertain time-varying piecewise continuous high-frequency gains, $\theta_{ai}(t) \in R^{p_i}$ are uncertain time-varying parameters, $d_i(t)$, $i = 1, \dots, n$, denote unknown time-varying bounded disturbances, ψ_{ai} and ϕ_{ai} are known smooth nonlinear functions in R^n. Similar class of systems was analyzed in [8].

In order to cope with the unknown sign of high-frequency gain, the Nussbaum gain technique is employed in this paper. A function $N(\chi)$ is called a Nussbaum-type function if it has the following properties [10]

$$\lim_{s\to\infty} \sup \frac{1}{s} \int_0^s N(\chi)d\chi = \infty, \tag{2}$$

$$\lim_{s\to\infty} \inf \frac{1}{s} \int_0^s N(\chi)d\chi = -\infty. \tag{3}$$

In this paper, the even Nussbaum function $\exp(\chi^2)\cos(\frac{\pi}{2}\chi)$ is exploited. As in [6] the following Lemma will be employed in later analysis.

Lemma 1 *Let $V(t)$ and $\chi(t)$ be a smooth function defined on $[0, t_f)$ with $V(t) \ge 0$, $\forall\, t \in [0, t_f)$, and $N(\chi) = \exp(\chi^2)\cos(\frac{\pi}{2}\chi)$ be an even smooth Nussbaum-type function. If the following inequality holds:*

$$V(t) \le f_0 + e^{-f_1 t} \int_0^t g_1 N(\chi)\dot{\chi} d\tau + e^{-f_1 t} \int_0^t \dot{\chi}(t) e^{f_1 \tau} d\tau \tag{4}$$

where constant $f_1 > 0$, g_1 is a parameter which takes values in the unknown closed intervals $I_1 = [l_1^-, l_1^+]$ with $0 \notin I_1$, and f_0 represents some suitable constant, then $V(t)$, $\chi(t)$ and $\int_0^t g_1 N(\chi)\dot{\chi} d\tau$ must be bounded on $[0, t_f)$.

For the considered system (1), the following assumptions are imposed.

Assumption 1 The uncertain parameter vector θ is inside a compact set Ω_θ, where $\theta = [b_m(t), \dots, b_0(t), \theta_{a1}(t), \dots, \theta_{an}(t)]^{\mathrm{T}}$. In addition, there exists an unknown bounded positive constant q so that $q \ge \|\dot{\theta}\|$. Also q is inside a compact intervals $\Omega_q = [I^-, I^+]$ and $b_m(t) \ne 0, \forall\, t$.

Assumption 2 The relative degree ρ is fixed and known. This is ensured by Assumption 1.

Assumption 3 The reference signal y_r and its $(\rho - 1)$-th order derivatives are also assumed to be known and bounded.

Assumption 4 The system is minimum phase in the sense defined in [8].

In order to design the desired adaptive control law with output via backstepping procedures, we now transform system (1) into the following form

$$\dot{x} = Ax + F(y,u)^{\mathrm{T}}\theta + \Phi_a(y)d(t)^{\mathrm{T}} + \psi_0(y) \tag{5}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{6}$$

$$F(y,u)^{\mathrm{T}} = \left[\begin{bmatrix} 0_{(\rho-1)\times(m+1)} \\ I_{m+1} \end{bmatrix} u, \quad \Psi_a(y) \right], \tag{7}$$

$$\Psi_a(y) = \begin{bmatrix} \psi_{a1}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \psi_{a2}^{\mathrm{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \psi_{an}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \Psi_{a1}(y) \\ \vdots \\ \Psi_{an}(y) \end{bmatrix}, \tag{8}$$

$$\Phi_a(y) = \begin{bmatrix} \phi_{a1}^{\mathrm{T}} & 0 & \dots & 0 \\ 0 & \phi_{a2}^{\mathrm{T}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \phi_{an}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \Phi_{a1}^{\mathrm{T}}(y) \\ \vdots \\ \Phi_{an}^{\mathrm{T}}(y) \end{bmatrix}, \tag{9}$$

$$\theta = [b_m(t), \dots, b_0(t), \theta_{a1}(t), \dots, \theta_{an}(t)]^{\mathrm{T}}, \tag{10}$$

$$d(t) = [d_1(t), \dots, d_n(t)], \tag{11}$$

$$\psi_0(y) = [\psi_{01}(y), \dots, \psi_{0n}(y)]^{\mathrm{T}}. \tag{12}$$

We employ the filters similar to those in [7], i.e.

$$\dot{\xi} = A_0\xi + ky + \psi_0(y) \tag{13}$$

$$\dot{\Omega}^{\mathrm{T}} = A_0\Omega^{\mathrm{T}} + F(y,u)^{\mathrm{T}} \tag{14}$$

where

$$k \triangleq [k_1, k_2, \dots, k_n]^{\mathrm{T}}, \tag{15}$$

$$A_0 = A - ke_1^{\mathrm{T}}. \tag{16}$$

The vector k in (15) is chosen such that the matrix A_0 is strictly stable. It can be shown that Ω obtained from (14) satisfies the following equations

$$\Omega^{\mathrm{T}} = [v_m, \dots, v_1, v_0, \Xi], \tag{17}$$

$$\dot{\Xi} = A_0\Xi + \Psi_a(y), \tag{18}$$

$$\dot{\lambda} = A_0\lambda + e_n u, \tag{19}$$

$$v_j = A_0^j\lambda. \tag{20}$$

From our designed filters, system (1) can be represented as

$$\dot{y} = b_m v_{m,2} + \beta + \bar{\omega}^{\mathrm{T}}\theta + \epsilon_2 + d(t)\Phi_{a1}(y), \tag{21}$$

$$\dot{v}_{m,i} = v_{m,i+1} - k_i v_{m,1}, \quad i = 2, 3, \dots, \rho - 1, \tag{22}$$

$$\dot{v}_{m,\rho} = v_{m,\rho+1} - k_\rho v_{m,1} + u, \tag{23}$$

where

$$\beta = \xi_2 + \psi_{01}, \tag{24}$$

$$\omega = [v_{m,2}, v_{m-1,2}, \dots, v_{0,2}, \Xi_2 + \Psi_{a1}]^{\mathrm{T}}, \tag{25}$$

$$\bar{\omega} = [0, v_{m-1,2}, \dots, v_{0,2}, \Xi_2 + \Psi_{a1}]^{\mathrm{T}}. \tag{26}$$

In the above equations, ϵ_2, $v_{i,2}$ and $\xi_{i,2}$ denote the second entries of ϵ, v_i and ξ_i respectively, ϵ is the estimation error defined in (28).

With the above filters, a state estimate is given by

$$\hat{x} = \xi + \Omega^{\mathrm{T}}\theta \tag{27}$$

and the estimation error ϵ is defined as

$$\epsilon = x - \hat{x} \tag{28}$$

From the equations (5), (13), (14), (27) and (28), the estimation error satisfies

$$\dot{\epsilon} = A_0\epsilon + \Phi_a(y)d(t)^{\mathrm{T}} - \Omega^{\mathrm{T}}\dot{\theta}. \tag{29}$$

Remark 1 The error ϵ will be used in our design and analysis given later. As the disturbances and derivatives of time-varying parameters appear in (29), their effects should be considered in controller design. However for the state-feedback control in [6], no filter is required for state estimation. Their effects may not be necessarily considered in controller design and this makes the problem much simpler.

We now divide the error ϵ into two parts, i.e. $\epsilon = \epsilon_a + \epsilon_b$, where ϵ_a satisfies

$$\dot{\epsilon}_a = A_0\epsilon_a + \Phi_a(y)d(t)^{\mathrm{T}} \tag{30}$$

with $\epsilon_a(0) = \epsilon(0)$, and $\epsilon_b = \int_0^t e^{A_0(t-\tau)}(-\Omega^{\mathrm{T}}\dot{\theta})d\tau$. It can be shown that

$$\begin{aligned} \|\epsilon_b\| &\le \int_0^t \|e^{A_0(t-\tau)}\|\,\|\Omega\|\,\|\dot{\theta}\|\,d\tau \\ &\le q\int_0^t \|e^{A_0(t-\tau)}\|\,\|\Omega\|\,d\tau \le q\int_0^t e^{-\lambda_\theta(t-\tau)}k_\theta\|\Omega\|\,d\tau, \end{aligned} \tag{31}$$

where λ_θ and k_θ are chosen positive parameters so that

$$k_\theta e^{-\lambda_\theta t} \ge \|e^{A_0 t}\|, \quad \forall\, t \ge 0. \tag{32}$$

Thus ϵ_b satisfies

$$|\epsilon_b| \le h(t)q, \tag{33}$$

where $h(t)$ is generated by

$$\dot{h} = -\lambda_\theta h + k_\theta\left(\|\Omega\|^2 + \frac{1}{4}\right). \tag{34}$$

Suppose $P \in R^{n\times n}$ is a positive definite matrix, satisfying $PA_0 + A_0^{\mathrm{T}}P \le -3I$ and let

$$V_\epsilon = \epsilon_a^{\mathrm{T}} P \epsilon_a. \tag{35}$$

It can be shown that

$$\begin{aligned}\dot{V}_\epsilon &= \epsilon_a^{\mathrm{T}}(PA_0 + A_0^{\mathrm{T}}P)\epsilon_a + 2\epsilon_a^{\mathrm{T}} P\Phi_a(y)d(t)^{\mathrm{T}} \\ &\le -2\|\epsilon_a\|^2 + \|P\Phi_a(y)d(t)^{\mathrm{T}}\|^2.\end{aligned} \tag{36}$$

The problem of this paper is to design an adaptive controller to make system (1) BIBO stable.

3 Control Design

In this section, we present the adaptive control design using the backstepping technique with tuning functions in ρ steps. In order to avoid using the sign of the high frequency gain, we take the change of coordinates

$$z_1 = y - y_r, \tag{37}$$

$$z_i = v_{m,i} - \alpha_{i-1},\ i = 2, 3, \ldots, \rho, \tag{38}$$

where α_{i-1} is the virtual control at each step and will be determined in later discussions. Before presenting the detail, a useful function is introduced. Firstly we define $s(x)$ as

$$s(x) = \begin{cases} x^2 & |x| \ge \delta, \\ (\delta^2 - x^2)^\rho + x^2 & |x| < \delta, \end{cases} \tag{39}$$

where δ is a positive design parameter. It can be shown that $s(x)$ is $(\rho-1)$-th order differentiable and bounded below for $|x| < \delta$. Based on $s(x)$, a function $H(z_1)$ is defined as follows

$$H(z_1) = \frac{\Phi_a(y)}{s(z_1)} = \begin{cases} \dfrac{\Phi_a(y)}{z_1^2} & |z_1| \ge \delta, \\ \dfrac{\Phi_a(y)}{(\delta^2 - z_1^2)^\rho + z_1^2} & |z_1| < \delta. \end{cases} \tag{40}$$

Clearly H is well defined and for $|z_1| < \delta$, H is bounded as $s(z_1)$ is bounded below.

Remark 2 In [3], a similar function to (40) was used to design controllers for disturbance decoupling. However, the function is undefined at the time instants when $y = y_r$. Thus, the controller presented is undefined at these time instants.

From (36) and (40) it can be shown that

$$\dot{V}_\epsilon \leq -2\|\epsilon_a\|^2 + \frac{1}{2}s^4\|PH\|^4 + \frac{1}{2}\|d(t)\|^4. \tag{41}$$

We now illustrate the backstepping design procedures using Nussbaum gain with details given for the first two steps.

Step 1 It follows from (21) and (37) that

$$\dot{z}_1 = b_m v_{m,2} + \beta + \bar{\omega}^{\mathrm{T}}\theta + \epsilon_2 + d(t)\Phi_{a1}(y) - \dot{y}_r. \tag{42}$$

Without using the sign of b_m, the following virtual control law α_1 is designed

$$\alpha_1 = N(\chi)\bar{\alpha}_1 e^{-ft}, \tag{43}$$

$$N(\chi) = \exp(\chi^2)\cos\frac{\pi}{2}\chi, \tag{44}$$

where f is a positive real design parameter, χ is generated by

$$\dot{\chi} = z_1\bar{\alpha}_1 \tag{45}$$

and $\bar{\alpha}_1$ is chosen to be

$$\begin{aligned}\bar{\alpha}_1 = {}& \big(c_1 + l_1 + (e_1^{\mathrm{T}}\hat{\theta})^2\big)z_1 + \beta + \bar{\omega}^{\mathrm{T}}\hat{\theta} - \dot{y}_r \\ & + z_1 h^2\hat{q} + \frac{1}{4}z_1\|\Phi_{a1}(y)\|^2 + \sum_{i=1}^{\rho}\frac{1}{8l_i}z_1 s^3(z_1)\|PH\|^4,\end{aligned} \tag{46}$$

where c_1 and l_1 are two positive real design parameters, $\hat{\theta}$ and $\hat{q}$ denote the estimates of θ and q. Notice that

$$b_m v_{m,2} = b_m(z_2 + \alpha_1) = \hat{b}_m z_2 + b_m\alpha_1 + \tilde{b}_m z_2, \tag{47}$$

where $\tilde{b}_m = b_m - \hat{b}_m$, $\hat{b}_m$ is the first element of $\hat{\theta}$, i.e. $\hat{b}_m = e_1^{\mathrm{T}}\hat{\theta}$. Then from (42) and (46) we have

$$\begin{aligned}\dot{z}_1 - \bar{\alpha}_1 = {}& -(c_1 + l_1 + \hat{b}_m^2)z_1 + (\bar{\omega}^{\mathrm{T}} + z_2 e_1^{\mathrm{T}})\tilde{\theta} + \epsilon_{a,2} + \epsilon_{b,2} - z_1 h^2\hat{q} + \hat{b}_m z_2 + b_m\alpha_1 \\ & + d(t)\Phi_{a1}(y) - \frac{1}{4}z_1\|\Phi_{a1}(y)\|^2 - \sum_{i=1}^{\rho}\frac{1}{8l_i}z_1 s^3\|PH\|^4,\end{aligned} \tag{48}$$

where $\tilde{\theta} = \theta - \hat{\theta}$, $\epsilon_{a,2}$ and $\epsilon_{b,2}$ represent the second entry of ϵ_a and ϵ_b. To proceed, we define the Lyapunov function

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \frac{1}{4l_1}V_\epsilon, \tag{49}$$

where Γ is a positive definite matrix of $R^{(n+2)\times(n+2)}$. Then the derivative of V_1 along with (41), (43) and (48) is given by

$$\begin{aligned}\dot{V}_1 &= z_1(\dot{z}_1-\bar{\alpha}_1)+z_1\bar{\alpha}_1+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\dot{\theta}-\dot{\hat{\theta}})+\tilde{q}\dot{\tilde{q}}+\frac{1}{4l_1}\dot{V}_\epsilon\\ &\le -(c_1+\hat{b}_m^2)z_1^2+\hat{b}_m z_1 z_2+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1-\dot{\hat{\theta}})-l_1 z_1^2+\epsilon_{a,2}z_1-\frac{1}{2l_1}\|\epsilon_a\|^2\\ &\quad+\epsilon_{b,2}z_1-\tilde{q}\dot{\hat{q}}-h^2\hat{q}z_1^2+d(t)\Phi_{a1}(y)z_1-\frac{1}{4}z_1^2\|\Phi_{a1}(y)\|^2+b_m\alpha_1 z_1+\bar{\alpha}_1 z_1\\ &\quad+\frac{1}{8l_1}s^4\|PH\|^4-\sum_{i=1}^{\rho}\frac{1}{8l_i}z_1^2 s^3\|PH\|^4+\frac{1}{8l_1}\|d(t)\|^4+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta},\end{aligned}\tag{50}$$

where

$$\tau_1=\Gamma z_1(\bar{\omega}+z_2 e_1).\tag{51}$$

Here we know that

$$\epsilon_{b,2}z_1-h^2\hat{q}z_1^2\le hq|z_1|-h^2\hat{q}z_1^2\le q(h^2z_1^2+1/4)-h^2\hat{q}z_1^2=h^2\tilde{q}z_1^2+\frac{q}{4}.$$

Then we can get

$$\begin{aligned}\dot{V}_1 &\le (b_m N(\chi)e^{-ft}+1)\dot{\chi}-c_1 z_1^2+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1-\dot{\hat{\theta}})\\ &\quad+\tilde{q}(\iota_1-\dot{\hat{q}})-\frac{1}{4l_1}\|\epsilon_a\|^2+\frac{1}{4}z_2^2+M_1,\end{aligned}\tag{52}$$

where

$$\iota_1=h^2 z_1^2,\tag{53}$$

$$M_1=\|d(t)\|^2+\frac{1}{8l_1}\|d(t)\|^4-\sum_{i=2}^{\rho}\frac{1}{8l_i}s^4\|PH\|^4+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta}+\frac{1}{4}q+\overline{N},\tag{54}$$

$$\overline{N}=\begin{cases}0, & |z_1|\ge\delta,\\ \sum_{i=1}^{\rho}\frac{1}{8l_i}(\delta^2-z_1^2)^{\rho}s^3\|PH\|^4, & |z_1|<\delta.\end{cases}\tag{55}$$

From (40) we know that $\overline{N}$ is bounded.

Step 2 Now, we evaluate the dynamics of the second state z_2. Differentiating (38) for $i=2$ and using (22), we have

$$\dot{z}_2=v_{m,3}-k_2 v_{m,1}-\dot{\alpha}_1.\tag{56}$$

Note that α_1 is a function of y, $\hat{\theta}$, $\hat{q}$, ξ, Ξ, λ, χ and y_r and following from similar analysis to [7] by substituting (38) with $i=3$ into (56), we get

$$\dot{z}_2=\alpha_2-\beta_2-\frac{\partial\alpha_1}{\partial y}\Big(\epsilon_2+\omega^{\mathrm{T}}\tilde{\theta}+d(t)\Phi_{a1}(y)\Big)+z_3-\frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta}-\frac{\partial\alpha_1}{\partial\hat{\theta}}\dot{\hat{\theta}}-\frac{\partial\alpha_1}{\partial\hat{q}}\dot{\hat{q}},\tag{57}$$

where

$$\beta_2 \triangleq k_2 v_{m,1} + \frac{\partial\alpha_1}{\partial y}\beta + \frac{\partial\alpha_1}{\partial\Pi}\dot{\Pi} + \sum_{j=1}^{m+1}\frac{\partial\alpha_1}{\partial\lambda_j}(-k_j\lambda_1+\lambda_{j+1}) + \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial\alpha_1}{\partial\chi}\dot{\chi} \tag{58}$$

where $\Pi = [\xi^{\mathrm{T}}, \mathrm{Vec}\,(\Xi)^{\mathrm{T}}]^{\mathrm{T}}$. Define the Lyapunov function and choose the virtual control for this step as

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{4l_2}V_\epsilon, \tag{59}$$

$$\begin{aligned}\alpha_2 = &-\Big(c_2+\frac{1}{4}\Big)z_2 + \frac{\partial\alpha_1}{\partial y}\omega^{\mathrm{T}}\hat{\theta} - z_2\Big\|\frac{\partial\alpha_1}{\partial\hat{\theta}}\Big\|^2\|\tau_2\|^2 - z_2h^2\hat{q}\Big\|\frac{\partial\alpha_1}{\partial y}\Big\|^2 \\ &- z_2\Big\|\frac{\partial\alpha_1}{\partial\hat{q}}\Big\|^2\iota_2^2 - l_2\Big\|\frac{\partial\alpha_1}{\partial y}\Big\|^2 z_2 + \beta_2 - \frac{z_2}{4}\Big\|\frac{\partial\alpha_1}{\partial y}\Phi_{a1}(y)\Big\|^2,\end{aligned} \tag{60}$$

$$\tau_2 = \tau_1 - \Gamma\,\frac{\partial\alpha_1}{\partial y}\omega z_2, \tag{61}$$

$$\iota_2 = \iota_1 + h^2\Big\|\frac{\partial\alpha_1}{\partial 21}\Big\|^2 z_2^2. \tag{62}$$

Using (52), (59) and (60), we have that

$$\begin{aligned}\dot{V}_2 \le\ &\dot{V}_1 + z_2\dot{z}_2 + \frac{1}{4l_2}\,\dot{V}_\epsilon \\ \le &-\sum_{i=1}^{2}c_iz_i^2 + (b_mN(\chi)e^{-ft}+1)\dot{\chi} + z_2z_3 - \sum_{i=1}^{2}\frac{1}{4l_i}\|\epsilon_a\|^2 + M_2 \\ &+\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_1-\dot{\hat{\theta}}) - z_2\,\frac{\partial\alpha_1}{\partial y}\,\omega^{\mathrm{T}}\tilde{\theta} + z_2^2\Big\|\frac{\partial\alpha_1}{\partial\hat{\theta}}\Big\|^2\|\dot{\hat{\theta}}\|^2 - z_2^2\Big\|\frac{\partial\alpha_1}{\partial\hat{\theta}}\Big\|^2\|\tau_2\|^2 \\ &+\tilde{q}(\iota_1-\dot{\hat{q}}) + h^2\tilde{q}\Big\|\frac{\partial\alpha_1}{\partial y}\Big\|^2 z_2^2 + z_2^2\Big\|\frac{\partial\alpha_1}{\partial\hat{q}}\Big\|^2\dot{\hat{q}}^2 - z_2^2\Big\|\frac{\partial\alpha_1}{\partial\hat{q}}\Big\|^2\iota^2 \\ \le &-\sum_{i=1}^{2}c_iz_i^2 + (b_mN(\chi)e^{-ft}+1)\dot{\chi} + z_2z_3 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_2-\dot{\hat{\theta}}) + \tilde{q}(\iota_2-\dot{\hat{q}}) + M_2 \\ &+ z_2^2\Big\|\frac{\partial\alpha_1}{\partial\hat{\theta}}\Big\|^2(\|\dot{\hat{\theta}}\|^2-\|\tau_2\|^2) + z_2^2\Big(\frac{\partial\alpha_1}{\partial\hat{q}}\Big)^2(\dot{\hat{q}}^2-\iota_2^2) - \sum_{i=1}^{2}\frac{1}{4l_i}\|\epsilon_a\|^2,\end{aligned} \tag{63}$$

where

$$M_2 = \sum_{i=1}^{2}\frac{1}{8l_i}\|d(t)\|^4 + 2\|d(t)\|^2 - \sum_{i=3}^{\rho}\frac{1}{8l_i}\,s^4\|PH\|^4 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta} + \frac{1}{2} + \frac{1}{2}\,q + \overline{N}. \tag{64}$$

Remark 3 Note that M_2 contains $s^4\|PH\|^4$ and this term may not be bounded. As seen from our analysis, $\frac{1}{8l_2}s^4\|PH\|^4$ disappears in M_2 due to the use of V_ϵ at step 2. If we use V_ϵ at each step, this term will disappear in M_ρ on the last step.

Step i $(i = 3, \ldots, \rho)$ These steps are similar to those in [7]. Define

$$V_i = V_{i-1} + \frac{1}{2} z_i^2 + \frac{1}{4l_i} V_\epsilon, \tag{65}$$

$$\begin{aligned} \alpha_i = {} & -c_i z_i - l_i \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i - z_{i-1} + \beta_i + \frac{\partial \alpha_{i-1}}{\partial y} \omega^{\mathrm{T}} \hat{\theta} - \frac{z_i}{4} \left\| \frac{\partial \alpha_{i-1}}{\partial y} \Phi_{a1}(y) \right\|^2 \\ & - z_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \right\|^2 \|\tau_i\|^2 + \left(\sum_{k=2}^{i-1} z_k^2 \left\| \frac{\partial \alpha_{k-1}}{\partial \hat{\theta}} \right\|^2 \right) (\tau_i + \tau_{i-1})^{\mathrm{T}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega \\ & - z_i \left\| \frac{\partial \alpha_{i-1}}{\partial \hat{q}} \right\|^2 \iota_i^2 - \left(\sum_{k=2}^{i-1} z_k^2 \left\| \frac{\partial \alpha_{k-1}}{\partial \hat{q}} \right\|^2 \right) (\iota_i + \iota_{i-1})^{\mathrm{T}} h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i \\ & - z_i h^2 \hat{q} \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2, \end{aligned} \tag{66}$$

$$\tau_i = \tau_{i-1} - \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i, \tag{67}$$

$$\iota_i = \iota_{i-1} + h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i^2, \tag{68}$$

where

$$\beta_i \triangleq k_i v_{m,1} + \frac{\partial \alpha_{i-1}}{\partial y} \beta + \frac{\partial \alpha_{i-1}}{\partial \Pi} \dot{\Pi} + \frac{\partial \alpha_{i-1}}{\partial y_r} \dot{y}_r + \sum_{j=1}^{m+1} \frac{\partial \alpha_{i-1}}{\partial \lambda_j} (-k_j \lambda_1 + \lambda_{j+1}) + \frac{\partial \alpha_{i-1}}{\partial \chi} \dot{\chi}. \tag{69}$$

Also note that

$$\begin{aligned} \|\tau_i\|^2 &= \tau_i^{\mathrm{T}} \tau_i = \tau_i^{\mathrm{T}} \tau_i - \tau_{i-1}^{\mathrm{T}} \tau_{i-1} + \tau_{i-1}^{\mathrm{T}} \tau_{i-1} = (\tau_i + \tau_{i-1})^{\mathrm{T}} (\tau_i - \tau_{i-1}) + \tau_{i-1}^{\mathrm{T}} \tau_{i-1} \\ &= -(\tau_i + \tau_{i-1})^{\mathrm{T}} \Gamma \frac{\partial \alpha_{i-1}}{\partial y} \omega z_i + \tau_{i-1}^{\mathrm{T}} \tau_{i-1}, \\ \iota_i^2 &= (\iota_i + \iota_{i-1})^{\mathrm{T}} h^2 \left\| \frac{\partial \alpha_{i-1}}{\partial y} \right\|^2 z_i^2 + \iota_{i-1}^2. \end{aligned} \tag{70}$$

Then the actual adaptive controller is obtained and given by

$$u(t) = \alpha_\rho - v_{m,\rho+1}, \tag{71}$$

$$\dot{\hat{\theta}} = \mathrm{Proj}\,(\tau_\rho), \tag{72}$$

$$\dot{\hat{q}} = \mathrm{Proj}\,(\iota_\rho), \tag{73}$$

where $\mathrm{Proj}\,(\cdot)$ is a smooth projection operation to ensure the estimates belong to compact sets for all time. Such an operation can be found in [7].

Remark 4 Note that the designed tuning functions are different from existing schemes in [7] as the projection operations are used in the parameter estimators.

By using the properties that $-\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\operatorname{Proj}(\tau) \le -\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tau$ and $\operatorname{Proj}(\tau)^{\mathrm{T}}\operatorname{Proj}(\tau) \le \tau^{\mathrm{T}}\tau$ the final Lyapunov function V_ρ satisfies

$$
\begin{aligned}
\dot{V}_\rho \le & -\sum_{k=1}^{\rho} c_k z_k^2 + (b_m N(\chi)e^{-ft} + 1)\dot{\chi} + M_\rho - \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2 \\
& + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}(\tau_\rho - \operatorname{Proj}(\tau_\rho)) + \left(\sum_{k=2}^{\rho} z_k^2\left\|\frac{\partial\alpha_{k-1}}{\partial\hat{\theta}}\right\|^2\right)(\operatorname{Proj}(\tau_\rho)^{\mathrm{T}}\operatorname{Proj}(\tau_\rho) - \|\tau_\rho\|^2) \\
& + \tilde{q}(\iota_\rho - \operatorname{Proj}(\iota_\rho)) + \left(\sum_{k=2}^{\rho} z_k^2\left(\frac{\partial\alpha_{k-1}}{\partial\hat{q}}\right)^2\right)(\operatorname{Proj}(\iota_\rho)^2 - \iota_\rho^2) \\
\le & -\sum_{k=1}^{\rho} c_k z_k^2 + b_m N(\chi)e^{-ft}\dot{\chi} + \dot{\chi} + M_\rho - \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2,
\end{aligned}
\tag{74}
$$

where

$$
M_\rho = \sum_{i=1}^{\rho}\frac{1}{8l_i}\|d(t)\|^4 + \rho\|d(t)\|^2 + \tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\dot{\theta} + \frac{\rho-1}{2} + \frac{\rho}{4}q + \overline{N} \tag{75}
$$

Integrating both sides of (74) over the interval $[0,t]$ gives

$$
\begin{aligned}
\int_0^t \dot{V}_\rho e^{f\tau}d\tau \le & -\int_0^t \sum_{k=1}^{\rho} c_k z_k^2 e^{f\tau}d\tau + \int_0^t b_m N(\chi)\dot{\chi}\,d\tau + \int_0^t \dot{\chi}e^{f\tau}d\tau \\
& + \int_0^t M_\rho e^{f\tau}d\tau - \int_0^t \sum_{i=1}^{\rho}\frac{1}{4l_i}\|\epsilon_a\|^2 e^{f\tau}d\tau.
\end{aligned}
\tag{76}
$$

Note that $V_\epsilon \le \|P\|\|\epsilon_a\|^2$. Then

$$
\begin{aligned}
V_\rho &= \sum_{k=1}^{\rho}\frac{1}{2}z_k^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \sum_{i=1}^{\rho}\frac{1}{4l_i}V_\epsilon \\
&\le \sum_{k=1}^{\rho}\frac{1}{2}z_k^2 + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \frac{1}{2}\tilde{q}^2 + \sum_{i=1}^{\rho}\frac{1}{4l_i}\|P\|\|\epsilon_a\|^2.
\end{aligned}
\tag{77}
$$

This yields

$$
\begin{aligned}
0 \le V_\rho(t) \le & V_\rho(0) + e^{-ft}\int_0^t b_m N(\chi)\dot{\chi}\,d\tau + \int_0^t \dot{\chi}e^{-f(t-\tau)}\,d\tau \\
& + \int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta}) + \tilde{q}^2)e^{-f(t-\tau)}\,d\tau + \int_0^t M_\rho e^{-f(t-\tau)}\,d\tau
\end{aligned}
\tag{78}
$$

where $f = \min\left\{\frac{1}{\|P\|_2}, 2c_1, 2c_2, \ldots, 2c_\rho,\right\} > 0$. Due to the utilization of projection operations for $\hat{\theta}$ and $\hat{q}$, the boundedness of $\tilde{\theta}$ and $\tilde{q}$ can be guaranteed. Together with the boundedness of $d(t)$, q and $\dot{\theta}$, the boundedness of M_ρ and

$$\int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \tilde{q}^2)e^{-f(t-\tau)}d\tau + \int_0^t M_\rho e^{-f(t-\tau)}d\tau$$

can be guaranteed. Thus by comparing with (4), f_0 is selected as the upper bound of

$$V_\rho(0) + \int_0^t \frac{f}{2}(\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} + \tilde{q}^2)e^{-f(t-\tau)}d\tau + \int_0^t M_\rho e^{-f(t-\tau)}d\tau, g_1 = b_m$$

and $f_1 = f$. Using Lemma 1, we can conclude that $V_\rho(t)$ and $\chi(t)$, hence z_i, $(i = 1, \ldots, \rho)$ are bounded. Finally, the stability of the whole system can be established as in [7].

To conclude this section, the results established are presented in the following theorem.

Theorem 1 *Consider the uncertain time-varying nonlinear system (1) satisfying Assumptions 1 – 4. With the application of the controller (71) and the parameter updating laws (72) and (73), the resulting closed loop system is BIBO stable.*

4 A Simulation Example

In this section, the proposed design method is illustrated on the following simple linear system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \theta_1(t)y(t), \\ \dot{x}_2(t) &= b(t)u(t) + d(t), \\ y(t) &= x_1(t), \end{aligned} \tag{79}$$

where $\theta_1(t) = 1 + \sin(t)$, $b(t) = 1 + \exp(-t)$, $d(t) = \cos(t)$ are unknown timevarying parameters in the controller design. The objective is to control the system output $y(t)$ to follow a desired trajectory $y_r(t) = \sin(t) + \sin(2t)$. The filters are implemented as

$$\dot{\xi} = A_0\xi + ky, \tag{80}$$

$$\dot{\lambda} = A_0\lambda + e_2 u, \tag{81}$$

$$\dot{\Xi} = A_0\Xi + \Psi, \quad \Psi = [y\ 0]^{\mathrm{T}}, \tag{82}$$

$$A_0 = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}. \tag{83}$$

The control law α_1 in (43), $u(t)$ in (71), and the parameter update law $\hat{\theta}$ in (72) are used with $\theta = [b\ \theta_1]^{\mathrm{T}}$. The design parameters are chosen as $c_1 = c_2 = 5$, $\Gamma = I_2$, $l_1 = l_2 = 2$, $k_1 = 6$, $k_2 = 8$. The initials $y(0) = 0.1$, $\hat{\theta}(0) = [0.2\ 0.5]^{\mathrm{T}}$ and others are set to zero. The simulation results presented in the Figure 4.1 show the system output $y(t)$ and the

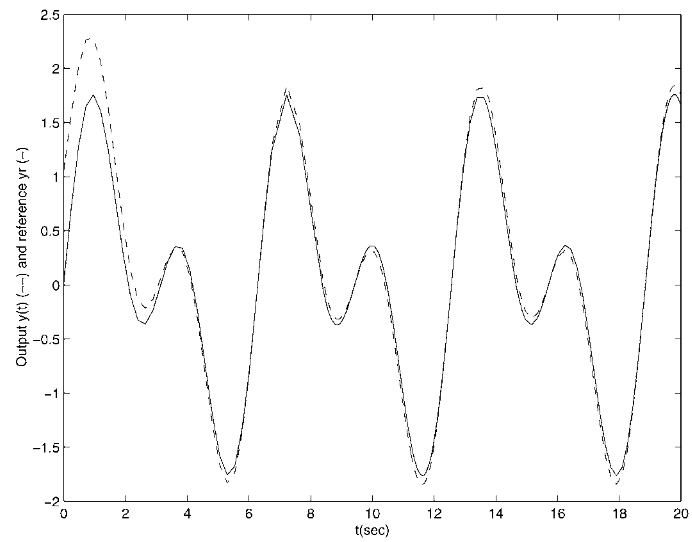

Figure 4.1. Output $y(--)$ and trajectory $y_r(-)$.

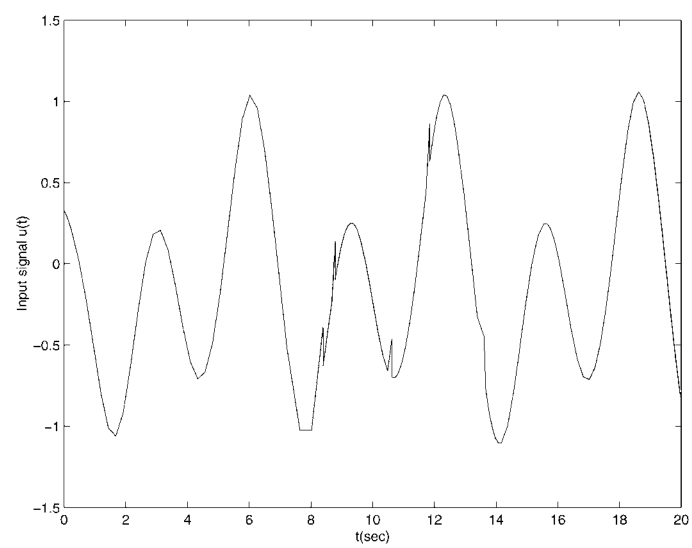

Figure 4.2. Control signal $u(t)$.

desired trajectory signal $y_r(t)$. Figure 4.2 shows the control signal $u(t)$. Clearly, these simulation results verify that our proposed scheme is effective.

5 Conclusion

In this paper, a scheme is proposed to design an adaptive output-feedback controller for uncertain time-varying nonlinear systems with unknown sign of high-frequency gains in the presence of disturbances. No growth conditions on system nonlinearities are imposed. In the design, certain well defined functions are used to cancel the effects of disturbances.

To deal with the time variation problem, an estimator is used to estimate the bound of the variation rates. Furthermore, the overparametrization problem is also solved by using the concept of tuning functions. It is shown that the controller obtained by the proposed design scheme can make the whole adaptive control system stable. Simulations performed on a simple system also verify the effectiveness of the proposed scheme.

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Stability of Nonautonomous Neutral Variable Delay Difference Equation

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Abstract: This paper studies the stability of a class of nonautonomous neutral delay difference equation. The case of several variable delays is mainly considered, and the sufficient conditions of uniform stability and uniform asymptotical stability are obtained. Some results with a constant delay in the literature are extended and improved.

Keywords: *Nonautonomous; neutral difference equations; stability.*

Mathematics Subject Classification (2000): 35D05, 35E05.

1 Introduction

Consider the nonautonomous neutral variable delay difference equation

$$\Delta(x(n) - cx(n-k)) + f(n, x(n-l_1(n)), \dots\ x(n-l_m(n))) = 0, \quad n \in N, \tag{1}$$

where $c \in (-1, 1)$; $k \in N$; $\{l_i(n)\}$ is a positive integer sequence and satisfies $l_i(n) \le l$, $i = 1, \dots, m$, $n \in N$; l is a given positive integer, $f(n, x_1, \dots x_m)\colon N \times R^m \to R$, and $f(n, 0, \dots 0)$ satisfies $f(n, x_1, \dots x_m) \equiv 0$, $n \in N$.

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In recent years there are lots of researches on stability of special-formed zero solution to the equation (1) (see [1−9]). In 1999, Z. Zhou and J.S. Yu studied the equation

$$\Delta(x(n) - cx(n-k)) + h(n, x(n-l)) = 0$$

where $c \in (-1, 1)$; $k \in N$; $l \in N$; $f(n, x)\colon\ N \times R \to R$ and $f(n, 0)$ satisfies $f(n, 0) \equiv 0$, $n \in N$, and obtained a sufficient condition of the stability and asymptotical stability for zero solution to this equation [7]. It will be more practical for the fact that if the function $f(n, x)$ is replaced by function $f(n, x_1, \dots x_m)$ and the constant delay is replaced by the variable delay. Based on the above-mentioned consideration, we studied the stability of equation (1) and discovered that the concerned conclusion can be extended to the more general equation (1) and obtained a sufficient condition of the stability and asymptotical stability of equation (1).

For simplicity, the basic conceptions and symbols which occur in the paper will be introduced as follows: "Δ" stands for the forward difference operator, say, $\Delta y(n) = y(n+1) - y(n)$; Z is the integer number set; R is the real number set. Suppose that $a \in Z$, let $N(a) = \{a,\, a+1,\, \dots\}$, $N = N(0)$. For any given $a, b \in Z$ and $a \le b$, let $N(a, b) = \{a,\, a+1,\, \dots,\, b\}$.

Definition 1.1 Sequence $\{x(n)\}$ is said to be *the solution of equation* (1) if for a certain $n_0 \in N$, the sequence is defined on the $N(n_0 - r)$, where $r = \max\{l, k\}$ and satisfies equation (1). Obviously, equation (1) has zero solution permanently.

Definition 1.2 If for any $\varepsilon > 0$ and $n_0 \in N$, there exists a $\delta(\varepsilon, n_0) > 0$, such that when $|x(n_0 + j)| < \delta$, $j \in N(-r, 0)$, the solution of equation (1) satisfies $|x(n)| < \varepsilon$, $n \in N(n_0)$, then the zero solution of equation (1) is said to be stable. If δ can be chosen independent of n_0, then the zero solution of equation (1) is said to be uniformly stable.

Definition 1.3 The *zero solution* of equation (1) is said to be *attractive*, if for any $n_0 \in N$, there exists a $\delta(\varepsilon, n_0) > 0$, such that when $\|x(n_0 + j)| < \delta$, $j \in N(-r, 0)$, the solution of equation (1) satisfies $\lim\limits_{n \to +\infty} x(n) = 0$, then the zero solution of equation (1) is said to be attractive. If δ can be chosen independent of n_0, the zero solution of equation (1) is said to be uniformly attractive.

Definition 1.4 The *zero solution* of equation (1) is said to be *uniformly asymptotically stable*, if its zero solution is uniformly stable and uniformly attractive.

Let

$$n - \alpha(n) = \min\{n - l_i(n)\colon x(n - l_i(n)) = \max\{x(n - l_1(n)), \dots, x(n - l_m(n))\}\}, \quad (2)$$
$$n - \beta(n) = \min\{n - l_i(n)\colon x(n - l_i(n)) = \min\{x(n - l_1(n)), \dots, x(n - l_m(n))\}\}, \quad (3)$$

S is a real number sequence, for any $x = \{x(1), \dots, x(n), \dots\} \in S$, let $\|x\| = \sup\{|x(i)|\}$, for a given $H > 0$, denote

$$S_H = \{x \in S\colon\ \|x\| < H\}. \quad (4)$$

If $m > n$, we assume that $C_n^m = 0$.

2 Main Results and Proofs

Theorem 2.1 *Suppose that there exists a nonnegative real number sequence $\{p(n)\}$, such that*

(1) *for positive constant H and any $x \in S_H$, when $n \in N$, we have*

$$p(n)x(n-\beta(n)) \le f(n,\, x(n-l_1(n)), \dots x(n-l_m(n))) \le p(n)x(n-\alpha(n)); \tag{5}$$

(2) *the following inequalities are satisfied*

$$2|c|(2-|c|) + \sum_{i=n-\alpha(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N, \tag{6}$$

$$2|c|(2-|c|) + \sum_{i=n-\beta(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N. \tag{7}$$

Then the zero solution of equation (1) is uniformly stable.

Theorem 2.2 *Suppose that there exists a nonnegative real number sequence $\{p(n)\}$, such that*

(1) *for positive constant H and any $x \in S_H$, when $n \in N$, we have*

$$p(n)x(n-\beta(n)) \le f(n, x(n-l_1(n)), \dots, x(n-l_m(n))) \le p(n)x(n-\alpha(n)); \tag{8}$$

(2)
$$\sum_{n=1}^{+\infty} p(n) = +\infty; \tag{9}$$

(3)
$$2|c|(2-|c|) + \sum_{i=n-\alpha(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N, \tag{10}$$

$$2|c|(2-|c|) + \sum_{i=n-\beta(n)}^{n} p(i) < \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)}, \quad n \in N. \tag{11}$$

Then the zero solution of equation (1) is uniformly asymptotically stable.

Proof of Theorem 2.1 For any $0 < \varepsilon < H$, $n_0 \in N$, there is a $\delta > 0$, when the solution $\{x(n)\}$ to the equation satisfies $|x(n_0+i)| < \delta$, $i = -r, -r+1, \dots, 0$, we get

$$|x(n)| < \varepsilon, \quad n \in N(n_0). \tag{12}$$

We select
$$\delta = \frac{(1-|c|)}{(1+|c|)(2|c|+3)^{3r}}\,\varepsilon.$$

In the following, we will prove that when $n \in N(n_0+1, n_0+3r)$, (12) holds. In fact, from(1), we can see that

$$\begin{aligned} |x(n_0+1)| = |&cx(n_0+1-k) - cx(n_0-k) + x(n_0) \\ &- f(n_0, x(n_0-l_1(n_0)), \dots, x(n_0-l_m(n_0)))| \\ &< (1+2|c|+p(n_0))\delta \le (2|c|+3)\delta < \varepsilon < H. \end{aligned}$$

Generally, when $i \in N(1, 3r)$, we have

$$|x(n_0 + i)| < (2|c| + 3)^i \delta < \varepsilon < H. \tag{13}$$

In the following, we will prove that when $n \in N(n_0 + 3r + 1)$, (12) holds. In fact, otherwise, there must be a $n_1 \in N(n_0 + 3r + 1)$ such that $|x(n_1)| \geq \varepsilon$ and when $n \in N(n_0, n_1 - 1)$, such that

$$|x(n)| < \varepsilon. \tag{14}$$

Suppose $x(n_1) > 0$, we then have $x(n_1) \geq \varepsilon$. Let

$$y(n) = x(n) - cx(n - k), \quad n \in N(n_0), \tag{15}$$

then

$$y(n_1) = x(n_1) - cx(n_1 - k) \geq (1 - |c|)\varepsilon. \tag{16}$$

Because

$$y(n_0 + 3r) \leq |x(n_0 + 3r)| + |c||x(n_0 + 3r - k)| < (1 + |c|)(2|c| + 3)^{3r}\delta = (1 - |c|)\varepsilon$$

then there is a $n^* \in N(n_0 + 3r + 1, n_1)$, such that

$$\begin{aligned} y(n^* - 1) &< (1 - |c|)\varepsilon, \\ y(n^*) &\geq (1 - |c|)\varepsilon, \end{aligned} \tag{17}$$

and when $n \in N(n^* + 1, n_1)$, we have $y(n) \geq (1 - |c|)\varepsilon$, thus we get

$$\Delta y(n^* - 1) > 0. \tag{18}$$

From (6) we can see that $|c| < \dfrac{1}{2}$, such that

$$x(n^*) = y(n^*) + cx(n^* - k) \geq y(n^*) - |c|\varepsilon \geq (1 - 2|c|)\varepsilon. \tag{19}$$

From (5) and (18) we can see that

$$\begin{aligned} &p(n^* - 1)x(n^* - 1 - \beta(n^* - 1)) \\ &\quad \leq f(n^* - 1, x(n^* - 1 - l_1(n^* - 1)), \ldots, x(n^* - 1 - l_m(n^* - 1))) = -\Delta y(n^* - 1) < 0, \end{aligned}$$

then we have

$$x(n^* - 1 - \beta(n^* - 1)) < 0. \tag{20}$$

Therefore from (19) and (20) we can see that there exists $n_2 \in N(n^* - \beta(n^* - 1), n^*)$ and $\xi \in (0, 1)$, such that $x(n_2 - 1) < 0$. And when $n \in N(n_2, n^*)$, we have

$$x(n) > 0, \tag{21}$$

$$x(n_2 - 1) + \xi(x(n_2 - x(n_2 - 1)) = 0, \tag{22}$$

then from (22) and (15), we get

$$-[y(n_2 - 1) + \xi(y(n_2 - y(n_2 - 1))] = -[(1 - \xi)x(n_2 - k - 1) + \xi x(n_2 - k)]c \leq |c|\varepsilon \tag{23}$$

and

$$[y(n_2-1)+\xi(y(n_2-y(n_2-1)] = [(1-\xi)x(n_2-k-1)+\xi x(n_2-k)]c \le |c|\varepsilon$$

that is

$$y(n_2-1) \le |c|\varepsilon - \xi(y(n_2-y(n_2-1)). \tag{24}$$

In the following we will prove that when $n \in N(n_0+r,\, n^*-1)$, we have

$$-x(n) \le \left(2|c| + \sum_{i=n}^{n_2-1} p(i) + \xi p(n_2-1)\right)\varepsilon. \tag{25}$$

In fact, from (21) we can see that when $n \in N(n_2,\, n^*-1)$, obviously the above inequality holds.

In the following we will prove that when $n \in N(n_0+r,\, n_2-1)$, inequality (25) holds.

From (5) we can see that when $n \in N(n_0+r)$, we have

$$\Delta y(n) \le -p(n)x(n-\beta(n)), \tag{26}$$

thus when $n \in N(n_0+r,\, n_2-1)$, we get

$$\Delta y(n) \le p(n)\varepsilon. \tag{27}$$

Then when $n \in N(n_0+r,\, n_2-1)$, we have

$$\begin{aligned} &-[y(n)-y(n_2-1)-\xi(y(n_2)-y(n_2-1))] \\ &\qquad = \sum_{i=n}^{n_2-2} \Delta y(i) + \xi \Delta y(n_2-1) \le \left(\sum_{i=n}^{n_2-2} p(i) + \xi p(n_2-1)\right)\xi. \end{aligned}$$

From (14) and (15), when $n \in N(n_0+r,\, n_2-1)$, we have

$$\begin{aligned} -x(n) &= -(y(n)+cx(n-k)) = -[y(n)-y(n_2-1)-\xi(y(n_2) \\ &\quad - y(n_2-1))] - [y(n_2-1)+\xi(y(n_2)-y(n_2-1))] - cx(n-k) \\ &\le \left[\sum_{i=n}^{n_2-2} p(i) + \xi p(n_2-1)\right]\varepsilon + 2|c|\varepsilon. \end{aligned}$$

Therefore, inequality (25) holds.

Suppose

$$\beta = \frac{2}{3} + \frac{(1-2|c|)^2}{2(l+1)} - 2|c(2-|c|)|. \tag{28}$$

Then from (7), we have

$$\sum_{i=n-\beta(n)}^{n} p(i) < \beta, \quad n \in N. \tag{29}$$

Let

$$d = \sum_{i=n_2}^{n^*-1} p(i) + (1-\xi)p(n_2-1). \tag{30}$$

There are two situations needed to be contemplated.

Case 1 $d \le 1 - 2|c|$.

From (24), (25) and (26), we can see that

$$
\begin{aligned}
y(n^*) &= y(n_2-1) + \sum_{n=n_2-1}^{n^*-1} \Delta y(n) \le |c|\varepsilon - \xi(y(n_2) - y(n_2-1)) + \sum_{n=n_2-1}^{n^*-1} \Delta y(n) \\
&= |c|\varepsilon + (1-\xi)\Delta y(n_2-1) + \sum_{n=n_2}^{n^*-1} \Delta y(n) \le |c|\varepsilon - (1-\xi)p(n_2-1) \\
&\times x(n_2-1-\beta(n_2-1)) - \sum_{n=n_2}^{n^*-1} p(n)x(n-\beta(n)) \\
&\le |c|\varepsilon + (1-\xi)p(n_2-1)\Bigg[\sum_{i=n_2-1-\beta(n_2-1)}^{n_2-2} p(i) + \xi p(n_2-1) + 2|c|\Bigg]\varepsilon \\
&+ \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2-1) + 2|c|\Bigg].\varepsilon
\end{aligned}
$$

From (29) we get

$$
\begin{aligned}
y(n^*) &< |c|\varepsilon + (1-\xi)p(n_2-1)[\beta - (1-\xi)p(n_2-1) + 2|c|]\varepsilon \\
&\quad + \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\sum_{i=n-\beta(n)}^{n} p(i) \sum_{i=n_2}^{n} p(i) - (1-\xi)p(n_2-1) + 2|c|\Bigg]\varepsilon \\
&< |c|\varepsilon + (1-\xi)p(n_2-1)[\beta - (1-\xi)p(n_2-1) \\
&\quad + 2|c|]\varepsilon + \sum_{n=n_2}^{n^*-1} p(n)\Bigg[\beta - \sum_{i=n_2}^{n} p(i) - (1-\xi)p(n_2-1) + 2|c|\Bigg]\varepsilon.
\end{aligned}
$$

From (30), we have

$$
\begin{aligned}
y(n^*) &< |c|\varepsilon + \Bigg[(\beta+2|c|)d - (1-\xi)^2p^2(n_2-1) - \sum_{n=n_2}^{n^*-1} p(n) \sum_{i=n_2}^{n} p(i) \\
&- (1-\xi)p(n_2-1)\sum_{n=n_2}^{n^*-1} p(n)\Bigg]\varepsilon = |c|\varepsilon + \Bigg[(\beta+2|c|)d - (1-\xi)^2p^2(n_2-1) \\
&- \frac{1}{2}\Bigg(\sum_{n=n_2}^{n^*-1} p(n)\Bigg)^2 - \frac{1}{2}\sum_{n=n_2}^{n^*-1} p^2(n) - (1-\xi)p(n_2-1)\sum_{n=n_2}^{n^*-1} p(n)\Bigg]\varepsilon.
\end{aligned}
$$

Because

$$
\begin{aligned}
\sum_{n=n_2}^{n^*-1} p(n)^2 + (1-\xi)^2p^2(n_2-1) &\ge \frac{1}{n^*-n_2+1}\Bigg(\sum_{n=n_2}^{n^*-1} p(n) + (1-\xi)p(n_2-1)\Bigg)^2 \\
&= \frac{1}{n^*-n_2+1}d^2 \ge \frac{1}{l+1}d^2
\end{aligned}
$$

we have

$$y(n^*) < \left[|c| + (\beta + 2|c|)d - \left(\frac{1}{2} + \frac{1}{2(l+1)}\right)d^2\right]\varepsilon. \tag{31}$$

Because the function $g(x) = |c| + (2|c| + \beta)x - \dfrac{l+2}{2(l+1)}x^2$ is monotonously increasing on the interval $[0,\, 1-2|c|]$, then we have

$$y(n^*) < \left[|c| + (\beta + 2|c|)(1-2|c|) - \left(\frac{1}{2} + \frac{1}{2(l+1)}\right)(1-2|c|)^2\right]\varepsilon$$
$$\le [1 - |c| - |c|(1-2|c|^2] \le (1-|c|)\varepsilon$$

which contradicts inequality (17). Therefore, Case 1 is impossible.

Case 2 $d > 1 - 2|c|$.

In this case there exists a positive integer $n_3 \in N(n_2, n^*)$, which satisfies

$$2|c| + \sum_{n=n_3}^{n^*-1} p(n) \le 1 \quad \text{and} \quad 2|c| + \sum_{n=n_3-1}^{n^*-1} p(n) > 1,$$

then there is a $\eta \in (0,1]$, such that

$$2|c| + \sum_{n=n_3}^{n^*-1} p(n) + (1-\eta)p(n_3-1) = 1. \tag{32}$$

Because

$$y(n^*) = y(n_2-1) + \sum_{n=n_2-1}^{n_3-2} \Delta y(n) + \eta\Delta y(n_3-1) + (1-\eta)\Delta y(n_3-1) + \sum_{n=n_3}^{n^*-1} \Delta y(n)$$

and making use of (24), we get

$$y(n^*) \le |c|\varepsilon + \eta\Delta y(n_3-1) + (1-\xi)\Delta y(n_2-1) + \sum_{n=n_2}^{n_3-2} \Delta y(n) + (1-\eta)\Delta y(n_3-1) + \sum_{n=n_3}^{n^*-1} \Delta y(n).$$

From (27), we get

$$\eta\Delta y(n_3-1) + (1-\xi)\Delta y(n_2-1) + \sum_{n=n_2}^{n_3-2} \Delta y(n) < \left[(1-\xi)p(n_2-1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3-1)\right]\varepsilon$$

and from (25) and (26), we have

$$(1-\eta)\Delta y(n_3-1) + \sum_{n=n_3}^{n^*-1} \Delta y(n) \le (1-\eta)p(n_3-1)\Bigg[2|c|$$
$$+ \sum_{i=n_3-1-\beta(n_3-1)}^{n_2-2} p(i) + \xi p(n_2-1)\Bigg]\varepsilon + \sum_{n=n_3}^{n^*-1} p(n)\left[2|c| + \sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2-1)\right]\varepsilon.$$

We then have

$$\begin{aligned}
y(n^*) \leq |c|\varepsilon &+ \Big[(1-\xi)p(n_2-1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3-1)\Big]\varepsilon \\
&+ (1-\eta)p(n_3-1)\Big[2|c| + \sum_{i=n_3-1-\beta(n_3-1)}^{n_2-2} p(i) + \xi p(n_2-1)\Big]\varepsilon \\
&+ \sum_{n=n_3}^{n^*-1} p(n)\Big[2|c| + \sum_{i=n-\beta(n)}^{n_2-2} p(i) + \xi p(n_2-1)\Big]\varepsilon.
\end{aligned}$$

From (29) and (32), we get

$$\begin{aligned}
y(n^*) &\leq |c|\varepsilon + \Big[(1-\xi)p(n_2-1) + \sum_{n=n_2}^{n_3-2} p(n) + \eta p(n_3-1)\Big]\varepsilon \\
&\quad + (1-2|c|)[2|c| - (1-\xi)p(n_2-1)]\varepsilon + (1-\eta)p(n_3-1)\Big[\sum_{\substack{i=n_3-1-\\ \beta(n_3-1)}}^{n_3-1} p(i) - \sum_{i=n_2}^{n_3-1} p(i)\Big]\varepsilon \\
&\quad + \sum_{n=n_3}^{n^*-1} p(n)\Big[\sum_{i=n-\beta(n)}^{n} p(i) - \sum_{i=n_3}^{n} p(i) - \sum_{i=n_2}^{n_3-1} p(i)\Big]\varepsilon \\
&\quad < |c|\varepsilon + 2|c|(1-2|c|)\varepsilon + 2|c|(1-\xi)p(n_2-1)\varepsilon + 2|c|\sum_{i=n_2}^{n_3-1} p(i)\varepsilon \\
&\quad - \varepsilon(1-\eta)p(n_3-1) + \varepsilon(1-\eta)\beta p(n_3-1) + \sum_{n=n_3}^{n^*-1} p(n)\Big[\beta - \sum_{i=n_3}^{n} p(i)\Big]\varepsilon \\
&\quad = |c|\varepsilon + 2|c|(1-2|c|)\varepsilon - 2|c|\xi p(n_2-1)\varepsilon + 2|c|\Big[\sum_{i=n_2-1}^{n^*-1} p(i) - \sum_{i=n_3}^{n^*-1} p(i)\Big]\varepsilon \\
&\quad - \varepsilon(1-\eta)p(n_3-1) + (1-2|c|)\beta\varepsilon - \frac{1}{2}\Big[\sum_{i=n_3}^{n^*-1} p(i)\Big]^2\varepsilon - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i)\varepsilon.
\end{aligned}$$

Because

$$\begin{aligned}
&-2|c|\sum_{i=n_3}^{n^*-1} p(i) - (1-\eta)p(n_3-1) - \frac{1}{2}\Big[\sum_{i=n_3}^{n^*-1} p(i)\Big]^2 - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i) \\
&\quad = -2|c|(1-2|c| - (1-\eta)p(n_3-1)) - (1-\eta)p(n_3-1) \\
&\quad - \frac{1}{2}[1-2|c| - (1-\eta)p(n_3-1)]^2 - \frac{1}{2}\sum_{i=n_3}^{n^*-1} p^2(i) \\
&\quad = -2|c|(1-2|c|) - \frac{1}{2}(1-2|c|)^2 - \frac{1}{2}\Big[\sum_{i=n_3}^{n^*-1} p^2(i) + (1-\eta)^2 p^2(n_3-1)\Big]
\end{aligned}$$

and

$$\sum_{i=n_3}^{n^*-1} p^2(i) + (1-\eta)^2 p^2(n_3-1) \geq \frac{1}{n^*-n_3}\Big[\sum_{i=n_3}^{n^*-1} p(i) + (1-\eta)p(n_3-1)\Big]^2$$
$$\geq \frac{1}{l+1}\Big[\sum_{i=n_3}^{n^*-1} p(i) + (1-\eta)p(n_3-1)\Big]^2$$

we get

$$y(n^*) < \varepsilon[|c| + 2|c|(1-2|c|) + 2|c|\beta + (1-2|c|)\beta - 2|c|(1-2|c|) - \frac{1}{2}(1-2|c|)^2 - \frac{1}{2}\frac{(1-2|c|)^2}{(l+1)}] = (1-|c|)\varepsilon.$$

This inequality contradicts (17). Therefore Case 2 is also impossible.

Based on the above two cases, we see that (12) holds. Hence the zero solution of equation (1) is uniformly stable.

Proof of Theorem 2.2 From Theorem 2.1, we see that the zero solution of equation (1) is uniformly stable, thus we only need to prove that the zero solution of equation (1) is uniformly attractive.

Select

$$\delta = \frac{(1-|c|)}{(1+|c|)(2|c|+3)^{3r}}\, H.$$

In the following, we prove that for any $n_0 \in N$, if the solution $\{x(n)\}$ of the equation satisfies $|x(n_0+i)| < \delta$, $i = -r, -r+1, \dots, 0$, we have

$$\lim_{n\to+\infty} x(n) = 0. \tag{33}$$

The following proof is similar to that of Theorem 2.1, so we have

$$x(n)| < H, \quad n \in N(n_0). \tag{34}$$

Let

$$y(n) = x(n) - cx(n-k), \quad n \in N(n_0), \tag{35}$$

then

$$|y(n)| < (1+|c|)H, \quad n \in N(n_0). \tag{36}$$

There are two situations that needed to be contemplated.

Case 1 $\{y(n)\}$ is eventually monotonous.

Let

$$A = \lim_{n\to+\infty} \inf x(n), \quad B = \lim_{n\to+\infty} \sup x(n). \tag{37}$$

We will prove that $A = B = 0$ and $A \leq 0$.

In fact, if $A > 0$, then for any $0 < \varepsilon < A$, there is $n_1 \in N(n_0+l)$, such that

$$x(n_1 - l) > A - \varepsilon > 0.$$

Hence, when $n \in N(n_1 - l)$, we have

$$x(n) > A - \varepsilon. \tag{38}$$

Therefore from (35), we get

$$\begin{aligned} y(n_1) - y(n_1+1) &= f(n_1, x(n_1 - l_1(n)), \ldots, x(n_1 - l_m(n))) \\ &\geq p(n_1)x(n_1 - \beta(n_1)) > p(n_1)(A - \varepsilon). \end{aligned}$$

In general, for $m = 0, 1, \ldots,$ we have

$$y(n_1 + m) - y(n_1 + m + 1) > p(n_1 + m)(A - \varepsilon).$$

Then we have

$$y(n_1) - y(n_1 + m + 1) > \sum_{i=0}^{m} p(n_1 + i)(A - \varepsilon).$$

From (36) and $\{y(n)\}$ being eventually monotonous, we can see that the limit value of $\{y(n)\}$ exists. Therefore from (9), we know that the above inequality doesn't hold and hence $A \leq 0$.

In the following we will prove $A = 0$. Suppose

$$\lim_{n \to +\infty} y(n) = y^*.$$

We will prove that

$$y^* = 0. \tag{39}$$

In fact, if (39) doesn't hold, we assume that $y^* > 0$, from the definition of A. We can see that there is a positive integer sequence $\{n_j\}$, such that

$$\lim_{j \to +\infty} n_j = +\infty, \quad \lim_{n \to +\infty} x(n_j) = A,$$

then when $j \to +\infty$, we have

$$cx(n_j - k) = x(n_j) - y(n_j) \to A - y^*, \tag{40}$$

and since

$$\lim_{n \to +\infty} f(n_1, x(n_1 - l_1(n), \ldots, x(n_1 - l_m(n))) = \lim_{n \to +\infty} (-\Delta y(n)) = 0 \tag{41}$$

from(40), we see that there must exist $c \neq 0$.

If $c = 0$, we must have

$$\lim_{j \to +\infty} cx(n_j - k) = 0 = A - y^*$$

that is $A = y^*$, which obviously doesn't hold.

Hence

$$\lim_{j \to +\infty} cx(n_j - k) = \frac{A - y^*}{c}. \tag{42}$$

From the definitions of A and B, we see that

$$A = \lim_{n\to+\infty} \inf x(n) \le \lim_{j\to+\infty} x(n_j - k) = \frac{A - y^*}{c} \le \lim_{n\to+\infty} \sup x(n) = B. \qquad (43)$$

If $c > 0$, from $A < (A - y^*)/c$ we have $(1 - c)A > y^*$, then we see that the inequality doesn't hold.

If $c < 0$, from $x(n) = y(n) + cx(n - k), n \in N(n_0)$, we get

$$\lim_{n\to+\infty} \sup x(n) = \lim_{n\to+\infty} \sup(y(n) + cx(n - k)),$$

then $B = y^* + cB$.

From (43), we can see that $cA \ge cB$, then we have $B \le y^* + cA$. Since $B \ge (A - y^*)/c$, we have $(1 + c)y^* \le (1 - c^2)A$ which can not hold. Therefore (39) must hold. Hence, $A = y^* + cA = cA$, that is $(1 - c)A = 0$ or $A = 0$.

In the following we will prove $B = 0$.

In fact, according to the definition of B, we can see that there is a positive integer sequence $\{l_j\}$, such that

$$\lim_{j\to+\infty} l_j = +\infty \quad \text{and} \quad \lim_{j\to+\infty} x(l_j) = B.$$

If $c = 0$, obviously, we get $B = 0$. If $c < 0$, while $j \to +\infty$, we get

$$y(l_j) - y(l_j - k) = x(l_j) - (1 + c)x(l_j - k) + cx(l_j - 2k) \to 0,$$

then for $j \to +\infty$, we have

$$(1 + c)x(l_j - k) - cx(l_j - k) \to B.$$

Since the line $(1 + c)x - cy = B$, $c > 0$ and the region $0 \le x, y \le B$ only have one crossover point (B, B), so

$$\lim_{j\to+\infty} x(l_j - k) = \lim_{j\to+\infty} x(l_j - 2k) = B.$$

Therefore

$$\lim_{j\to+\infty} y(l_j - k) = (1 - c)B = 0,$$

that is $B = 0$.

If $c > 0$, we can similarly prove that $B = 0$.

In conclusion, if $\{y(n)\}$ is eventually monotonous, then

$$\lim_{n\to+\infty} \inf x(n) = \lim_{n\to+\infty} \sup x(n) = 0,$$

that is

$$\lim_{n\to+\infty} x(n) = 0.$$

Case 2 $\{y(n)\}$ is not eventually monotonous. Let

$$M = \lim_{n\to+\infty} \sup |x(n)|, \quad N = \lim_{n\to+\infty} \sup |y(n)|.$$

If (33) doesn't hold, we must have $M > 0$. Then for any $\varepsilon > 0$, and ε satisfies $\varepsilon < \dfrac{1-2|c|}{1+|c|}M$, $\varepsilon < I$, there must exist a $n_2 \in N(n_0+r)$, such that when $n \in N(n_2-r)$, we have

$$|x(n)| < M + \varepsilon. \tag{44}$$

Therefore, when $n \in N(n_2)$, we get

$$y(n) \geq |x(n)| - |c|(M+\varepsilon_1) \tag{45}$$

and we have $I \geq M - |c|(M+\varepsilon_1)$. Because of the arbitrariness of ε, we have

$$I \geq (1-|c|)M. \tag{46}$$

Since $\{y(n)\}$ is not eventually monotonous, for the above ε, there must exist a $n^* \in N(n_2+2r+1)$, which satisfies

$$y(n^*) > I - \varepsilon, \tag{47}$$

such that

$$y(n^*) > y(n^*+1), \quad y(n^*) \geq y(n^*-1). \tag{48}$$

Therefore, we have

$$x(n^*) = y(n^*) - cx(n^*-k) \geq I - \varepsilon - |c|(M+\varepsilon) \geq (1-|c|)M - \varepsilon - |c|(M+\varepsilon) > 0$$

and

$$x(n^*-1-\beta(n^*-1)) \leq 0. \tag{49}$$

Thus there must be a $n_3 \in N(n^* - \beta(n^*-1), n^*)$ and a $\xi \in [0,1)$, such that

$$x(n_3-1) \leq 0, \quad x(n) > 0, \quad \text{where} \quad n \in N(n_3, n^*), \tag{50}$$

$$x(n_3-1) + \xi(x(n_3) - x(n_3-1)) = 0. \tag{51}$$

Then from (35) and (44), we have

$$\begin{gathered} -[y(n_3-1) + (y(n_3) - y(n_3-1))] = c[(1-\xi)x(n_3-1-k) + \xi x(n_3-1)] < |c|(M+\varepsilon), \\ y(n_3-1) + \xi(y(n_3) - y(n_3-1)) = -c[(1-\xi)x(n_3-1-k) + \xi x(n_3-1)] < |c|(M+\varepsilon). \end{gathered} \tag{52}$$

That is

$$y(n_3-1) < |c|(M+\varepsilon) - \xi \Delta y(n_3-1). \tag{53}$$

Now we will prove that, when $n \in N(n_2, n^*)$, we have

$$-x(n) \leq \left[2|c| + \sum_{i=n}^{n_3-2} p(i) + \xi p(n_3-1)\right](M+\varepsilon). \tag{54}$$

In fact, when $n \in N(n_3, n^*)$, from (50) we can see that the above equality holds. In the following, we will prove that, when $n \in N(n_2,\, n_3-1)$, (54) holds. From (1) and (6), we see that, when $n \in N(n_2)$, we have

$$\Delta y(n) < -p(n)x(n-\beta(n)). \tag{55}$$

Thus, from (44) we see that, when $n \in N(n_2)$, we have

$$\Delta y(n) < p(n)(M+\varepsilon). \tag{56}$$

Therefore, for any $n \in N(n_2, n_3-1)$, we have

$$\begin{aligned}
-[y(n) - y(n_3-1) - \xi(y(n_3) - y(n_3-1))] &= \sum_{i=n}^{n_3-2} \Delta y(i) + \xi \Delta y(n_3-1) \\
&< -\sum_{i=n}^{n_3-2} p(i)x(i-\beta(i)) - \xi p(n_3-1)x(n_3-1-\beta(n_3-1)) \\
&\le \left[\sum_{i=n}^{n_3-2} p(i) + \xi p(n_3-1)\right](M+\varepsilon).
\end{aligned} \tag{57}$$

Then from (35), (44), (52) and (57), we know that if $n \in N(n_2)$, we have

$$\begin{aligned}
-x(n) = -(y(n) + cx(n-k)) &= -[y(n) - y(n_3-1) + \xi(y(n_3) - y(n_3-1))] \\
&\quad - y(n_3-1) - \xi(y(n_3) - y(n_3-1)) - cx(n-k) \\
&\le \left[\sum_{i=n}^{n_3-2} p(i) + \xi p(n_3-1) + 2|c|\right](M+\varepsilon).
\end{aligned}$$

Therefore (54) holds.

Suppose

$$\beta = \frac{3}{2} + \frac{(1-2|c|)^2}{2(l+1)} - 2|c|(2-|c|).$$

Then from (11), we have

$$\sum_{i=n-\beta(n)}^{n} p(i) \le \beta, \quad n \in N. \tag{58}$$

Denote

$$d = \sum_{i=n_3}^{n^*-1} p(i) + (1-\xi)p(n_3-1). \tag{59}$$

In the following, we have two situations to contemplate.

Case 2-a $d \le 1-2|c|$.

From (53), we obtain

$$y(n^*) = y(n_3-1) + \sum_{i=n_3-1}^{n^*-1} \Delta y(i) \le |c|(M+\varepsilon) - \xi\Delta y(n_3-1) + \sum_{i=n_3-1}^{n^*-1} \Delta y(i).$$

From (54) and (55), we get

$$\begin{aligned} y(n^*) &\le |c|(M+\varepsilon) + (1-\xi)\Delta y(n_3-1) + \sum_{i=n_3}^{n^*-1} \Delta y(i) \\ &< |c|(M+\varepsilon) + (1-\xi)p(n_3-1)\Bigg[\sum_{\substack{i=n_3-1-\\ \beta(n_3-1)}}^{n_3-2} p(i) + \xi p(n_3-1) + 2|c|\Bigg](M+\varepsilon) \\ &+ \sum_{i=n_3}^{n^*-1} p(i)\Bigg[\sum_{j=i-\beta(i)}^{n_3-2} p(j) + \xi p(n_3-1) + 2|c|\Bigg](M+\varepsilon). \end{aligned}$$

The following proof is similar to Case 1 of Theorem 2.1, we have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Case 2-b $d > 1-2|c|$.

Now there exists a positive integer $n_4 \in N(n_3, n^*)$, such that

$$2|c| + \sum_{i=n_4}^{n^*-1} p(i) < 1 \quad \text{and} \quad 2|c| + \sum_{i=n_4-1}^{n^*-1} p(i) > 1.$$

Therefore there is a $\eta \in (0,1)$, such that

$$2|c| + \sum_{i=n_4}^{n^*-1} p(i) + (1-\eta)p(n_4-1) = 1. \tag{60}$$

Since

$$y(n^*) = y(n_3+1) + \sum_{n=n_4-1}^{n4-2} \Delta y(n) + \eta\Delta y(n_4-1) + (1-\eta)\Delta y(n_4-1) + \sum_{n=n_4}^{n^*-1} \Delta y(n),$$

then from (53), (54) and (56), we obtain

$$\begin{aligned} y(n^*) &< |c|(M+\varepsilon)(1-\xi)\Delta y(n_3-1) + \sum_{n=n_3-2}^{n_4-2} \Delta y(n) + \eta\Delta y(n_4-1) \\ &+ (1+\eta)\Delta y(n_4-1) + \sum_{n=n_4}^{n^*-1} \Delta y(n) \\ &\le |c|(M+\varepsilon) + \Bigg[(1-\xi)p(n_3-1) + \sum_{n=n_3-2}^{n_4-2} p(n) + \eta p(n_4-1)\Bigg](M+\varepsilon) \\ &+ (1-\eta)p(n_4-1)\Bigg[2|c| + \sum_{i=n_4-1-\beta(n_4-1)}^{n_3-2} p(i) + \xi p(n_3-1)\Bigg](M+\varepsilon) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=n_4}^{n^*-1} p(n)\Big[2|c|+\sum_{i=n-\beta(n)}^{n_3-2} p(i)+\xi p(n_3-1)\Big](M+\varepsilon)\\
&=|c|(M+\varepsilon)+\Big[(1-\xi)p(n_3-1)+\sum_{n=n_3-2}^{n_4-2} p(n)+\eta p(n_4-1)\Big](M+\varepsilon)\\
&+(1-2|c|)[2|c|-(1-\xi)p(n_3-1)](M+\varepsilon)\\
&+(1-\eta)p(n_4-1)\Big[\sum_{i=n_4-1-\beta(n_4-1)}^{n_4-1} p(i)-\sum_{i=n_3}^{n_4-1} p(i)\Big](M+\varepsilon)\\
&+\sum_{n=n_4}^{n^*-1} p(n)\Big[\sum_{i=n-\beta(n)}^{n} p(i)-\sum_{i=n_4}^{n} p(i)-\sum_{i=n_3}^{n_4-1} p(i)\Big](M+\varepsilon).
\end{aligned}
$$

The following proof is similar to Theorem 2.1. We have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Based on the two cases a and b, we have

$$y(n^*) < (1-|c|)(M+\varepsilon).$$

Hence, from (47), we have

$$I-\varepsilon < y(n^*) < (1-|c|)(M+\varepsilon).$$

From the arbitrariness of ε, we have

$$I < (1-|c|)M,$$

which contradicts (46). Therefore Case 2 is impossible. Thus when $\{y(n)\}$ is not eventually monotonous, (33) also holds.

Based on these two cases, we can see that (33) must hold. Thus the zero solution of the equation is uniformly attractive. Therefore the zero solution of equation (1) is said to be uniformly asymptotically stable.

3 Conclusions

According to the above analysis, in the cases of several variable delay, we have obtained the sufficient conditions of uniform stability and uniform asymptotical stability. These results extent and improve the relative theorem in the literature [7]. And the methods used in this paper can have important significances in the studies of the stabilities of difference equation with several variable delays.

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