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Robust Dynamic Parameter-Dependent Output Feedback Control of Uncertain Parameter-Dependent State-Delayed Systems

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Abstract: In this paper, we investigate the problem of robust dynamic parameter-dependent output feedback (RDP-DOF) stabilization under H_∞ performance index for a class of linear time invariant parameter-dependent (LTIPD) systems with multi-time delays in the state vector and in the presence of norm-bounded non-linear uncertainties. Using Hamiltonian–Jacobi–Isaac (HJI) method and the idea of polynomial parameter-dependent quadratic (PPDQ) Lyapunov–Krasovskii functions, a new sufficient condition is derived to ensure robust asymptotic stability and robust disturbance attenuation of the closed-loop system. Finally, an example is included that demonstrates the application of the results.

Keywords: *Parameter-dependent systems; multi-time delays; linear matrix inequality; robust dynamic parameter-dependent.*

Mathematics Subject Classification (2000): 34D20, 93A30, 93B36.

1 Introduction

The stability analysis and control design of linear time invariant parameter-dependent (LTIPD) systems where the state-space matrices depend affinely on parameter vector, whose values are not known *a priori*, but can be measured online for control process, have received considerable attention recently (see for instance [1, 2, 3, 5, 6, 18, 23, 25, 26, 28, 31] and the references therein). In many industrial applications, like flight control and process control, the operating point can indeed be determined from measurement, making the LTIPD approach viable, see for example [21, 24]. Establishing stability via the use of

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classical quadratic Lyapunov function is conservative for the LTIPD systems. To investigate the stability of LTIPD systems one needs to resort the use of parameter-dependent Lyapunov functions to achieve necessary and sufficient conditions of system stability, see [7, 10, 11, 14, 16, 30]. However, Bliman in [10] proposed robust stability analysis for LTIPD systems with polytopic uncertain parameters. He also developed some conditions for robust stability in terms of solvability of some linear matrix inequalities (LMIs) without conservatism. Moreover, the existence of a polynomial parameter-dependent quadratic (PPDQ) Lyapunov function for parameter-dependent systems, which are robustly stable, is stated in [11]. Recently, sufficient conditions for robust stability of the linear state-space models affected by polytopic uncertainty have been provided in [12] using homogeneous polynomial parameter-dependent quadratic Lyapunov functions, which are formulated in terms of LMI feasibility tests.

On the other hand, time delays are often present in engineering systems, which have been generally regarded as a main source on instability and poor performance. Therefore, the stabilization of LTIPD state-delayed systems is a field of intense research. Generally, a way to ensure stability robustness with respect to the uncertainty in the delays is to employ stability criteria valid for any nonnegative value of the delays that is *delay-independent results*. This assumption that no information on the value of the delay is known is often coarse in practice. Recently, a systematic way for the use of PPDQ Lyapunov functions in the *state feedback control* of the LTIPD systems with time-delay in the state vector was proposed in [19]. It was also shown that the PPDQ Lyapunov-Krasovskii functions make some *sufficient conditions* under the form of linear matrix inequalities (LMIs).

In this paper, we extend the robust parameter-dependent state-feedback stabilization problem of the LTIPD state-delayed systems in [9, 19] to *robust dynamic parameter-dependent output feedback* (RDP-DOF) control synthesis problem for the LTIPD systems with multi-time delays in the state vector and in the presence of norm-bounded non-linear uncertainties based on the Hamiltonian–Jacoby–Isaac (HJI) method. It is provided a systematic framework for the use of the PPDQ Lyapunov functions in the issue of RDP-DOF stabilization with preserving H_∞ performance criteria. Delay-independent stabilization problem of the system is stated in terms of some LMIs. It would be shown that the use of HJI method makes a *sufficient condition* to have a parameter-dependent bilinear matrix inequality (BMI) optimization problem; thereafter, parameter-independent BMI optimization problem is derived utilizing the PPDQ Lyapunov functions. Therefore, a complete synthesis technique is developed and solving a parameter-independent LMI and a set of linear algebraic equations can construct the RDP-DOF matrices. The simulation results show that the obtained RDP-DOF control can achieve the delay-independent stability and disturbance attenuation of the closed-loop system, simultaneously.

The notations used throughout the paper are fairly standard. The matrices I_n, 0_n and $0_{n\times p}$ are the identity matrix, the $n\times n$ and $n\times p$ zero matrices, respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning $M^{0\otimes}=1$, $M^{p\otimes}=M^{(p-1)\otimes}\otimes M$. Let $\hat{J}_k, \tilde{J}_k \in R^{k\times(k+1)}$ and $u^{[k]}$ be defined by $\hat{J}_k=[I_k,\, 0_{k\times 1}]$, $\tilde{J}_k=[0_{k\times 1},\, I_k]$ and $u^{[k]}=[1, u, \ldots, u^{k-1}]^{\mathrm{T}}$, respectively, which have essential roles for polynomial manipulations [10]. Finally given a signal $x(t)$, $\|x(t)\|_2$ denotes the L_2 norm of $x(t)$; i.e., $\|x(t)\|_2^2=\int_0^\infty x(t)^{\mathrm{T}}x(t)\,dt$.

2 Problem Description

In this paper, we consider a class of LTIPD systems with multi-time delays in the state vector and in the presence of norm-bounded nonlinear uncertainties in which the state-space matrices depend affinely on the constant vector $\rho = [\rho_1, \rho_2, \dots, \rho_m]^{\mathrm{T}} \in \zeta \subset R^m$ (with ζ being a compact set) as follows:

$$
\begin{aligned}
&\dot{x}(t) = A(\rho)x(t) + \sum_{i=1}^{r} A_d^{(i)}(\rho)x(t-h_i) + B_1u(t) + E_1(\rho)w(t) + \Delta(x(t)),\\
&x(t) = \varphi(t), \qquad t \in [-h, 0],\\
&z(t) = C_1x(t),\\
&y(t) = C_2x(t) + E_2w(t)
\end{aligned}
\tag{1}
$$

where the constant parameter h_i is time-delay, $h = \max_i\{h_i\}$ for $i = 1, 2, \dots, r$, and $\varphi(t)$ is the continuous vector valued initial function, also $x(t) \in R^n$, $u(t) \in R^l$, $w(t) \in R^s$, $z(t) \in R^z$ and $y(t) \in R^p$ are the state vector, the control input, the disturbance vector, the controlled output and the output vector, respectively. Moreover, the parameter-dependent matrices $A(\rho)$, $A_d^{(i)}(\rho)$ and $E_1(\rho)$ are expressed as $A(\rho) = A_0 + \sum_{i=1}^{m} \rho_i A_i$, $A_d^{(i)}(\rho) = A_{0d}^{(i)} + \sum_{j=1}^{m} \rho_j A_{jd}^{(i)}$ and $E_1(\rho) = E_{01} + \sum_{i=1}^{m} \rho_i E_{i1}$, respectively, and the vector function $\Delta(x(t))$ is non-linear term of uncertainty set. Furthermore, it is known that the vector ρ is contained in a priori given set whereas the actual curve of the vector ρ is unknown but can be measured online for control process.

Assumption 1 There exists a known real constant matrix $H \in R^{n\times n}$ for the non-linear uncertainty vector $\Delta(\cdot) \in \Omega(\cdot)$ such that $\|\Delta(x(t))\|_2 \le \|Hx(t)\|_2$ for any $x(t) \in R^n$. Denote the corresponding uncertainty set by $\Omega(x(t)) = \{\Delta(x(t))\colon \|\Delta(x(t))\|_2 \le \|Hx(t)\|_2\}$.

The robust dynamic parameter-dependent output feedback (RDP-DOF) control problem that we address in this paper is of the form

$$
\begin{aligned}
\dot{x}_c(t) &= A_K(\rho)x_c(t) + B_K(\rho)y(t),\\
u(t) &= C_K(\rho)x_c(t),
\end{aligned}
\tag{2}
$$

where $x_c(t) \in R^{n_c}$ and the parameter-dependent matrices of $A_K(\rho)$, $B_K(\rho)$ and $C_K(\rho)$ are defined as $A_K(\rho) = A_{0K} + \sum_{i=1}^{m} \rho_i A_{iK} \in R^{n_c\times n_c}$, $B_K(\rho) = B_{0K} + \sum_{i=1}^{m} \rho_i B_{iK} \in R^{n_c\times p}$ and $C_K(\rho) = C_{0K} + \sum_{i=1}^{m} \rho_i C_{iK} \in R^{l\times n_c}$, respectively. In the sequel, the RDP-DOF control state-space matrices will be determined.

Applying the RDP-DOF control (2) into the system (1), we obtain the following augmented closed-loop system

$$
\begin{aligned}
&\dot{X}(t) = \bar{A}_\rho X(t) + \sum_{i=1}^{r} \bar{A}_{d\rho}^{(i)} X(t-h_i) + \overline{E}_\rho w(t) + \overline{\Delta}(SX(t)),\\
&X(t) = \overline{\varphi}(t), \quad t \in [-h,\, 0],\\
&z(t) = \overline{C}_1X(t),\\
&y(t) = \overline{C}_2X(t) + E_2w(t),
\end{aligned}
\tag{3}
$$

where $X(t) = [x^{\mathrm{T}}(t), x_c^{\mathrm{T}}(t)]^{\mathrm{T}}$, $S = [I_n, 0_{n\times n_c}]$, $\overline{C}_1 = [C_1, 0_{z\times n_c}]$, $\overline{C}_2 = [C_2, 0_{p\times n_c}]$,

$$\overline{\Delta}(\cdot) = \begin{bmatrix} \Delta(\cdot) \\ 0_{n_c\times 1} \end{bmatrix}, \quad \bar{A}_\rho = \tilde{A}_\rho + F_1\Gamma_\rho F_2, \quad \bar{A}_{d\rho}^{(i)} = \begin{bmatrix} A_{d\rho}^{(i)} & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c} \end{bmatrix}, \quad \overline{E}_\rho = \widetilde{E}_\rho + \widehat{S}\Gamma_\rho \widehat{E}$$

and

$$\tilde{A}_\rho = \begin{bmatrix} A(\rho) & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c} \end{bmatrix}, \quad F_1 = \begin{bmatrix} B_1 & 0_{n\times n_c} \\ 0_{n_c\times l} & I_{n_c} \end{bmatrix}, \quad F_2 = \begin{bmatrix} C_2 & 0_{p\times n_c} \\ 0_{n_c\times n} & I_{n_c} \end{bmatrix},$$

$$\Gamma_\rho = \begin{bmatrix} 0_{l\times p} & C_k(\rho) \\ B_k(\rho) & A_k(\rho) \end{bmatrix}, \quad \widetilde{E}_\rho = \begin{bmatrix} E_1(\rho) \\ 0_{n_c\times s} \end{bmatrix}, \quad \widehat{E} = \begin{bmatrix} E_2 \\ 0_{n_c\times s} \end{bmatrix},$$

$$\widehat{S} = \begin{cases} I_{n+n_c} & \text{for} \quad n = l, \\ \begin{bmatrix} 0_{(n-l)\times(l+n_c)} \\ I_{l+n_c} \end{bmatrix} & \text{for} \quad n > l. \end{cases}$$

The main objective of the paper is to seek the state-space matrices of the RDP-DOF control (2) that asymptotically stabilizes the closed-loop system (3) with multi-time delays and norm-bounded nonlinear uncertainties as well as guarantees a prescribed H_∞ performance, i.e.,

$$\|z(t)\|_2^2 < \gamma^2 \|w(t)\|_2^2 \tag{4}$$

for all nonzero $w(t) \in L_2(0,\infty)$ under zero initial conditions and a positive scalar γ.

Definition 1 We call a polynomial parameter-dependent quadratic (PPDQ) Lyapunov function any quadratic function $x^{\mathrm{T}}(t)S(\rho)x(t)$ such that

$$S(\rho) = (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)^{\mathrm{T}} S_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_n)$$

for every $x(t) \in R^n$ and a certain $S_k \in R^{k^m n}$. The integer $k-1$ is called the degree of the PPDQ function $S(\rho)$.

3 Delay-Independent Stability Analysis

In this section, assuming that the structure of the RDP-DOF control (2) is known and we will investigate the conditions under which the closed-loop system (3) is asymptotically stable for all admissible vectors $\rho \in \zeta$ and any nonlinear function $\Delta(\cdot) \in \Omega(\cdot)$ independent of time delay parameters h_i for $i = 1, 2, \ldots, r$.

The approach employed here is to investigate the delay-independent stability analysis of the closed-loop system (3) in the presence of the disturbance (exogenous input) and norm-bounded nonlinear uncertainties based on the standard HJI method. In the literature, extensions of the Lyapunov method to the Lyapunov–Krasovskii method have been proposed for time-delayed systems [8, 20]. Hence, we define a class of PPDQ Lyapunov–Krasovskii functions of the degree $k-1$ for this purpose in the following form

$$V(X(t)) = X(t)^{\mathrm{T}} P_\rho X(t) + \sum_{i=1}^{r} \int_{t-h_i}^{t} X(\sigma)^{\mathrm{T}} Q_\rho^{(i)} X(\sigma)\, d\sigma \tag{5}$$

where the positive definite matrices $P_\rho = P(\rho) \in R^{n+n_c}$ and $Q_\rho^{(i)} = Q^{(i)}(\rho) \in R^{n+n_c}$ for $i = 1, 2, \dots, r$ are expressed as

$$P_\rho = (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_{n+n_c})^{\mathrm{T}} P_k (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_{n+n_c}), \tag{6}$$

$$Q_\rho^{(i)} = (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_{n+n_c})^{\mathrm{T}} Q_k^{(i)} (\rho_m^{[k]} \otimes \cdots \otimes \rho_1^{[k]} \otimes I_{n+n_c}) \tag{7}$$

with $P_k, Q_k^{(i)} \in R^{k^m(n+n_c)}$ for $i = 1, 2, \dots, r$. Therefore, the following HJI function is considered as

$$J[w(t),\, \Delta(\cdot)] = \frac{dV\,(X(t))}{dt} + z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) \tag{8}$$

where derivative of $V(X(t))$ is evaluated along the trajectory of the closed-loop system (3). It is well known that a *sufficient condition* for achieving robust disturbance attenuation is that the inequality $J[w(t),\, \Delta(\cdot)] < 0$ for every $w \in L^2$, $\rho \in \zeta$ and $\Delta(\cdot) \in \Omega(\cdot)$ results in a function $V(X(t))$, which is strictly radially unbounded (see, for example, [27, 29]). Therefore, we will establish conditions under which

$$\sup_{\Delta \in \Omega} \sup_{w \in L^2} J[w(t),\, \Delta(\cdot)] < 0, \tag{9}$$

then for every T, taking the definite integral from 0 to T of both sides of (8) gives

$$\int_0^{\mathrm{T}} z^{\mathrm{T}}(t)\, z(t)\, dt - \gamma^2 \int_0^{\mathrm{T}} w^{\mathrm{T}}(t)\, w(t)\, dt < V(X(0)) - V(X(T)) \le V(X(0)) = 0$$

i.e., constraint of disturbance attenuation (4).

From (5) – (8), we find

$$\begin{aligned} J[w(t), \Delta(\cdot)] &= X(t)^{\mathrm{T}}(\bar{A}_\rho^{\mathrm{T}} P_\rho + P_\rho \bar{A}_\rho + \sum_{i=1}^{r} Q_\rho^{(i)} + \overline{C}_1^{\mathrm{T}} \overline{C}_1) X(t) \\ &\quad + X(t)^{\mathrm{T}} P_\rho \sum_{i=1}^{r} \bar{A}_{d\rho}^{(i)} X(t-h_i) + \left(\sum_{i=1}^{r} \bar{A}_{d\rho}^{(i)} X(t-h_i) \right)^{\mathrm{T}} P_\rho X(t) \\ &\quad - \sum_{i=1}^{r} X(t-h_i)^{\mathrm{T}} Q_\rho^{(i)} X(t-h_i) + \overline{\Delta}(SX(t))^{\mathrm{T}} P_\rho X(t) + X(t)^{\mathrm{T}} P_\rho \overline{\Delta}(S\,X(t)) \\ &\quad + w(t)^{\mathrm{T}} \overline{E}_\rho^{\mathrm{T}} P_\rho X(t) + X(t)^{\mathrm{T}} P_\rho \overline{E}_\rho w(t) - \gamma^2 w(t)^{\mathrm{T}} w(t). \end{aligned} \tag{10}$$

It is easy to show that the worst-case disturbance in (10) occurs when

$$w^*(t) = \gamma^{-2} \overline{E}_\rho^{\mathrm{T}} P_\rho X(t). \tag{11}$$

By substituting (11) into (10), we obtain

$$
\begin{aligned}
\sup_{w\in L^2} J(w(t),\Delta) &= J(w^*,\Delta)\\
&= X(t)^{\mathrm{T}}\Big(\bar{A}_\rho^{\mathrm{T}}P_\rho + P_\rho\bar{A}_\rho + \gamma^{-2}P_\rho\overline{E}_\rho\overline{E}_\rho^{\mathrm{T}}P_\rho + \sum_{i=1}^{r} Q_\rho^{(i)} + \overline{C}_1^{\mathrm{T}}\overline{C}_1\Big)X(t)\\
&\quad + X(t)^{\mathrm{T}}P_\rho\sum_{i=1}^{r}\bar{A}_{d\rho}^{(i)}X(t-h_i) + \Big(\sum_{i=1}^{r}\bar{A}_{d\rho}^{(i)}X(t-h_i)\Big)^{\mathrm{T}}P_\rho X(t)\\
&\quad - \sum_{i=1}^{r}X(t-h_i)^{\mathrm{T}}Q_\rho^{(i)}X(t-h_i) + \overline{\Delta}(SX(t))^{\mathrm{T}}P_\rho X(t) + X(t)^{\mathrm{T}}P_\rho\overline{\Delta}(S\,X(t)).
\end{aligned}
\tag{12}
$$

Now, by utilizing Lemma 2 and Assumption 1, it is trivial to show that for any positive scalar ε the following matrix inequality holds

$$
\begin{aligned}
\overline{\Delta}(SX(t))^{\mathrm{T}}P_\rho X(t) + X(t)^{\mathrm{T}}P_\rho\overline{\Delta}(S\,X(t)) &\le \varepsilon X(t)^{\mathrm{T}}P_\rho^2X(t) + \varepsilon^{-1}\overline{\Delta}(S\,X(t))^{\mathrm{T}}\overline{\Delta}(SX(t))\\
&\le X(t)^{\mathrm{T}}(\varepsilon P_\rho^2 + \varepsilon^{-1}(HS)^{\mathrm{T}}(HS))X(t),
\end{aligned}
\tag{13}
$$

then from (12)–(13), the following inequality is obtained

$$
\sup_{\Delta\in\Omega}\sup_{w\in L^2} J[w(t),\,\Delta(\cdot)] = \sup_{\Delta\in\Omega} J(w^*,\Delta) \le \bar{X}(t)^{\mathrm{T}}M_\rho\bar{X}(t) \tag{14}
$$

where the vector $\bar{X}(t) = [X(t)^{\mathrm{T}},\,X(t-h_1)^{\mathrm{T}},\,\dots,\,X(t-h_r)^{\mathrm{T}}]^{\mathrm{T}}$ is an augmented state and the parameter-dependent matrix M_ρ is defined in the form

$$
\begin{bmatrix}
\begin{matrix}\bar{A}_\rho^{\mathrm{T}}P_\rho + P_\rho\bar{A}_\rho + \gamma^{-2}P_\rho\overline{E}_\rho\overline{E}_\rho^{\mathrm{T}}P_\rho + \varepsilon P_\rho^2 + \\ \sum_{i=1}^{r} Q_\rho^{(i)} + \varepsilon^{-1}(HS)^{\mathrm{T}}(HS) + \overline{C}_1^{\mathrm{T}}\overline{C}_1\end{matrix} & P_\rho\bar{A}_{d\rho}^{(1)} & \dots & P_\rho\bar{A}_{d\rho}^{(r)}\\
(P_\rho\bar{A}_{d\rho}^{(1)})^{\mathrm{T}} & -Q_\rho^{(1)} & \dots & 0\\
\cdots & \cdots & \cdots & \cdots\\
(P_\rho\bar{A}_{d\rho}^{(r)})^{\mathrm{T}} & 0 & \dots & -Q_\rho^{(r)}
\end{bmatrix}. \tag{15}
$$

Consequently, if there exist the positive scalar ε and the positive definite solutions P_ρ and $Q_\rho^{(i)}$ for $i = 1, 2, \dots, r$ to the parameter-dependent matrix inequality $M_\rho < 0$, then we have

$$
J[w(t),\,\Delta(\cdot)] < 0, \quad \forall\, w(t) \in L^2, \quad \rho \in \zeta, \quad \Delta(\cdot) \in \Omega(\cdot). \tag{16}
$$

Using Schur Complement Lemma, the parameter-dependent inequality $M_\rho < 0$ can be represented as

$$
\begin{bmatrix}
\begin{matrix}\bar{A}_\rho^{\mathrm{T}}P_\rho + P_\rho\bar{A}_\rho + \sum_{i=1}^{r} Q_\rho^{(i)} + \\ \varepsilon^{-1}(HS)^{\mathrm{T}}(HS) + \overline{C}_1^{\mathrm{T}}\overline{C}_1\end{matrix} & P_\rho & P_\rho\overline{E}_\rho & P_\rho\bar{A}_{d\rho}^{(1)} & \dots & P_\rho\bar{A}_{d\rho}^{(r)}\\
P_\rho & -\varepsilon^{-1}I_{n+n_c} & 0 & 0 & \dots & 0\\
(P_\rho\overline{E}_\rho)^{\mathrm{T}} & 0 & -\gamma^2I_s & 0 & \dots & 0\\
(P_\rho\bar{A}_{d\rho}^{(1)})^{\mathrm{T}} & 0 & 0 & -Q_\rho^{(1)} & \dots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
(P_\rho\bar{A}_{d\rho}^{(r)})^{\mathrm{T}} & 0 & 0 & 0 & \dots & -Q_\rho^{(r)}
\end{bmatrix} < 0. \tag{17}
$$

The following result is now concluded for the delay-independent stability analysis of the uncertain parameter-dependent state-delayed system (1).

Theorem 1 *Let the parameters $\gamma > 0$, $k > 1$ (degree of the PPDQ Lyapunov–Krasovskii functions) and the RDP-DOF control matrices $A_K(\rho)$, $B_K(\rho)$ and $C_K(\rho)$ are given. If there exist positive parameter ε and positive definite matrices P_ρ and $Q_\rho^{(i)}$ for $i = 1, 2, \ldots, r$ to the parameter-dependent matrix inequality (17), then the augmented closed-loop system (3) is asymptotically stable and preserves the H_∞ performance for all admissible vectors $\rho \in \zeta$ and any $\Delta(\cdot) \in \Omega(\cdot)$, independent of the time delay parameters h_i for $i = 1, 2, \ldots, r$.*

Remark 1 A general framework for relaxing parameter-dependent matrix inequality problems into parameter-independent matrix inequalities (conventional form) has been investigated in [4]. However, application of the PPDQ Lyapunov functions as a new tool for relaxing parameter dependency of the matrix inequalities will be stated in the next section.

4 RDP-DOF Control Design

This section is devoted to design of the state-space matrices $A_K(\rho)$, $B_K(\rho)$ and $C_K(\rho)$ for the RDP-DOF control (2) by using the result of Theorem 1 in the previous section.

In Theorem 1, the parameter-dependent inequality (17) can be written in the following from

$$\begin{bmatrix} \begin{array}{c}\tilde{A}_\rho^{\mathrm{T}} P_\rho + P_\rho \tilde{A}_\rho + (F_1\Gamma_\rho F_2)^{\mathrm{T}} P_\rho + \\ P_\rho (F_1\Gamma_\rho F_2) + \sum\limits_{i=1}^{r} Q_\rho^{(i)} + \\ \varepsilon^{-1}(HS)^{\mathrm{T}}(HS) + \overline{C}_1^{\mathrm{T}}\overline{C}_1\end{array} & P_\rho & P_\rho\widetilde{E}_\rho + P_\rho\widehat{S}\Gamma_\rho\widehat{E} & P_\rho\bar{A}_{d\rho}^{(1)} & \ldots & P_\rho\bar{A}_{d\rho}^{(r)} \\ P_\rho & -\varepsilon^{-1} I_{n+n_c} & 0 & 0 & \ldots & 0 \\ (P_\rho\widetilde{E}_\rho + P_\rho\widehat{S}\Gamma_\rho\widehat{E})^{\mathrm{T}} & 0 & -\gamma^2 I_s & 0 & \ldots & 0 \\ (P_\rho\bar{A}_{d\rho}^{(1)})^{\mathrm{T}} & 0 & 0 & -Q_\rho^{(1)} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (P_\rho\bar{A}_{d\rho}^{(r)})^{\mathrm{T}} & 0 & 0 & 0 & \ldots & -Q_\rho^{(r)} \end{bmatrix} < 0 \tag{18}$$

and it is clear that the above constraint is however not simultaneously *convex* in the parameter P_ρ and the controller parameters Γ_ρ. In the literature, more attention has been paid to the problems having this nature, which called bilinear matrix inequality (BMI) problems [22].

In the sequel, we state application of the PPDQ Lyapunov functions to relax dependency of the BMI (18) into the parameter vector ρ. At first, for *parameter-dependent* matrix $R_\rho = \tilde{A}_\rho^{\mathrm{T}} P_\rho + P_\rho \tilde{A}_\rho$, the PPDQ Lyapunov function of degree k is expressed in the form

$$R_\rho = (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} R_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c}) \tag{19}$$

and by some matrix manipulations, in (19) the parameter-independent matrix $R_k \in R^{(k+1)^m(n+n_c)}$ which depends on matrix P_k linearly is obtained as follows

$$\begin{aligned} R_k = {} & \left((\hat{J}_k^{m\otimes} \otimes \tilde{A}_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \tilde{A}_i) \right)^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) \\ & + (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \left((\hat{J}_k^{m\otimes} \otimes \tilde{A}_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \tilde{A}_i) \right) \end{aligned} \tag{20}$$

where

$$\tilde{A}_\rho = \tilde{A}_0 + \sum_{i=1}^{m} \rho_i \tilde{A}_i \quad \text{and} \quad \tilde{A}_i = \begin{bmatrix} A_i & 0_{n\times n_c} \\ 0_{n_c\times n} & 0_{n_c} \end{bmatrix} \quad \text{for} \quad i = 0, 1, \dots, m.$$

Similarly, the PPDQ Lyapunov function of degree k for the parameter-dependent matrix $\Sigma_\rho = (F_1\Gamma_\rho F_2)^{\mathrm{T}} P_\rho + P_\rho(F_1\Gamma_\rho F_2)$ will be as

$$\Sigma_\rho = (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} \Sigma_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c}) \tag{21}$$

where the parameter-independent matrix $\Sigma_k \in R^{(k+1)^m(n+n_c)}$ is shown as follows

$$\begin{aligned} \Sigma_k = &\left((\hat{J}_k^{m\otimes} \otimes F_1\Gamma_0 F_2) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1\Gamma_i F_2) \right)^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) \\ &+ (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \left((\hat{J}_k^{m\otimes} \otimes F_1\Gamma_0 F_2) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1\Gamma_i F_2) \right) \end{aligned} \tag{22}$$

where

$$\Gamma_\rho = \Gamma_0 + \sum_{i=1}^{m} \rho_i \Gamma_i \quad \text{with} \quad \Gamma_j = \begin{bmatrix} 0_{l\times p} & C_{jk} \\ B_{jk} & A_{jk} \end{bmatrix} \quad \text{for} \quad j = 1, 2, \dots, m.$$

Lemma 4 *Let the degree of the PPDQ Lyapunov function P_ρ be $k-1$. The parameter-dependent matrix $P_\rho \mathrm{T}_\rho$ satisfies the following representation form*

$$P_\rho \mathrm{T}_\rho = (\rho_m^{[k+1t]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} H_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_q), \tag{23}$$

where $\mathrm{T}_\rho = \mathrm{T}_0 + \sum_{i=1}^{m} \rho_i \mathrm{T}_i$ *and* $\mathrm{T}_i \in R^{(n+n_c)\times q}$, *then the matrix*

$$H_k \in R^{((k+1)^m(n+n_c))\times((k+1)^m q)}$$

which depends on the matrix P_k linearly is defined as

$$H_k = (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \left((\hat{J}_k^{m\otimes} \otimes \mathrm{T}_0) + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \mathrm{T}_i) \right). \tag{24}$$

According to Lemma 4 for the parameter-dependent matrices $\widetilde{E}_\rho = \widetilde{E}_0 + \sum_{j=1}^{m} \rho_j \widetilde{E}_j$, $\bar{A}_{d\rho}^{(i)} = \bar{A}_{0d}^{(i)} + \sum_{j=1}^{m} \rho_j \bar{A}_{jd}^{(i)}$ and $\widehat{S}\Gamma_\rho \widehat{E} = \widehat{E}_0 + \sum_{j=1}^{m} \rho_j \widehat{E}_j$, we obtain

$$\begin{aligned} P_\rho \widetilde{E}_\rho &= (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} \widetilde{\Xi}_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_s), \\ P_\rho \bar{A}_{d\rho}^{(i)} &= (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} \Xi_k^{(i)} (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c}), \\ P_\rho \widehat{S}\Gamma_\rho \widehat{E} &= (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} \widehat{\Xi}_k (\rho_m^{[k+1]} \otimes \cdots \otimes \rho_1^{[k+1]} \otimes I_s), \end{aligned} \tag{25}$$

where the parameter-independent matrices $\widetilde{\Xi}_k$, $\overline{\Xi}_k^{(i)}$ and $\widehat{\Xi}_k$ are represented in the forms

$$\begin{aligned}
\widetilde{\Xi}_k &= (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \Big((\hat{J}_k^{m\otimes} \otimes \widetilde{E}_0) + \sum_{j=1}^{m} (\hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes \widetilde{E}_j) \Big), \\
\overline{\Xi}_k^{(i)} &= (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \Big((\hat{J}_k^{m\otimes} \otimes \bar{A}_{0d}^{(i)}) + \sum_{j=1}^{m} (\hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes \bar{A}_{jd}^{(i)}) \Big), \\
\widehat{\Xi}_k &= (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \Big((\hat{J}_k^{m\otimes} \otimes \widehat{E}_0) + \sum_{j=1}^{m} (\hat{J}_k^{(m-j)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(j-1)\otimes} \otimes \widehat{E}_j) \Big)
\end{aligned} \tag{26}$$

with

$$\bar{A}_{jd}^{(i)} = \begin{bmatrix} A_{jd}^{(i)} & 0 \\ 0 & 0 \end{bmatrix}, \quad \widetilde{E}_j = \begin{bmatrix} E_{j1} \\ 0_{n_c \times s} \end{bmatrix} \quad \text{and} \quad \widehat{E}_j = \widehat{S}\Gamma_j \begin{bmatrix} E_2 \\ 0_{n_c \times s} \end{bmatrix}$$

for $j = 1, 2, \dots, m$ and $i = 1, 2, \dots, r$.

Similarly, the parameter-independent matrices $\overline{C}_1^{\mathrm{T}} \overline{C}_1$, $(HS)^{\mathrm{T}}(HS)$ and I_s can be also represented as

$$\begin{aligned}
\overline{C}_1^{\mathrm{T}} \overline{C}_1 &= (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_{n+n_c})^{\mathrm{T}} \overline{C}_k (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_{n+n_c}) \\
&= (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \overline{C}_k \\
&\quad \times (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c}),
\end{aligned} \tag{27}$$

$$\begin{aligned}
(HS)^{\mathrm{T}}(HS) &= (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_{n+n_c})^{\mathrm{T}} \bar{H}_k (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_{n+n_c}) \\
&= (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c})^{\mathrm{T}} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \bar{H}_k \\
&\quad \times (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_{n+n_c}),
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
I_s &= (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_s)^{\mathrm{T}} \bar{I}_k^s (\rho_m^{[k]} \otimes \dots \otimes \rho_1^{[k]} \otimes I_s) \\
&= (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_s)^{\mathrm{T}} (\hat{J}_k^{m\otimes} \otimes I_s)^{\mathrm{T}} \bar{I}_k^s \\
&\quad \times (\hat{J}_k^{m\otimes} \otimes I_s) (\rho_m^{[k+1]} \otimes \dots \otimes \rho_1^{[k+1]} \otimes I_s)
\end{aligned} \tag{29}$$

where the certain matrices $\overline{C}_k$, $\bar{H}_k$ and $\bar{I}_k^s$ are defined, respectively, as

$$\overline{C}_k = \mathrm{diag}\,(\overline{C}_1^{\mathrm{T}} \overline{C}_1, \underbrace{0_{n+n_c}, \dots, 0_{n+n_c}}_{(k^m-1)\ \text{elements}}), \quad \bar{H}_k = \mathrm{diag}\,((HS)^{\mathrm{T}}(HS), \underbrace{0_{n+n_c}, \dots, 0_{n+n_c}}_{(k^m-1)\ \text{elements}}),$$

$$\text{and} \quad \bar{I}_k^s = \mathrm{diag}\,(I_s, \underbrace{0_s, \dots, 0_s}_{(k^m-1)\ \text{elements}}).$$

Therefore using the defined notations as well as the definition

$$\bar{I}_k^{n+n_c} = \mathrm{diag}\,(I_{n+n_c}, \underbrace{0_{n+n_c}, \dots, 0_{n+n_c}}_{(k^m-1)\ \text{elements}})$$

and some matrix manipulations, the following parameter-independent BMI form can be obtained from the parameter-dependent inequality (18),

$$
\begin{bmatrix}
R_k + \Sigma_k + (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \left(\varepsilon^{-1} \bar{H}_k + \overline{C}_k + \sum\limits_{i=1}^{r} Q_k^{(i)} \right) (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) & (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) & \widetilde{\Xi}_k + \widehat{\Xi}_k & \overline{\Xi}_k^{(1)} & \cdots & \overline{\Xi}_k^{(r)} \\
(\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) & -\varepsilon^{-1} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \bar{I}_k^{n+n_c} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) & 0 & 0 & \cdots & 0 \\
\widetilde{\Xi}_k^{\mathrm{T}} + \widehat{\Xi}_k^{\mathrm{T}} & 0 & -\gamma^2 (\hat{J}_k^{m\otimes} \otimes I_s)^{\mathrm{T}} \bar{I}_k^{s} (\hat{J}_k^{m\otimes} \otimes I_s) & 0 & \cdots & 0 \\
\overline{\Xi}_k^{(1)^{\mathrm{T}}} & 0 & 0 & -(\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} Q_k^{(1)} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\overline{\Xi}_k^{(r)^{\mathrm{T}}} & 0 & 0 & 0 & \cdots & -(\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} Q_k^{(r)} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})
\end{bmatrix} < 0. \tag{30}
$$

Remark 2 Using the property of $AC \otimes BD = (A \otimes B)(C \otimes D)$, the defined matrices $\widehat{\Xi}_k$ and Σ_k can be shown in the following forms

$$
\begin{aligned}
\widehat{\Xi}_k = {} & (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes \widehat{S})(I_{(k+1)^m} \otimes \Gamma_i) \left(I_{(k+1)^m} \otimes \begin{bmatrix} E_2 \\ 0_{n_c \times s} \end{bmatrix} \right) \\
& + \sum_{i=1}^{m} (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \widehat{S}) \\
& \quad \times (I_{(k+1)^m} \otimes \Gamma_i) \left(I_{(k+1)^m} \otimes \begin{bmatrix} E_2 \\ 0_{n_c \times s} \end{bmatrix} \right)
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
\Sigma_k = {} & \Bigg((\hat{J}_k^{m\otimes} \otimes F_1)(I_{(k+1t)^m} \otimes \Gamma_0)(I_{(k+1)^m} \otimes F_2) \\
& + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1)(I_{(k+1)^m} \otimes \Gamma_i)(I_{(k+1)^m} \otimes F_2) \Bigg)^{\mathrm{T}} P_k (\hat{J}_k^{m\otimes} \otimes I_{n+n_c}) \\
& + (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} P_k \Bigg((\hat{J}_k^{m\otimes} \otimes F_1)(I_{(k+1)^m} \otimes \Gamma_0)(I_{(k+1)^m} \otimes F_2) \\
& + \sum_{i=1}^{m} (\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1)(I_{(k+1)^m} \otimes \Gamma_i)(I_{(k+1)^m} \otimes F_2) \Bigg).
\end{aligned} \tag{32}
$$

The constraint (30) is not convex in terms of the parameter P_k and the controller parameters $\Gamma_0, \Gamma_1, \dots, \Gamma_m$. Consequently, it cannot be used directly for synthesis. It is clear that constraint (30) includes multiplication of control matrices and Lyapunov function matrix. In the sequel, we will simplify and restate the BMI (30) along with the robust performance satisfaction to derive tractable solvability conditions.

Define new matrices as

$$\begin{aligned}
&\Omega_0 = P_k(\hat{J}_k^{m\otimes} \otimes F_1)(I_{(k+1)^m} \otimes \Gamma_0),\\
&\Omega_i = P_k(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1)(I_{(k+1)^m} \otimes \Gamma_i), \quad i = 1, 2, \dots, m,
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
&\Pi_0 = P_k(\hat{J}_k^{m\otimes} \otimes \widehat{S})(I_{(k+1)^m} \otimes \Gamma_0),\\
&\Pi_i = P_k(\hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \widehat{S})(I_{(k+1)^m} \otimes \Gamma_i), \quad i = 1, 2, \dots, m.
\end{aligned} \tag{34}$$

From the above definitions, the following algebraic equations can be concluded

$$\begin{bmatrix} \hat{J}_k^{m\otimes} \otimes F_1 \\ \hat{J}_k^{m\otimes} \otimes \widehat{S} \end{bmatrix} (I_{(k+1)^m} \otimes \Gamma_0) = P_k^{-1} \begin{bmatrix} \Omega_0 \\ \Pi_0 \end{bmatrix} \tag{35}$$

and

$$\begin{bmatrix} \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes F_1 \\ \hat{J}_k^{(m-i)\otimes} \otimes \tilde{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes \widehat{S} \end{bmatrix} (I_{(k+1)^m} \otimes \Gamma_i) = P_k^{-1} \begin{bmatrix} \Omega_i \\ \Pi_i \end{bmatrix}, \quad i = 1, 2, \dots, m, \tag{36}$$

in the case of the matrix F_1 or equivalently the matrix B_1 has the *full column rank*, it can be concluded from the linear algebra theory that the set of algebraic equations (35) and (36) has *at most* one solution $\Gamma_0, \Gamma_1, \dots, \Gamma_m$.

According to (33) and (34), the matrices Σ_k and $\widehat{\Xi}_k$ in the BMI (30) can be represented in the forms

$$\begin{aligned}
\Sigma_k = &\left(\left(\Omega_0 + \sum_{i=1}^{m} \Omega_i\right)(I_{(k+1)^m} \otimes F_2)\right)^{\mathrm{T}} P_k(\hat{J}_k^{m\otimes} \otimes I_{n+n_c})\\
&+ (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \left(\Omega_0 + \sum_{i=1}^{m} \Omega_i\right)(I_{(k+1)^m} \otimes F_2)
\end{aligned} \tag{37}$$

and

$$\widehat{\Xi}_k = (\hat{J}_k^{m\otimes} \otimes I_{n+n_c})^{\mathrm{T}} \left(\Pi_0 + \sum_{i=1}^{m} \Pi_i\right)\left(I_{(k+1)^m} \otimes \begin{bmatrix} E_2 \\ 0_{n_c \times s} \end{bmatrix}\right). \tag{38}$$

Then, from (33)–(37) the solutions of the BMI (30) can be stated as the solutions of an LMI and a set of algebraic equations. Finally, we summarize our result as follows.

Theorem 2 (Delay-independent stabilization) *Let the positive scalar $k-1$ as the degree of the PPDQ Lyapunov–Krasovskii functions is given. Consider the uncertain parameter-dependent system (1) with the constant time delay parameters h_i for $i = 1, 2, \dots, r$ and full column rank of the matrix B_1. For a given performance bound γ, if there exist positive parameter ε and the positive definite matrices P_k, $Q_k^{(i)} \in R^{k^m(n+n_c)}$ for $i = 1, 2, \dots, r$ as well as the matrices $\Omega_i, \Pi_i \in R^{k^m(n+n_c)\times(k+1)^m(p+n_c)}$ for $i =$*

$0, 1, \ldots, m$ to the parameter-independent BMI (30), then the sub-optimal RDP-DOF control law (2) with the following state-space matrices

$$\Gamma_\rho = \Gamma_0 + \sum_{i=1}^{m} \rho_i \Gamma_i \tag{39}$$

may be obtained from the linear algebraic equations (35) and (36) to achieve robust delay-independent asymptotic stability and disturbance attenuation for all admissible vector $\rho \in \zeta$ and any $\Delta(\cdot) \in \Omega(\cdot)$.

Theorem 2 gives a solution to the sub-optimal RDP-DOF control problem. Note that this result can be reformulated as an optimal controller synthesis procedure by solving the following optimization problem

$$\begin{gathered} \text{Min} \ \ \gamma \\ \text{subject to (30), (35) and (36).} \end{gathered} \tag{40}$$

Remark 3 It is observed that the inequality (30) is linear in P_k, $Q_k^{(1)}$, $Q_k^{(2)}$, $\ldots$, $Q_k^{(r)}$, Ω_0, Ω_1, $\ldots$, Ω_m and Π_0, Π_1, $\ldots$, Π_m which are calculated independently from the vector ρ. It is also seen from the above results that there exists some freedoms contained in the design of control law, such as the choices of appropriate the positive scalar ε and the degree of PPDQ Lyapunov function. These degrees of freedoms can be exploited to achieve other desired closed-loop properties.

5 Example

In this section, we illustrate the proposed methodology on a simple system. The state-space form of the uncertain parameter-dependent state-delayed plant is considered as

$$\begin{aligned} \dot{x}(t) &= (-5 - 2\,\rho_1)x(t) + (2 + \rho_1)x(t - h_1) + u(t) + (1 + \rho_1)w(t) + \Delta(x(t)), \\ x(t) &= 2, \quad t \in [-h_1, 0], \\ z(t) &= x(t), \\ y(t) &= 2\,x(t) + w(t), \end{aligned} \tag{41}$$

with $h_1 = 10$ seconds and $\sigma^2 = 0.5$ as the constant time delay and noise variance, respectively. The compact set of the parameter ρ_1 is considered as $\rho_1 \in (-1, 1)$. The non-linear uncertain term $\Delta(x(t))$ is assumed to be norm-bounded with the matrix bound $H = 1$. Using the definitions (33) and (34), solving the LMI (30) and the set of algebraic equations (35) and (36) for the performance bound $\gamma = 1.5$ by the Lmitool toolbox of the Matlab software [17] gives the following positive definite matrices P_k, $Q_k^{(1)}$ for $k = 2$,

$$P_k = \begin{bmatrix} 0.2256 & 0.0103 & -0.0264 & 0.0009 \\ 0.0103 & 0.0771 & -0.0846 & 0.0020 \\ -0.0264 & -0.0846 & 0.2001 & 0.0096 \\ 0.0009 & 0.0020 & 0.0096 & 0.0542 \end{bmatrix},$$

$$Q_k^{(1)} = \begin{bmatrix} 0.4484 & -0.0111 & 0.2732 & 0.0022 \\ -0.0111 & 0.5251 & 0.0047 & -0.0230 \\ 0.2732 & 0.0047 & 1.2472 & -0.0070 \\ 0.0022 & -0.0230 & -0.0070 & 0.6286 \end{bmatrix}.$$

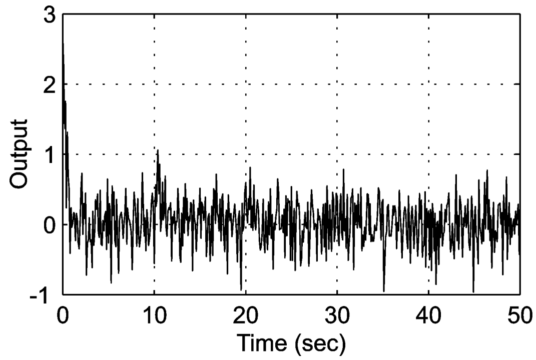

Figure 5.1. Time behavior of $y(t)$.

By considering the parameter $\rho_1 = 0.2225,$ time behavior of the system dynamic (41) has been depicted in Figure 5.1.

The sub-optimal RDP-DOF control (2) with the following state-space matrices

$$\Gamma_0 = \begin{bmatrix} 0 & 0.0771 \\ -0.0264 & -0.0846 \end{bmatrix} \quad \text{and} \quad \Gamma_1 = \begin{bmatrix} 0 & 0.0020 \\ 0.0096 & 0.0542 \end{bmatrix}$$

ensures the asymptotic stability of the closed-loop system (3) which has been shown in Figure 5.2.

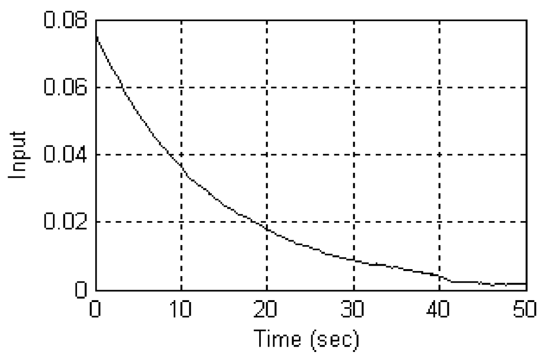

Figure 5.2. The sub-optimal RDP-DOF control.

Moreover, the correctness of disturbance attenuation on the controlled output, i.e. $\|z(t)\|_2^2 - \gamma^2\|w(t)\|_2^2 < 0,$ has been depicted in Figure 5.3.

6 Conclusion

In this paper, we have presented a systematic framework for the RDP-DOF stabilization under H_∞ performance index for a class of LTIPD systems with multi-time delays in the state vector and in the presence of norm-bounded non-linear uncertainties. Our main contribution consists in providing a new sufficient condition as QMIs formulations for the existence of the RDP-DOF control using the PPDQ Lyapunov–Krasovskii functions

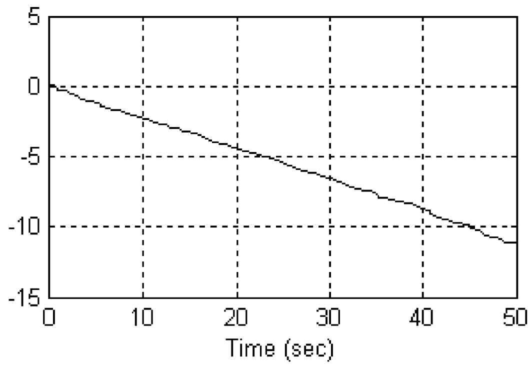

Figure 5.3. The plot of $\|z(t)\|_2^2 - \gamma^2\|w(t)\|_2^2$.

and HJI method. The applicability of the proposed method was illustrated on a simple example.

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Appendix

Lemma 1 (Schur Complement Lemma) *Given constant matrices* Ψ_1, Ψ_2 *and* Ψ_3, *where* $\Psi_1 = \Psi_1^{\mathrm T}$ *and* $\Psi_2 = \Psi_2^{\mathrm T} > 0$, *then* $\Psi_1 + \Psi_3^{\mathrm T}\Psi_2^{-1}\Psi_3 < 0$ *if and only if*

$$\begin{bmatrix} \Psi_1 & \Psi_3^{\mathrm T} \\ \Psi_3 & -\Psi_2 \end{bmatrix} < 0 \quad \textit{or equivalently,} \quad \begin{bmatrix} -\Psi_2 & \Psi_3 \\ \Psi_3^{\mathrm T} & \Psi_1 \end{bmatrix} < 0.$$

Lemma 2 [28] *For any matrix X and Y with appropriate dimensions and for any constant $\eta > 0$, we have*

$$X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \leq \eta X^{\mathrm{T}}X + \frac{1}{\eta}Y^{\mathrm{T}}Y.$$

Lemma 3 (Projection Lemma [13, 15]) *Given a symmetric matrix $H \in R^{h\times h}$ and two matrices $N \in R^{q\times h}$ and $M \in R^{p\times h}$, consider the problem of finding some matrices $X \in R^{p\times q}$ such that*

$$H + N^{\mathrm{T}}X^{\mathrm{T}}M + M^{\mathrm{T}}XN < 0$$

then, the inequality above is solvable for X if and only if

$$N^{\perp T}HN^{\perp} < 0 \quad \textit{and} \quad M^{T\perp T}HM^{T\perp} < 0.$$