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An Analysis of Clattering Impacts of a Falling Rod

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Abstract: This paper deals with both analytical and quantitative analysis of multiple impacts of a two-dimensional rod. The successions of clattering sequence of a rod dropping to the floor are modeled and analyzed to find out the impact responses as it collides with the ground. The model is described by a system of ordinary differential equations, with a classical contact problem. We conduct a comparison study of the cases where the effect of the gravity is neglected, versus the cases where the gravity is considered. This mathematical analysis can further provide useful information for durability study of the impact on mobile electronic device.

Keywords: *Two-dimensional rod; clattering impacts; analytical and quantitative analysis.*

Mathematics Subject Classification (2000): 70E18, 70F40, 70B10, 70G10.

1 Introduction

In a pioneering study of Goyal, *et al.* [1, 2], it was found that when a two-dimensional rod was dropped at a small angle to the ground, the second impact might be as large as twice of the initial impact under some assumptions. For its consequence in applications, their surprising result stirred some interest on this otherwise classical problem.

In the related literature, mathematical issues of one impact or first impact have been considered in a number of papers, see for example, [3 − 5] for rigid body collisions. Even in single-impact cases, the topic remains a focus of much discussion [6 − 8] as many theoretical contact dynamics issues involving frictions started to get resolved recently. Recent attention has been directed to detect and calculate the micro-collisions that occur

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in a short time interval, when the bodies are allowed to be flexible [9, 10]. These micro-collisions are consequence of the elastic oscillations during one impact, and occur in a relatively short period of time. During the sequence of micro-collisions, the location and posture of the bodies change very little.

The study of multiple-impacts, however, is only an emerging area. Goyal, *et al.* (1998) [1, 2] used transition matrix method to calculate the clattering sequence and its impacts. In a surprising way, they showed that when a two dimensional rod with uniform density is dropped to the ground at a very small angle, the second impact can be as large as twice of the first impact. Of course, this result is derived based on a number of assumption and simplifications such as full restitution and ignoring the effect of gravity, etc.

In this paper, we provide a study of the entire multiple-impact sequence of a two-dimensional rod with/without consideration of gravity, and using a general restitution coefficient. Our methodology allows us to consider a prototype problem for cell phone multi-impact dropping by several initial postures. We prove a number of assumptions required in Goyal's study are in fact valid, and interesting application is found in studying of clattering phenomenon of falling rigid bodies referred in [1, 2]. This model is a first step towards model study for the design and optimization of electronic components for mobile electronic product, future modeling considerations will involve flexible or multiple-body impacts.

We outline our article as follows. In Section 2, we state the basic rigid body dynamics equation. Section 3 includes impacts of analysis in absence of gravity. We give a comparison study to see the effect of gravity in Section 4. Discussion and conclusion are in Section 5.

2 Collision Equations for a Falling Rod

The model presented in this section is based on the linear impulse-momentum principle, the angular impulse-momentum principle for the rigid body, and some impact parameters that relates the pre- and post-impact variables, such as the coefficient of restitution, which is defined as the ratio of the post-impact relative normal velocity to the pre-impact relative normal velocity at the impact location. The limitation of the model is such that only sliding friction can occur. We assume that there is no sticking during the impact process. When sticking does occur, the situation becomes very complex. We defer discussion to Section 5.

We consider two rigid bodies having masses m_1 and m_2 respectively. We denote the initial velocities, before collision, in lower cases, and after collision, with capital letters. Collision equations are the following:

$$m_i\,(\overrightarrow{V}_i - \overrightarrow{v}_i) = \overrightarrow{P}_i, \quad i = 1, 2, \tag{1}$$

$$\overrightarrow{H}_i - \overrightarrow{h}_i = \overrightarrow{d}_i \times \overrightarrow{P}_i, \quad i = 1, 2, \tag{2}$$

where for body $i = 1, 2$, we denoted: m_i is the mass, $\overrightarrow{v}_i$ and $\overrightarrow{V}_i$ are the pre- and post-impact velocity, $\overrightarrow{P}_i$ is the impulse, $\overrightarrow{h}_i$ and $\overrightarrow{H}_i$ are the pre- and post- impact angular momentum, $\overrightarrow{d}_i$ is the position vector from the mass center to the collision contact point.

We can write:

$$\overrightarrow{P}_i = P_n\,(\overrightarrow{n} + \mu\,\overrightarrow{t}\,), \tag{3}$$

where μ is the sliding friction coefficient, $\overrightarrow{n}$ and $\overrightarrow{t}$ are the normal and tangential unit vectors of the contact surface.

The post-impact relative velocity $\overrightarrow{V}_r$ and pre-impact relative velocity $\overrightarrow{v}_r$ at the collision contact point are related by:

$$\overrightarrow{V}_r \cdot \overrightarrow{n} = -e\, \overrightarrow{v}_r \cdot \overrightarrow{n}, \tag{4}$$

where e is the coefficient of restitution.

Related to the center of mass, velocity and angular velocity, $\overrightarrow{V}_r$ and $\overrightarrow{v}_r$ can be written as:

$$\overrightarrow{V}_r = \overrightarrow{V}_1 + \overrightarrow{\Omega}_1 \times \overrightarrow{d}_1 - (\overrightarrow{V}_2 + \overrightarrow{\Omega}_2 \times \overrightarrow{d}_2), \tag{5}$$

$$\overrightarrow{v}_r = \overrightarrow{v}_1 + \overrightarrow{\omega}_1 \times \overrightarrow{d}_1 - (\overrightarrow{v}_2 + \overrightarrow{\omega}_2 \times \overrightarrow{d}_2), \tag{6}$$

where $\overrightarrow{\omega}_i$ and $\overrightarrow{\Omega}_i$ are the vectors of the pre- and post- impact angular velocities, respectively. For two-dimensional case, $\overrightarrow{\omega}_i = \omega_i \overrightarrow{k}$ and $\overrightarrow{\Omega}_i = \Omega_i \overrightarrow{k}$, where $\overrightarrow{k}$ is the unit vector normal to the two-dimensional work plane.

The equations (1)–(6) form a closed system. Solving the equations above, we derive (see [4] for example):

$$\begin{aligned}
V_{1n} &= v_{1n} + \frac{\overline{m}(1+e)q}{m_1} v_{rn}, & V_{1t} &= v_{1t} + \frac{\mu\overline{m}(1+e)q}{m_1} v_{rn},\\
V_{2n} &= v_{2n} - \frac{\overline{m}(1+e)q}{m_2} v_{rn}, & V_{2t} &= v_{2t} - \frac{\mu\overline{m}(1+e)q}{m_2} v_{rn},\\
\Omega_1 &= \omega_1 + \frac{\overline{m}(1+e)q(d_{1t} - \mu d_{1n})}{I_1} v_{rn}, & \Omega_2 &= \omega_2 - \frac{\overline{m}(1+e)q(d_{2t} - \mu d_{2n})}{I_2} v_{rn},
\end{aligned} \tag{7}$$

In the above solution, we denoted:

$$\overline{m} = \frac{m_1 m_2}{m_1 + m_2}, \qquad v_{rn} = (v_{2n} - d_{2t}\omega_2) - (v_{1n} + d_{1t}\omega_1),$$

$$q = \left[1 + \frac{\overline{m}d_{1t}^2}{I_1} + \frac{\overline{m}d_{2t}^2}{I_2} - \mu\left(\frac{\overline{m}d_{1t}d_{1n}}{I_1} + \frac{\overline{m}d_{2t}d_{2n}}{I_2}\right)\right]^{-1},$$

$$e = -\frac{V_{2n} - V_{1n}}{v_{2n} - v_{1n}}, \qquad \mu = \frac{P_t}{P_n}.$$

The formula for e is called the Newton's Law of Restitution. The value μ is the relative ratio of impulses (tangential over normal), and it reflects the friction coefficient, as long as no sticking is happening during the impact. The terms I_1 and I_2 represent the mass moment of inertia with respect to center of mass, for the two rigid bodies. The subscripts "n" and "t" in the equations (7) stand for the normal and tangential components of the velocity vector and the position vectors respectively. The Figure 2.1 shows the position vectors from the mass center to the collision contact point, $\overrightarrow{d}_1$ and $\overrightarrow{d}_2$, together with their normal and tangential components.

If a planar barrier collision occurs, for simplicity, let the moving body be the body 1 and the barrier be the body 2. All velocities related to body 2 are set to zero. The above approach is now applied to the multiple impacts of a falling rod, see Figure 2.2. In this

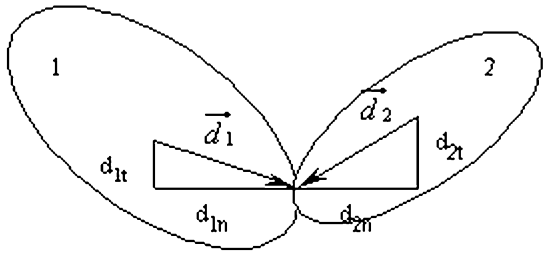

Figure 2.1. Rigid collision between two bodies.

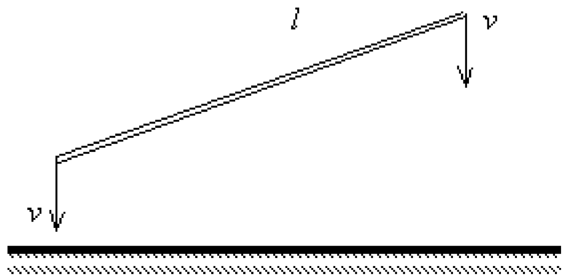

Figure 2.2. A rod colliding with the ground.

study, we consider a rod with uniform density. The mass of the rod $m_1 = 1$, the length of rod $l = 1$, the moment of inertia of the rod $I_1 = 1/12$, the friction coefficient $\mu = 0$, and the restitution coefficient $e \in [0, 1]$. The mass of the ground $m_2 = \infty$.

Hence, for our case, the equations (7) will reduce to

$$V_n = v_n + (1+e)qv_{rn}, \quad \Omega = \omega + 12(1+e)qd_tv_{rn}$$

with

$$q = \frac{1}{1+12d_t^2}\,, \quad v_{rn} = -(v_n + d_t\omega), \quad d_t - \mu d_n = d_t = -\frac{\cos\alpha}{2}\,.$$

We dropped the index $\{1, 2\}$ in the previous text because we will refer just to the normal and angular velocity of the rod relative to the ground. The tangential velocity remains zero at all the time. Further, we will be interested in the angle at the moment of the impact, and a qualitative estimation of the impact. We will be having the initial velocity v at the moment right before the first impact, as a unit.

3 The First Three Impacts, Disregarding the Effect of Gravity

We assume the impact sequence occurs without gravity. The clattering sequence terminates when the rod will no longer collide with the ground. The impact contact angles at the first three impacts are denoted as α, β and γ, as shown in Figure 3.1.

Following from the equations (7), for the first bounce, the quantities can be calculated as

$$V_n^I = \frac{e - 3\cos^2\alpha}{1+3\cos^2\alpha}\,v, \qquad \Omega^I = -\frac{6(1+e)\cos\alpha}{1+3\cos^2\alpha}\,v, \tag{8}$$

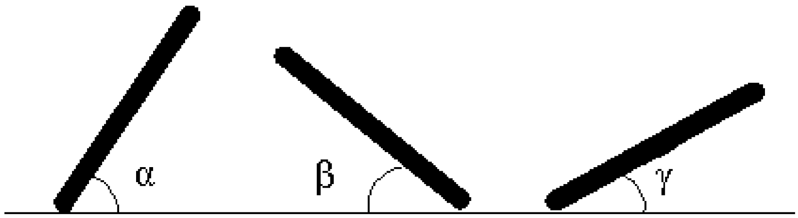

Figure 3.1. The succession of the first three impacts of the falling rod. The acute angle between the rod and the ground will be considered at all the times.

where α is the initial drop angle. Then the first impact is

$$P_n^I = V_n^I + v = \frac{1+e}{1+3\cos^2\alpha}\, v_{rn}^I, \tag{9}$$

where $v_{rn}^I = v$.

Let us consider h^I the vertical height of the rod's center of mass at first impact, and h^{II} the vertical height at the center of mass at second impact, without considering the gravity. These heights are related to the contact angles as

$$h^I = \frac{\sin\alpha}{2}\,, \qquad h^{II} = \frac{\sin\beta}{2}\,.$$

For the second impact, we have the equation

$$h^I + V_n^I T^I = h^{II},$$

where T^I is the duration of airborne. It can be analytically written as

$$T^I = \frac{-(\alpha+\beta)}{\Omega^I}\,.$$

We can determine the angle β numerically, for a given initial angle α, using the height relation, so that

$$\sin\alpha + \frac{e-3\cos^2\alpha}{3(1+e)\cos\alpha}\,(\alpha+\beta) = \sin\beta. \tag{10}$$

The new velocities for the second bounce are

$$V_n^{II} = V_n^I + \frac{1+e}{1+3\cos^2\beta}\, v_{rn}^{II}, \qquad \Omega^{II} = \Omega^I + \frac{6(1+e)\cos\beta}{1+3\cos^2\beta}\, v_{rn}^{II},$$

where

$$v_{rn}^{II} = -V_n^I - \frac{\cos\beta}{2}\,\Omega^I.$$

This gives the relation between the velocities of first two impacts

$$\begin{aligned} V_n^{II} &= \frac{-e+3\cos^2\beta}{1+3\cos^2\beta}\, V_n^I + \frac{-\frac{1+e}{2}\cos\beta}{1+3\cos^2\beta}\,\Omega^I, \\ \Omega^{II} &= \frac{-6(1+e)\cos\beta}{1+3\cos^2\beta}\, V_n^I + \frac{1-3e\cos^2\beta}{1+3\cos^2\beta}\,\Omega^I. \end{aligned} \tag{11}$$

Hence, by substituting equations (8) into equations (11), we derive

$$V_n^{II} = \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v,$$
$$\Omega^{II} = \frac{-6(1+e)(\cos\alpha+e\cos\beta)(1-3\cos\alpha\cos\beta)}{(1+3\cos^2\alpha)(1+3\cos^2,\beta)}\,v,$$
$$v_{rn}^{II} = v + (1+e)\,\frac{-1+3\cos\alpha\cos\beta}{(1+3\cos^2\alpha)}\,v.$$

The second angle, β, is numerically determined by solving equation (10) using Mathematica [11], and the impulse for second impact is

$$P_n^{II} = V_n^{II} - V_n^{I} = \frac{1+e}{1+3\cos^2\beta}\,v_{rn}^{II}.$$

The third impact can be calculated in a similar way. The height at the center of mass at the third impact will be $h^{III} = \frac{sin\gamma}{2}$, where γ is the third impact angle between the rod and the floor.

At the third impact

$$h^{II} + V_n^{II}T^{II} = h^{III}, \tag{12}$$

where $T^{II} = \dfrac{\beta+\gamma}{\Omega^{II}}$ is the elapsed time between the second and the third impacts. Therefore, we obtain that

$$V_n^{II}T^{II} = \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{6(1+e)(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}\,(\beta+\gamma). \tag{13}$$

Using the relations in equations (12) and (13), we obtain the following equation that relates α, β and γ for a general value of the restitution coefficient e

$$\sin\beta + \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta)+3(1+e)^2\cos\alpha\cos\beta}{3(1+e)(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}\,(\beta+\gamma) = \sin\gamma. \tag{14}$$

Once the angle β is obtained by equation (10) for any given α, the angle γ can be computed numerically by equation (14).

Now we find the center of mass' velocity and angular velocity, V_n and Ω, for the third bounce:

$$V_n^{III} = V_n^{II} + \frac{1+e}{1+3\cos^2\gamma}\,v_{rn}^{III}, \qquad \Omega^{III} = \Omega^{II} + \frac{6(1+e)\cos\gamma}{1+3\cos^2\gamma}\,v_{rn}^{III},$$

where

$$v_{rn}^{III} = -V_n^{II} + \frac{cos\gamma}{2}\,\Omega^{II}.$$

Hence,

$$\begin{aligned} V_n^{III} &= \frac{-e+3\cos^2\gamma}{1+3\cos^2\gamma}\,V_n^{II} + \frac{\frac{1+e}{2}\cos\gamma}{1+3\cos^2\gamma}\,\Omega^{II}, \\ \Omega^{III} &= \frac{-6(1+e)\cos\gamma}{1+3\cos^2\gamma}\,V_n^{II} + \frac{1+3(e+2)\cos^2\gamma}{1+3\cos^2\gamma}\,\Omega^{II}. \end{aligned} \tag{15}$$

To derive an explicit expression of V_n^{III} and Ω^{III}, we substitute the expression of V_n^{II} and Ω^{II} to get

$$V_n^{III} = \frac{1}{(1+3\cos^2\alpha)(1+3\cos^2\beta)(1+3\cos^2\gamma)}\,[(e-3\cos^2\alpha)(e-3\cos^2\beta)(e-3\cos^2\gamma)$$
$$-3(1+e)^2(\cos\alpha\cos\beta(e-3\cos^2\gamma)+\cos\beta\cos\gamma(e-3\cos^2\alpha)$$
$$+\cos\gamma\cos\alpha(e-3\cos^2\beta))]\,v,$$
$$\Omega^{III} = \frac{6(1+e)}{(1+3\cos^2\alpha)(1+3\cos^2\beta)(1+3\cos^2\gamma)}\,[-3(1+e)^2\cos\alpha\cos\beta\cos\gamma$$
$$+\cos\gamma(e-3\cos^2\alpha)(e-3\cos^2\beta)+\cos\alpha(1-3e\cos^2\beta)(1+3(1+e)\cos^2\gamma)$$
$$+\cos\beta(1+3(2+e)\cos^2\gamma)(e-3\cos^2\alpha)]v.$$

Also, the contact velocity at the third impact is

$$v_{rn}^{III} = v + \frac{(e^2-1)-3(e+1)(\cos^2\alpha+\cos^2\beta)-3(e+1)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v$$
$$+3(e+1)\frac{\cos\gamma(\cos\alpha+e\cos\beta)(-1+3\cos\alpha\cos\beta)}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,v,$$

and the impulse at the third impact is

$$P_n^{II} = V_n^{III} - V_n^{II} = \frac{1+e}{1+3\cos^2\gamma}\,v_{rn}^{III}.$$

We give numerical examples of the formulae for the impact sequence.

For complete restitution case with $e = 1$, given a small angle α, the angle β should be less than or equal to α, as long as $1-3\cos^2\alpha < 0$. We have the equality $\alpha = \beta$ at $54.74°$. Numerically, solution β exists until the rod drops on an angle of $\alpha = 58.49°$. Also, up to this value, the impulse keeps a positive value. There is no solution for β afterwards. From physical point of view, the rod impact sequence ends with just one impact for $\alpha > 58.49°$.

The impulse for the third impact decreases from 0.5 to 0, and it reaches the zero value for $\alpha = 24.79°$. Afterwards, the third impact ceases to exist. The results for full restitution are expressed graphically in Figure 3.2 and Figure 3.3.

In engineering applications it was found the restitution $e = 0.5$ is of significance. We show the impact results for half restitution $(e = 0.5)$ in a comparison study below.

The results when the restitution coefficient is 0.5 are similar to the full restitution case, although the rebounds at both ends are slower due to energy loss. We can obtain solution for β until the rod drops on an angle of $\alpha = 67.21°$. There is no solution for β afterwards.

The impulse for the third impact reaches the zero value for $\alpha = 35.00°$. The results for half restitution are expressed graphically in Figure 3.4 and Figure 3.5.

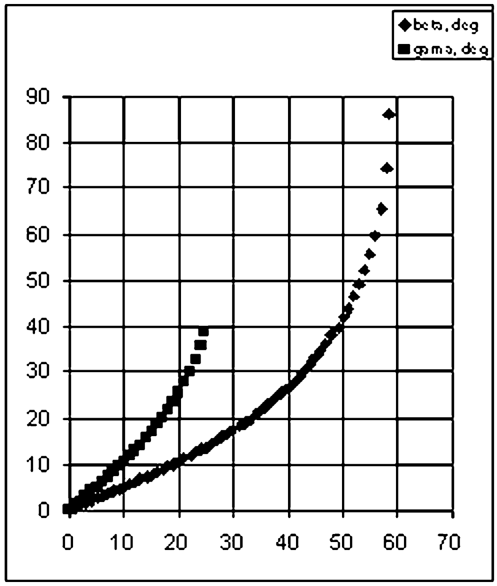

Figure 3.2. The dropping angles at the second and the third impact are shown as functions of the angle α, when e=1. When α is small, β is roughly half of angle α, and γ is nearly the same as angle α.

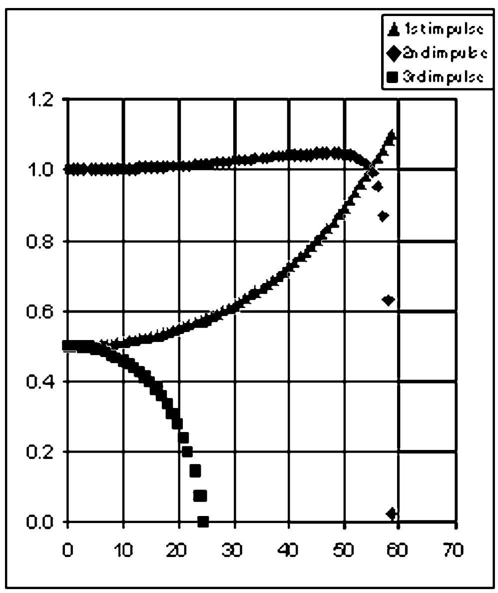

Figure 3.3. The impulses at the first, second and third impacts are shown as functions of the initial angle α, when e=1. When α is small, the second impact is nearly twice of first one, and the third impact is about the same as the first one. The first two impulses become equal at $\alpha = 54.74°$.

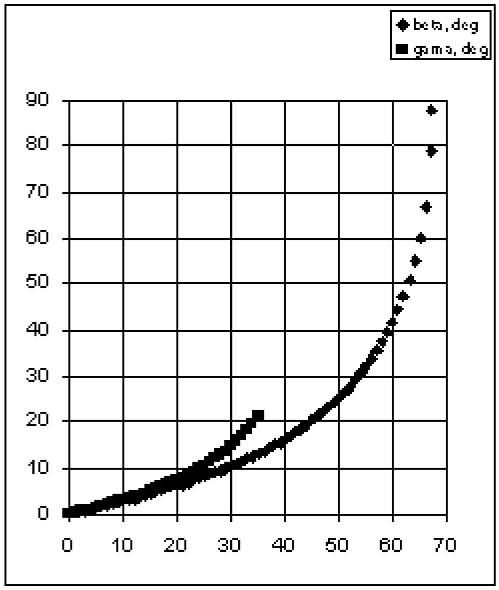

Figure 3.4. The dropping angle at the second and the third impact as function of initial angle, when e=0.5. They are smaller than those for full restitution. The angles where second and third impact terminate are relative higher values, when e=0.5.

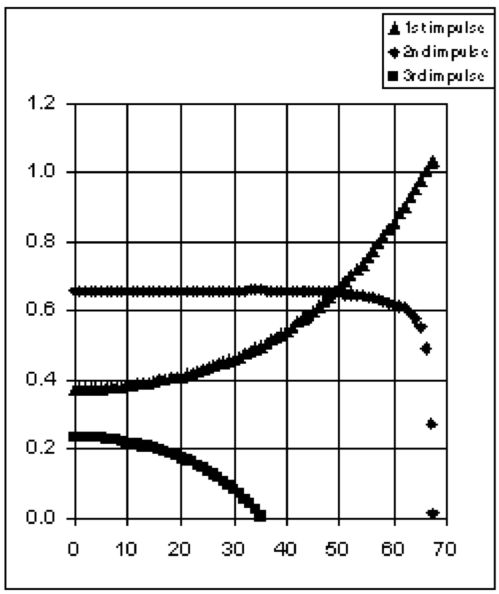

Figure 3.5. The impulses at the first, second and third impact are presented as function of initial angle α, when e=0.5. The impact with half restitution involves energy loss during the impact process. Still, the second impact shows much larger impulse when the angle α is relatively small.

4 The First Three Impacts, with the Gravity

In previous studies [1, 2], it is generally assumed there is no gravity. The validity of such an assumption needs to be checked. In this section, we compare quantitatively the effect of gravity for the impacts sequence. Now, with gravitational force, the impact sequence does not end in finite number, as the rod will fall back again and again. We will still define the clattering sequence as the same number of impact as the case without gravity.

In order to determine the new angles β and γ, we will use the following equations

$$\begin{aligned} h^I + V_n^I T^I - \frac{1}{2}\, g {T^I}^2 &= h^{II}, \\ h^{II} + V_n^{II} T^{II} - \frac{1}{2}\, g {T^{II}}^2 &= h^{III}, \end{aligned} \tag{16}$$

respectively.

From (16), we use

$$V_n^I = \frac{e - 3\cos^2\alpha}{1 + 3\cos^2\alpha}\, v,$$

and

$$T^I = \frac{-(\alpha+\beta)}{\Omega^I} = \frac{1 + 3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\,\frac{1}{v}.$$

Hence the new angle relation for first and second impact is expressed as

$$\sin\alpha + 2\left(\frac{e - 3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\right) - \frac{1}{2}\frac{g}{v^2}\left(\frac{1+3\cos^2\alpha}{6(1+e)\cos\alpha}\,(\alpha+\beta)\right)^2 = \sin\beta. \tag{17}$$

To derive the relation from second angle to third angle, we use

$$V_n^{II} = \frac{-(e-3\cos^2\alpha)(e-3\cos^2\beta) + 3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\, v$$

and

$$T^{II} = \frac{\beta+\gamma}{\Omega^{II}} = \frac{-(1+3\cos^2\alpha)(1+3\cos^2\beta)}{6(1+e)(\cos\alpha + e\cos\beta)(1-3\cos\alpha\cos\beta)}\,(\beta+\gamma)\,\frac{1}{v}\,.$$

Hence

$$\begin{aligned} \sin\beta + 2\left(\frac{(e-3\cos^2\alpha)(e-3\cos^2\beta) - 3(1+e)^2\cos\alpha\cos\beta}{(1+3\cos^2\alpha)(1+3\cos^2\beta)}\,(\beta+\gamma)\right) \\ -\frac{1}{2}\frac{g}{v^2}\left(\frac{-(1+3\cos^2\alpha)(1+3\cos^2\beta)}{6(1+e)(\cos\alpha+e\cos\beta)(1-3\cos\alpha\cos\beta)}\,(\beta+\gamma)\right)^2 = \sin\gamma. \end{aligned} \tag{18}$$

Using the equations (17) and (18), we can find the angles β and γ, respectively, given velocity v.

For example, as we are motivated by the cell phone dropping problem, that phone typically starts a free fall from the pocket. Supposing it drops from a height of one meter, we can find v and go on to find the impact angles

$$\begin{aligned} \frac{1}{2} g\, t^2 = 1 \;\Rightarrow\; t = \sqrt{\frac{2}{g}}, \\ v = g\, t \;\Rightarrow\; v = \sqrt{2\, g} \;\Rightarrow\; \frac{g}{v^2} = \frac{1}{2}. \end{aligned}$$

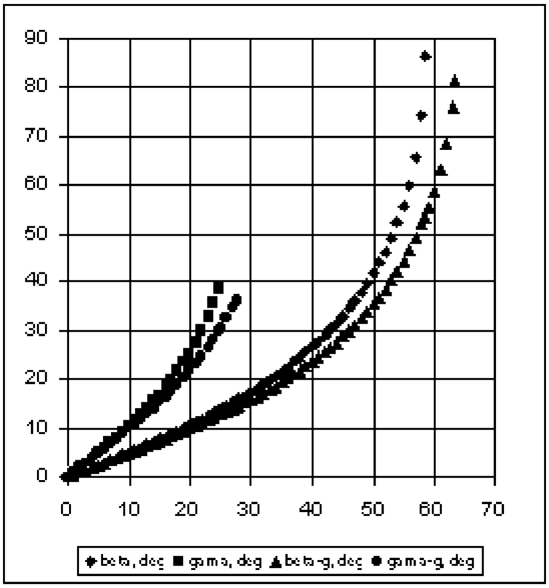

Figure 4.1. The second and the third angles, for total restitution and in both cases, with and without gravity, as a function of initial angle α.

So, by plugging in 1/2 for g/v^2 value in the above equations, we find the impact angles and impulses as shown in Figures 4.1 − 4.4 below.

As we observe in Figure 4.1 that the second and third angles change very little for small initial angles by the effect of gravity. Both angles β and γ are smaller in the case with gravity, and also the second and third clattering moment exists for slightly wider ranges of intervals of α, than in the case when gravity is not considered. The difference between the values for β and also the difference of the values for γ, in the cases without and with gravity, is less than one degree for roughly half of the interval of existence of β and γ respectively, which is 12 and 25 degrees respectively.

The results for the impulse are similar, in the sense that for the same landmarks (say at 12 degree and 25 degree), the difference between the values of impulse in the two cases is less than 0.003 for the second impact, and less than 0.005 for the third impact, while the ranges of the impulses for both cases are at (1.000, 1.018) for the second impact when $0^\circ \le \beta \le 12^\circ$, and are at (0.435, 0.500) for the third impact when $0^\circ \le \gamma \le 25^\circ$, as we see in Figure 4.2.

For both figures, the discrepancy is present when the clattering sequence takes longer time to finish.

When the restitution coefficient equals 0.5, we also compare the results.

As we observe in Figure 4.3, that is similar to the cases with total restitution, the angles β and γ change very little for small angles of α by the gravity effect. Both impact angles β and γ are smaller in the case with gravity though, and also the second and third clattering moment exists for a wider interval for α than in the case without gravity. The difference between the values for β and also the difference for the values for γ, in the cases without and with gravity, is less than one degree for roughly half of the interval of existence of β and γ respectively, which is 17 and 23 degrees respectively, comparing to 12 and 25 in the case with total restitution.

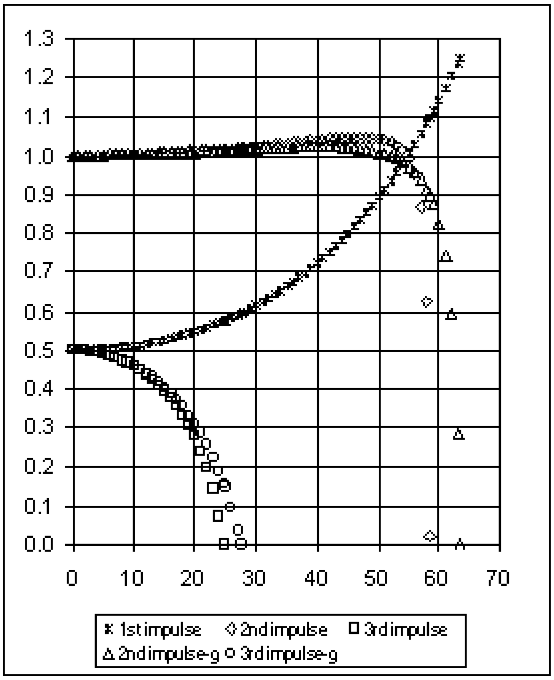

Figure 4.2. The impulses of the first, second and third impact, for total restitution and in both cases, with and without gravity, are shown as a function of initial angle α.

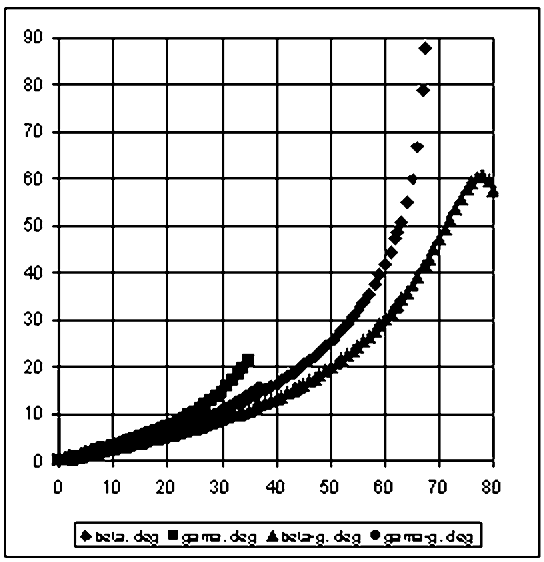

Figure 4.3. The second and the third angle for e=0.5, in both cases, with and without gravity, are shown as a function of initial angle α.

The results for impulses are also similar, in the sense that for the same landmarks (at 17 and 23 degree respectively), the difference between the values in the without gravity/with gravity cases is less than 0.001 for the second impact, and less than 0.01 for the third impact. The ranges of the impulses are both at (0.6563, 0.6593) for the second impact when $0° \leq \beta \leq 17°$, and at (0.1930, 0.2344) for the third impact when $0° \leq \beta \leq 23°$, as we observe in Figure 4.4.

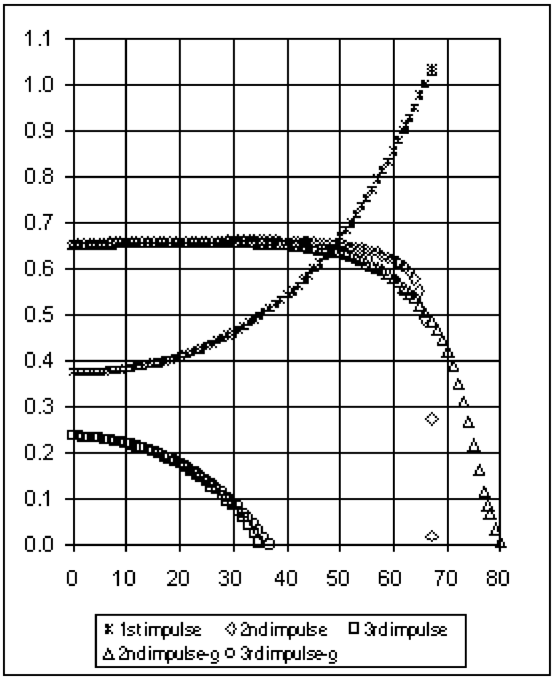

Figure 4.4. The impulse of the first, second and third impact, for e=0.5, in both cases, with and without gravity, is shown as a function of initial angle α.

5 Discussions

The overall aim of this article is to study analytically the issues surrounding clattering. Our discussions are limited to a rod with a uniformly distributed mass. Our study confirms the results of Goyal, *et al.* [1, 2] that if a rod falls to ground in a small angle, then its clattering impact series has a much larger second impact than the initial one. Furthermore, our analytic study finds that same phenomenon is happening to angles as large as 54 degree. In realistic situations, the range might be small when energy dissipation and softness of the ground are included in consideration as we indicated in the case study of $e = 0.5$.

In both situations of $e = 0.5$ and $e = 1.0$ without gravity, there is no forth impact. With gravity, the forth impact will occur, but it does not belong to the same clattering sequence of the first three impacts. So we restrict our discussion to first three impacts.

Through the comparison study at Section 4, we find that gravity plays only a minor role in our clattering problems. Though friction is not considered in this study, we understand that the fiction is a much complex issues. Some initial study indicated that with a certain friction on the ground, when drop angle is small, sticking might occur during the impact process. If the initial rotation is also included, then there is possibility of revered sliding as well as sticking, as discussed in [8]. These topics as well as the clattering of multiple-body and flexible body remain subject of further study.

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