Nonlinear Dynamics and Systems Theory, **6**(3) (2006) 257–279

Duality in Distributed-Parameter Control of Nonconvex and Nonconservative Dynamical Systems with Applications

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Received: May 7, 2005; Revised: June 18, 2006

Abstract: Based on a newly developed canonical dual transformation methodology, this paper presents a potentially useful duality theory and method for solving fully nonlinear distributed-parameter control problems. The extended Lagrange duality and the interesting triality theory proposed recently in finite deformation theory are generalized into nonconvex dissipative Hamiltonian systems. It is shown that in canonical dual phase space, the solutions of chaotic systems form an invariant set. Thus, an important bifurcation criterion is proposed, which leads to an effective dual feedback control against chaotic vibrations. Applications are illustrated by a large deformation "smart" beam structure with both shear/damping actuators, and a dissipative Duffing system.

Keywords: *Duality; control theory; chaos; nonconvex analysis; Hamiltonian system.*

Mathematics Subject Classification (2000): 37K05, 37K45.

1 Problems and Motivations

We shall study a duality approach for solving the following very general abstract distributed parameter problem (($\mathcal{P}$) for short),

$$(\mathcal{P}): \quad \rho(u_{,tt} + \nu u_{,t}) + A(u, \mu) = 0 \ \ \forall u \in \mathcal{U}_k, \tag{1}$$

where the feasible space $\mathcal{U}_k$ is a convex, non-empty subset of a reflexive Banach space $\mathcal{U}$ over an open space-time domain $\Omega_t = \Omega \times (0, t_c) \subset R^n \times R^+$, in which certain essential boundary-initial conditions are prescribed. We assume that for a given distributed parameter control field $\mu(x, t)$ over Ω_t, the mapping $A(u, \mu)$ is a potential operator from

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$\mathcal{U}_k$ into its dual space $\mathcal{U}^*$, i.e., there exists a Gâteaux differentiable potential functional $P_\mu(u) = P(u;\mu)$, such that the directional derivative of P_μ at $\bar{u} \in \mathcal{U}_k$ in the direction δu can be written as

$$\delta P_\mu(\bar{u};\delta u) = \langle DP_\mu(\bar{u}), \delta u\rangle \quad \forall \delta u \in \mathcal{U}_k,$$

where the operator $DP_\mu(\bar{u}) = A(\bar{u},\mu)$ is the Gâteaux derivative of P_μ at the point $\bar{u}$; the bilinear form $\langle\cdot,\cdot\rangle : \mathcal{U} \times \mathcal{U}^* \to R$ places $\mathcal{U}$ and $\mathcal{U}^*$ in duality. By nonlinear operator theory we know that the mapping $A : \mathcal{U}_k \to \mathcal{U}^*$ is monotone if P is convex on $\mathcal{U}_k$.

The problem $(\mathcal{P})$ is said to be *exactly controllable* if for certain given initial data $(u_0(x), v_0(x))$ in $\mathcal{U}_k$ and the final state $(\bar{u}_c(x), \bar{v}_c(x))$ there exists suitable control function $\mu(x,t)$ such that the solution $u(x,t)$ of the problem $(\mathcal{P})$ satisfies

$$u(x,t_c) = \bar{u}_c(x), \quad u_{,t}(x,t_c) = \bar{v}_c(x) \quad \forall x \in \Omega. \tag{2}$$

Dually, the problem $(\mathcal{P})$ is said to be *observable* if, for certain given input control $\mu(x,t)$, there exists an output function $h(u)$ such that the initial state $(u_0(x), v_0(x))$ can be uniquely determined from the output $h(u(x,t))$ over any interval $0 < t < t_c$. These dual concepts play a crucial role in many control system design methodologies that have evolved since the early 1960's, such as pole placement, LQG (H^2), H^∞ and minimum time optimization, realization theory, adaptive control, and system identification.

The abstract form of problem $(\mathcal{P})$ covers a great variety of situations. Very often, the total potential $P_\mu(u)$ can be written as

$$P_\mu(u) = \Phi_\mu(u, \Lambda(u)) = W_\mu(\Lambda(u)) - F_\mu(u),$$

where Λ is a Gâteaux differentiable operator from $\mathcal{U}$ into another Banach space $\mathcal{E}$; the functional $W_\mu(\xi)$ is the so-called stored (or internal) potential; while the functional $F_\mu(u)$ represents the external potential of the system.

In convex Hamiltonian systems, the total potential $P_\mu(u)$ is convex and its Gâteaux derivative $A(u,\mu) = DP_\mu(u)$ is usually an elliptic operator in conservative problems. In linear field theory of mathematical physics, Λ is usually a gradient-like operator, say $\Lambda = \text{grad}$, and $W_\mu(\xi)$ is a quadratic functional, for example,

$$P_\mu(u) = \int_\Omega \frac{1}{2} a(x)|\nabla u|^2 \, \mathrm{d}\Omega - F_\mu(u),$$

where $a(x) > 0 \ \ \forall x \in \Omega$. In this case, the governing equation (1) reads

$$\rho(u_{,tt} + \nu u_{,t}) = \nabla \cdot (a(x)\nabla u) + DF_\mu(u) \quad \forall (x,t) \in \Omega_t. \tag{3}$$

It is a linear wave equation if $F_\mu(u)$ is a linear functional, say $F_\mu(\mu) = \langle u \ , \ u^*(\mu)\rangle$, where $u^*(\mu)$ is a given function of the input control field $\mu(x,t)$. If $F_\mu(u)$ is nonlinear, then the governing equation (3) is semi-linear. Due to the efforts of more than thirty years research by many well-known mathematicians and scientists, the mathematical theory for distributed-parameter control systems have been well-established for convex Hamiltonian systems governed by partial differential equations with substantial applications in mechanics and structures (see, for examples, Lasiecka and Triggiani, 1999). In linear systems, there exists a very elegant duality relationship between the controllability and observability (see Dolecki and Russell, 1977).

Duality is a fundamental concept that underlies almost all natural phenomena. In classical optimization and calculus of variation, duality methods possess beautiful theoretical properties, potentially powerful alternative performances and wonderful relationships to many other fields. The associated theory and extremality principles have been well studied for convex static and Hamiltonian systems (cf. e.g., Toland, 1978, 1979; Auchmuty, 1983-2000; Strang, 1986; Rockafellar and Wets, 1997). There is a rapidly growing interest in studying and applications of convex duality theory in optimal control (cf., e.g., Mossino (1975), Chan and Ho (1979), Chan (1985), Chan and Yung (1987), Barron (1990), Tanimoto (1992), Lee and Yung (1997), Bergounioux *et al.* (1999), Arada and Raymond (1999) and many others). The interesting one-to-one analogy between the optimal control and engineering structural mechanics was discussed by Zhong *et al.* (1993, 1999). Recently, the so-called primal-dual interior-point (PDIP) method has been considered as a revolution in linear constrained optimization problems (cf. e.g., Gay *et al.*, 1998; Wright, 1998). It was shown by Helton *et al.* (1998) that the fundamental H^∞ optimization problem of control can be naturally treated with the PDIP methods.

However, the beautiful duality relationship in convex Hamiltonian systems is broken in nonconvex problems. In many applications of engineering and sciences, the total potential of system is usually nonconvex and even nonsmooth. The exact controllability and stability for nonconvex/nonsmooth systems are fundamentally difficult. For example, in the shear-damping control of large deformed beam structures, the actuators could be certain piezoelectric materials attached to the upper and lower beam surfaces, or distributed "smart" dampers (see Figure 1.1). The external signals effect changes of the properties of these actuators in such way that they produce shear forces $\mu^{\pm}(x,t)$ and damping force $\nu w_{,t}$. Thus, $\mu^{\pm}(x,t)$ and ν are, in effect, the applied distributed-control, and the composite beam/actuator system is then an instance of an active, or "smart" structure.

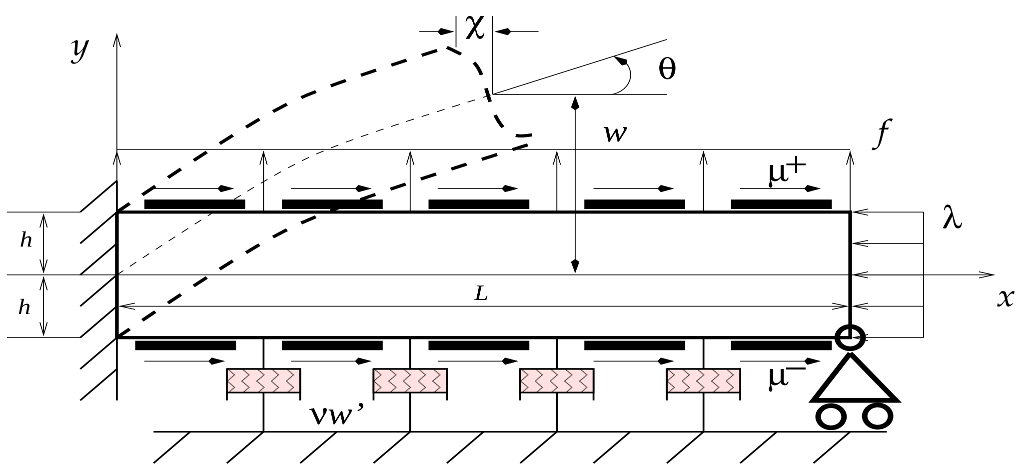


Figure 1.1. Large deformed beam with shear actuators and dampers.

Since the repeated operation of these actuator devices results large shear deformations, the traditional Timoshenko beam model can not be used to the study of these phenomena because it assumes that the shear deformation is a function of x and t alone and does not vary in the lateral beam direction. In order to study control problems of smart structures, several extended beams models have been proposed recently (see Gao *et al.*, 1997-2000), where the state variable space $\mathcal{U} = C^1(\Omega_t; R^2)$ is a displacement space over the space time domain $\Omega_t = (0,\ell) \times (-h,h) \times (0,t_c)$. The element

$u = \{\chi(x,y,t), w(x,t)\} \in \mathcal{U}$ is a continuous, differentiable vector in R^2 with domain Ω_t, where $\chi(x,y,t)$ measures the shear deformation of the beam at the point (x,y), while $w(x,t)$ is the deflection of the beam. In the case that the elastic beam subjected to the transverse load $f(x,t)$ undergone moderately large deformation, the total potential is a nonconvex functional (Gao, 2000a)

$$\begin{aligned} P_\mu(\chi, w) &= \frac{1}{2}\int_\Omega [(\chi_{,x}^2 + \frac{1}{2}\alpha w_{,x}^2 - \lambda)^2 + \beta(\chi_{,y} + w_{,x})^2]\, d\Omega \\ &\quad - \int_0^\ell (\mu^+(x,t)\chi(x,h,t) + \mu^-(x,t)\chi(x,-h,t) + f(x,t)w)\, dx. \end{aligned}$$

If the beam is clamped at $x = 0$, simply supported at $x = \ell$, and is subjected to a compressive load at $x = \ell$, the kinematical admissible space $\mathcal{U}_k \subset \mathcal{U}$ can be defined as

$$\mathcal{U}_k = \left\{ \begin{pmatrix} \chi \\ w \end{pmatrix} \in \mathcal{U} \;\middle|\; \begin{array}{l} w(0,t) = w(\ell,t) = 0,\ \chi(0,y,t) = \chi_{,x}(\ell,y,t) = 0; \\ (\chi, w) = (\chi_0, w_0),\ \ (\chi_{,t}, w_{,t}) = (\dot\chi_0, \dot w_0) \text{ at } t = 0 \end{array} \right\},$$

where (χ_0, w_0) and $(\dot\chi_0, \dot w_0)$ are initial conditions. In this case, the abstract governing equation (1) is a coupled nonlinear partial differential system

$$\begin{aligned} &\rho(w_{,tt} + \nu w_{,t}) = \left(\tfrac{3\alpha^2}{2} w_{,x}^2 + \beta - \lambda\alpha\right) w_{,xx} + \tfrac{\beta}{2h}|\chi_{,x}|_{\pm h} + f\ \forall (x,t) \in (0,\ell)\times(0,t_c), \\ &\chi_{,xx} + \beta\chi_{,yy} = 0,\ \ \forall (x,y,t) \in \Omega_t, \\ &\chi_{,y}(x,\pm h,t) + w_{,x}(x,t) = \pm\mu^\pm(x,t),\ \ \forall (x,t) \in (0,\ell)\times(0,t_c), \end{aligned} \tag{4}$$

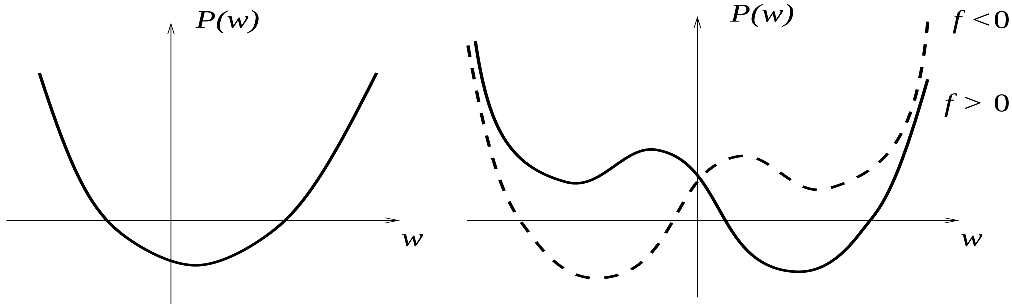
where $\alpha, \beta > 0$ are given material constants, $\lambda \in R$ represents the axial load, and $|\chi_{,x}|_{\pm h} = \chi_{,x}(x,h,t) - \chi_{,x}(x,-h,t)$ is the difference of the top and bottom shear displacements. This coupled nonlinear partial differential system is a typical example in finite deformation mechanics. Since the total potential of this system is nonconvex, the system is very sensitive to initial conditions, driving forces and numerical methods adopted. If the shear deformation can be ignored, the total potential can simply be written as

$$P(w) = \int_I \frac{1}{2}\left(\frac{1}{2}w_{,x}^2 - \lambda\right)^2 dx - \int_I f w\ dx. \tag{5}$$

Clearly, if the beam is subjected to extension, then $\lambda < 0$ and the total potential $P(w)$ is strictly convex (see Figure 1.2 a). It possesses at most one global minimizer. In this case, the system is stable. However, for compressive axial load, $\lambda > 0$, the total potential $P(w)$ is a so-called double-well energy (see Figure 1.2 b). In static buckling problem, this nonconvex potential has three critical points: two local minimizers, corresponding to two possible stable buckled states, and one local maximizer, corresponding to an unstable buckled state. The global minimizer depends on the lateral load f.

If the compressed beam is subjected to a periodically dynamical load $f(x,t)$, the two local minimizers of P_μ become extremely unstable, and the beam is in dynamical post-buckling state. If the deflection $w(x,t)$ can be separated variables such that $w(x,t) = u(t)v(x)$, this post-buckling dynamical beam model leads to the well-known Duffing equation

$$u_{,tt} + \nu u_{,t} = au(\lambda - \frac{1}{2}u^2) + \mu(t), \tag{6}$$



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 1.2. Convex and double-well potentials.

where $a > 0$ is a parameter. It is known that this equation is extremely sensitive to the initial conditions. For certain given parameters λ, ν, and driving force $\mu(t)$, this equation possesses the so-called chaotic solutions. Figure 5.4 shows that even for the same given data, different numerical methods produce totally different results.

Control theory in finite deformation mechanics has emerged as the most challenging and active research field in recent years. Mathematically speaking, the total potentials of large deformed structures are generally nonconvex, or even nonsmooth. Very small perturbations of the system's initial conditions and parameters may lead the system to different operating points with significantly different performance characteristics. This is the one of main reasons why the traditional perturbation analysis, the direct approaches and many standard control techniques cannot successfully be applied to nonconvex systems. Based upon these observations and in order to handle the nonlinear problem, a school of new techniques has been developed (see, e.g., Fowler, 1989; Ott *et al.*, 1990; Chen and Dong, 1993; Ogorzalek, 1993; Antoniou *et al.*, 1996; Ghezzi and Piccardi, 1997; Koumboulis and Mertzios, 1996, 2000).

Duality theory in fully nonlinear variational problems was originally studied by Gao and Strang (1989) for large deformation nonsmooth mechanics. In order to recover the broken symmetry in fully nonlinear systems (see Definition 2.2), a so-called *complementary gap function* was introduced. It was realized in post-buckling analysis of nonlinear beam theory (Gao, 1997) that this function recovered the duality gap between the nonconvex primal problems and the Fenchel-Rockafellar dual problems. A self-contained comprehensive presentation of the mathematical theory in general nonconvex systems was given recently by Gao (2000d), wherein, a so-called *canonical dual transformation method* and associated triality theory have been proposed for solving nonconvex/nonsmooth variational-boundary value problems. Recent results show that certain very difficult constrained nonconvex problems in global optimization can be solved completely by this method (see Gao, 2003, 2005). Compared with the traditional analytic methods and direct approaches, the main advantages of this canonical dual transformation method are the following:

(1) it converts nonconvex/nonsmooth constrained variational problems into smooth unconstrained dual problems;

(2) it transforms certain fully nonlinear partial differential equations into algebraic

systems;

(3) it provides powerful and efficient primal-dual alternative approaches.

The aim of the present article is to generalize the author's previous results on nonconvex variational problems into nonconservative distributed-parameter control systems. The rest of this paper is divided into four main sections. The next section set up notations used in the paper. A general framework in fully nonlinear systems are discussed. Section 3 presents an extended Lagrangian critical point theorem and the associated triality theory in general nonconvex, nonconservative dynamical systems. The critical points in fully nonlinear systems are classified. Section 4 is devoted mainly to the construction of dual action in nonconvex dissipative Hamiltonian systems. The tri-duality proposed in static boundary value problems is generalized into control problems. Section 5 discusses application in dissipative Duffing system. A bifurcation criterion is proposed which can be used for feedback controlling against chaotic vibrations.

2 Framework for Canonical Systems and Classification

Let $\mathcal{U}$ and $\mathcal{U}^*$ be two real linear spaces, placed in duality by a bilinear form $\langle u, u^* \rangle : \mathcal{U} \times \mathcal{U}^* \to R$. Let $P : \mathcal{U}_s \to R$ be a given functional, well-defined on a convex domain $\mathcal{U}_s \subset \mathcal{U}$ such that for any given $u \in \mathcal{U}_s$, $P(u)$ is Gâteaux differentiable. Thus, the Gâteaux derivative DP of P at $u \in \mathcal{U}_s$ is a mapping from $\mathcal{U}_s$ into $\mathcal{U}^*$. Let $\mathcal{U}_s^* \subset \mathcal{U}^*$ be the range of the mapping $DP : \mathcal{U}_s \to \mathcal{U}^*$. If the relation $u^* = DP(u)$ is reversible on $\mathcal{U}_s$, then for any given $u^* \in \mathcal{U}_s^*$, the classical Legendre conjugate functional $P^* : \mathcal{U}_s^* \to R$ of $P(u)$ is defined by

$$P^*(u^*) = \{\langle u, u^* \rangle - P(u) | \ u^* = DP(u)\}.$$

The conjugate pair (u, u^*) is called the *canonical duality pair* on $\mathcal{U}_s \times \mathcal{U}_s^* \subset \mathcal{U} \times \mathcal{U}^*$ if and only if the equivalent relations

$$u^* = DP(u) \quad \Leftrightarrow \quad u = DP^*(u^*) \quad \Leftrightarrow \quad P(u) + P^*(u^*) = \langle u, u^* \rangle. \tag{7}$$

hold on $\mathcal{U}_s \times \mathcal{U}_s^*$.

The following notations and definitions, used in Gao (2000c,d), will be of convenience in nonconvex control problems.

Definition 2.1 The set of functionals $P : \mathcal{U} \to R$ which are either convex or concave is denoted by $\Gamma(\mathcal{U})$. In particular, let $\check{\Gamma}(\mathcal{U})$ denote the subset of functionals $P \in \Gamma(\mathcal{U})$ which are convex and $\hat{\Gamma}(\mathcal{U})$ the subset of $P \in \Gamma(\mathcal{U})$ which are concave.

The *canonical functional space* $\Gamma_G(\mathcal{U}_s)$ is a subset of functionals $P \in \Gamma(\mathcal{U}_s)$ which are Gâteaux differentiable on $\mathcal{U}_s \subset \mathcal{U}$, such that the relation $u^* = DP(u)$ is reversible for any given $u \in \mathcal{U}_s$. $\diamondsuit$

Clearly, if $P \in \Gamma_G(\mathcal{U}_s)$ and $\mathcal{U}_s^*$ is the range of the mapping $DP : \mathcal{U}_s \to \mathcal{U}^*$, then the canonical duality relations (7) hold on $\mathcal{U}_s \times \mathcal{U}_s^*$.

Let $(\mathcal{E}, \mathcal{E}^*)$ be an another pair of real linear spaces paired in duality by the second bilinear form $\langle \cdot \, ; \, \cdot \rangle : \mathcal{E} \times \mathcal{E}^* \to R$. The so-called *geometrical operator* $\Lambda : \mathcal{U} \to \mathcal{E}$ is a continuous, Gâteaux differentiable operator such that for any given $u \in \mathcal{U}_a \subset \mathcal{U}$, there exists an element $\xi \in \mathcal{E}_a \subset \mathcal{E}$ satisfying the *geometrical equation*

$$\xi = \Lambda(u).$$

The directional derivative of ξ at $\bar{u}$ in the direction $u \in \mathcal{U}$ is then defined by

$$\delta\xi(\bar{u}; u) := \lim_{\theta \to 0^+} \frac{\xi(\bar{u} + \theta u) - \xi(\bar{u})}{\theta} = \Lambda_t(\bar{u})u, \tag{8}$$

where $\Lambda_t(\bar{u}) = D\Lambda(\bar{u}) : \mathcal{U} \to \mathcal{E}$ denotes the Gâteaux derivative of the operator Λ at $\bar{u}$. For a given $\xi^* \in \mathcal{E}^*$, $G_\Lambda(u) = \langle \Lambda(u) \; ; \; \xi^* \rangle$ is a real-valued functional of u on $\mathcal{U}$. Its Gâteaux derivative at $\bar{u} \in \mathcal{U}$ in the direction $u \in \mathcal{U}$ reads

$$\delta G_\Lambda(\bar{u}; u) = \langle \Lambda_t(\bar{u})u \; ; \; \xi^* \rangle = \langle u \; , \; \Lambda_t^*(\bar{u})\xi^* \rangle,$$

where $\Lambda_t^*(\bar{u}) : \mathcal{E}^* \to \mathcal{U}^*$ is the adjoint operator of Λ_t associated with the two bilinear forms.

Let $\mathcal{V}$ and $\mathcal{V}^*$ be velocity and momentum spaces, respectively, placed in duality by the third bilinear form $\langle * \; , \; * \rangle : \mathcal{V} \times \mathcal{V}^* \to R$. For Newtonian systems, the kinetic energy $K : \mathcal{V} \to R$ and its Legendre conjugate $K^* : \mathcal{V}^* \to R$ are quadratic forms

$$K(v) = \int_\Omega \frac{1}{2} \rho v^2 \, \mathrm{d}\Omega, \quad K^*(p) = \int_\Omega \frac{1}{2} \rho^{-1} p^2 \, \mathrm{d}\Omega.$$

Thus the canonical physical relations between $\mathcal{V}$ and $\mathcal{V}^*$ are linear:

$$p = DK(v) = \rho v \quad \Leftrightarrow \quad v = DK^*(p) = \rho^{-1} p.$$

Let $\mathcal{V}_a \subset \mathcal{V}$ be an admissible velocity space, in which certain essential initial/boundary conditions are given, say

$$\mathcal{V}_a = \{ v \in \mathcal{V} | \;\; v(x, 0) = v_0 \;\; \forall x \in \Omega \}. \tag{9}$$

Finally, we let $\mathcal{M}$ be an admissible control space over Ω_t. For any given $\mu \in \mathcal{M}$, we assume that there exists a Gâteaux differentiable functional $\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \subset \mathcal{U} \times \mathcal{E} \to R$, such that the total potential $P(u; \mu)$ of the system can be written as

$$P_\mu(u) = P(u; \mu) = \Phi_\mu(u, \Lambda(u)). \tag{10}$$

Thus, for a dissipative dynamical system with linear damping, the total action of the system is a weighted nonconvex functional

$$\Pi_\mu(u) = \int_0^{t_c} e^{\nu t} \left[K(\partial_t u) - \Phi_\mu(u, \Lambda(u)) \right] \mathrm{d}t, \tag{11}$$

which is well-defined on the feasible space $\mathcal{U}_k$ given by

$$\mathcal{U}_k = \{ u \in \mathcal{U}_a | \;\; \Lambda(u) \in \mathcal{E}_a, \;\; \partial_t u \in \mathcal{V}_a \}. \tag{12}$$

For the linear time-differential operator $\partial_t = \partial / \partial t$, its formal adjoint associated with this weighted functional is an affine operator $\partial_t^* = -\partial / \partial t - \nu$ (see Gao (2000d)).

The following classification for distributed parameter control systems was originally introduced in nonlinear variational/boundary value problems by Gao (1998, 2000d,2000).

Definition 2.2 Suppose that for the problem $(\mathcal{P})$ given in (1), the associated total potential $P_\mu(u)$ is well-defined on its domain $\mathcal{U}_s \subset \mathcal{U}$. If the geometrical operator Λ :

$\mathcal{U} \to \mathcal{E}$ can be chosen such that $P_\mu(u) = \Phi_\mu(u, \Lambda(u))$, $\Phi_\mu \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{E}_a)$ and $\mathcal{U}_s = \{u \in \mathcal{U}_a | \ \Lambda(u) \in \mathcal{E}_a\}$, then

(1) the transformation $\{P;\mathcal{U}_s\} \to \{\Phi_\mu;\mathcal{U}_a \times \mathcal{E}_a\}$ is called the *canonical transformation*, and $\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to R$ is called the *canonical functional associated with* Λ;

(2) the problem $(\mathcal{P})$ is called *geometrically nonlinear (or linear)* if $\Lambda : \mathcal{U} \to \mathcal{E}$ is nonlinear (or linear); it is called *physically nonlinear* (resp. linear) if the duality mapping $D\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{E}_a^*$ is nonlinear (resp. linear); it is called *fully nonlinear* if it is both geometrically and physically nonlinear. $\diamondsuit$

The canonical transformation plays a fundamental role in duality theory of nonconvex systems. Clearly, if $\Phi_\mu \in \Gamma_G(\mathcal{U}_a) \times \Gamma_G(\mathcal{E}_a)$ is a canonical functional, the Gâteaux derivative $D\Phi_\mu : \mathcal{U}_a \times \mathcal{E}_a \to \mathcal{U}_a^* \times \mathcal{E}_a^* \subset \mathcal{U}^* \times \mathcal{E}^*$ is a monotone mapping, i.e., the duality relations

$$u^* = D_u \Phi_\mu(u, \xi), \quad \xi^* = D_\xi \Phi_\mu(u, \xi) \tag{13}$$

are reversible between the paired spaces $(\mathcal{U}_a, \mathcal{U}_a^*)$ and $(\mathcal{E}_a, \mathcal{E}_a^*)$, where $D_u\Phi_\mu$ and $D_\xi\Phi_\mu$ denote partial Gâteaux derivatives of Φ_μ with respect to u and ξ, respectively. Thus, on $\mathcal{U}_k$ the directional derivative of P_μ at $\bar{u}$ in the direction $u \in \mathcal{U}_k$ can be written as

$$\begin{aligned} \delta P_\mu(\bar{u}; u) &= \langle u \ , \ D_u \Phi_\mu(\bar{u}, \Lambda(\bar{u}))\rangle + \langle \Lambda_t(\bar{u})u \ ; \ D_\xi \Phi_\mu(\bar{u}, \Lambda(\bar{u}))\rangle \\ &= \langle u \ , \ \bar{u}^*\rangle + \langle u \ ; \ \Lambda_t^*(\bar{u})\bar{\xi}^*\rangle \quad \forall u \in \mathcal{U}_k. \end{aligned}$$

In terms of canonical variables, the governing equation (1) for fully nonlinear problems can be written in the *tri-canonical forms*, namely,

$$\begin{array}{ll} \text{(1) geometrical equations:} & v = \partial_t u, \ \ \xi = \Lambda(u), \\ \text{(2) physical relations:} & p = \rho v, \ \ (u^*, \xi^*) = D\Phi_\mu(u, \xi), \\ \text{(3) balance equation:} & \partial_t^* p - u^* - \Lambda_t^*(u)\xi^* = 0. \end{array} \tag{14}$$

The framework for fully nonlinear systems is shown in Figure 2.1. Extensive illustrations of the canonical transformation and the tri-canonical forms in mathematical physics and variational analysis were given in the monograph by Gao (2000).

$$\begin{tikzcd}
v \in \mathcal{V} & \langle v \ , \ p\rangle \arrow[l] \arrow[r] & \mathcal{V}^* \ni p \arrow[d, "\partial_t^* = -\frac{\partial}{\partial t} - \nu"] \\
u \in \mathcal{U} \arrow[u, "\frac{\partial}{\partial t} = \partial_t"'] \arrow[d, "\Lambda_t + \Lambda_c = \Lambda"'] & \langle u \ , \ u^*\rangle \arrow[l] \arrow[r] & \mathcal{U}^* \ni u^* \\
\xi \in \mathcal{E} & \langle \xi \ ; \ \xi^*\rangle \arrow[l] \arrow[r] & \mathcal{E}^* \ni \xi^* \arrow[u, "\Lambda_t^* = (\Lambda - \Lambda_c)^*"']
\end{tikzcd}$$

Figure 2.1. Framework in fully nonlinear Newtonian systems with linear damping.

In geometrically linear systems, where $\Lambda : \mathcal{U} \to \mathcal{E}$ is linear, we have $\Lambda = \Lambda_t$. For dynamical problems, if the total potential P_μ is convex, the total action associated with the problem $(\mathcal{P})$ is a *d.c. functional*, i.e., the difference of convex functionals:

$$\Pi_\mu(u) = \int_0^{t_c} e^{\nu t}[K(\partial_t u) - P_\mu(u)]\,\mathrm{d}t.$$

It was shown by Gao (2000d) that the critical point of Π_μ either minimizes or maximizes Π_μ over the kinetically admissible space. The classical Hamiltonian associated with this d.c. functional Π_μ is a convex functional on the phase space $\mathcal{U} \times \mathcal{V}^*$, i.e.

$$H(u,p) = K^*(p) + P_\mu(u), \tag{15}$$

The classical canonical forms for convex Hamilton systems are well-known

$$\partial_t u = D_p H(u,p), \quad \partial_t^* p = D_u H(u,p).$$

Furthermore, if the canonical functional Φ_μ can be written in the form $\Phi_\mu(u,\xi) = \frac{1}{2}\langle \xi\,;\, C\xi\rangle - F_\mu(u)$, where $C : \mathcal{E} \to \mathcal{E}^*$ is a linear symmetrical operator, then the governing equations for linear system can be written as

$$\rho(u_{,tt} + \nu u_{,t}) + \Lambda^* C \Lambda u = DF_\mu(u).$$

In mathematical physics, the geometrical mapping Λ is usually a gradient-like operator. Then $A = \Lambda^* C \Lambda$ is an elliptic operator if C is positive-definite.

In geometrically nonlinear systems, $\Lambda \neq \Lambda_t$, and the total potential $P_\mu(u)$ is usually a nonconvex functional. In this case, we have the following operator decomposition

$$\Lambda(u) = \Lambda_t(u)u + \Lambda_c(u), \tag{16}$$

where $\Lambda_c : \mathcal{U} \to \mathcal{E}$ is called the complementary operator of the Gâteaux derivative operator Λ_t. By this decomposition, we have

$$\langle \Lambda(u)\,;\, \xi^*\rangle = \langle u\,,\, \Lambda_t^*(u)\xi^*\rangle - G(u,\xi^*), \tag{17}$$

where $G : \mathcal{U} \times \mathcal{E}^* \to R$ is so-called *complementary gap functional*, defined by

$$G(u,\xi^*) = \langle -\Lambda_c(u)\,;\, \xi^*\rangle : \mathcal{U} \times \mathcal{E}^* \to R. \tag{18}$$

This functional was first introduced by Gao and Strang (1989) in finite deformation theory to recover a broken symmetry in geometrical nonlinear systems. It is now understood that this gap functional plays a key role in extremality analysis of nonconvex variational problems.

As a typical example in nonconvex dynamical systems, let us consider the following nonconvex variational problem over the domain $\Omega_t = (0,\ell) \times (0,t_c)$:

$$\Pi_\mu(u) = \int_{\Omega_t} e^{\nu t}\left[\frac{1}{2}\rho u_{,t}^2 - \frac{1}{2}a(\frac{1}{2}u_{,x}^2 - \mu)^2 + uf\right]\,\mathrm{d}x\,\mathrm{d}t \quad \to \mathrm{sta} \ \ \forall u \in \mathcal{U}_k, \tag{19}$$

where $a,\ \mu$ are given positive constants. This nonconvex problem also appears very often in phase transitions and hysteresis.

First, we let $\Lambda = \partial/\partial x$ be a linear operator, and $P_\mu(u) = W_\mu(\Lambda u) - F_\mu(u)$ with

$$W_\mu(\epsilon) = \int_0^\ell \frac{1}{2}a(\frac{1}{2}\epsilon^2 - \mu)^2\,\mathrm{d}x, \quad F_\mu(u) = \int_0^\ell uf\,\mathrm{d}x.$$

Thus, $W_\mu(\epsilon)$ is the so-called van der Waals' double-well function of the linear "strain" $\epsilon = u_{,x}$. Since $W_\mu(\epsilon)$ is not a canonical functional, the constitutive equation $\epsilon^* = DW_\mu(\epsilon)$ is not one-to-one. Thus, the Legendre conjugate of $W_\mu(\epsilon)$ does not have a simple algebraic expression. The Fenchel conjugate $W_\mu^*(\epsilon^*)$ of the double-well energy $W_\mu(\epsilon)$, defined by

$$W_\mu^*(\epsilon^*) = \sup_\epsilon \{\langle \epsilon \ ; \ \epsilon^* \rangle - W_\mu(\epsilon)\},$$

is always a convex, lower semi-continuous functional. However, the well-known Fenchel-Young inequality

$$W_\mu(u_{,x}) \geq \langle u_{,x} \ ; \ \epsilon^* \rangle - W_\mu^*(\epsilon^*)$$

leads to a so-called duality gap between the primal problem and the Fenchel-Rockafellar dual problem (see Gao, 2000d). This nonzero duality gap indicates that the well-established Fenchel-Rockafellar duality theory can be used only for solving convex variational problems.

From the theory of continuum mechanics we know that in finite deformation problems, $\epsilon = u_{,x}$ is not a strain measure (it does not satisfy the *axiom of material frame-indifference* (cf. e.g., Gao, 2000d)). In order to recover this duality gap, we need to choose a suitable geometrical operator Λ, say, $\Lambda(u) = \frac{1}{2}u_{,x}^2 - \mu$, so that the nonconvex problem (19) can be put in our framework. In continuum mechanics, this quadratic measure $\xi = \Lambda(u)$ is a Cauchy-Green type strain. Thus, in terms of u and ξ, $\Phi_\mu(u,\xi) = W_\mu(\xi) - F_\mu(u) = \frac{1}{2}\langle \xi \ ; \ a\xi \rangle - \langle u \ , \ f \rangle$ is a canonical functional. The Legendre conjugate of the quadratic functional $W_\mu(\xi) = \frac{1}{2}\langle \xi \ ; \ a\xi \rangle$ is simply defined by $W^*(\xi^*) = \frac{1}{2}\langle a^{-1}\xi^* \ ; \ \xi^* \rangle$. The operator decomposition (16) for this quadratic operator reads

$$\Lambda(u) = \Lambda_t(u)u + \Lambda_c(u), \ \ \Lambda_t(u)u = u_{,x}u_{,x}, \ \ \Lambda_c(u) = -\frac{1}{2}u_{,x}^2 - \mu.$$

The complementary gap functional associated with this quadratic operator is a quadratic functional of u

$$G(u,\xi^*) = \langle -\Lambda_c(u) \ ; \ \xi^* \rangle = \int_0^\ell \frac{1}{2}u_{,x}^2 \xi^* \ \mathrm{d}x.$$

For homogeneous boundary conditions, we have

$$\langle \Lambda_t(u)u \ ; \ \xi^* \rangle = \int_0^\ell u_{,x}u_{,x}\xi^* \ \mathrm{d}x = -\int_0^\ell u(u_{,x}\xi^*)_{,x} \ \mathrm{d}x = \langle u \ , \ \Lambda_t^*(u)\xi^* \rangle,$$

which leads to the adjoint operator Λ_t^* of Λ_t. Thus, the tri-canonical equations for this nonconvex problem can be listed as the following.

$$v = \partial_t u, \ \ \xi = \frac{1}{2}au_{,x}^2 - \mu,$$

$$p = \rho v, \ \ \xi^* = DW_\mu(\xi) = a\xi, \ \ u^* = DF_\mu(u) = f,$$

$$p_{,t} + \nu p = -\Lambda_t^*(u)\xi^* + u^* = (u_{,x}\xi^*)_{,x} + f.$$

Since the geometrical operator Λ is nonlinear, and the canonical constitutive equations are linear, the nonconvex problem (19) is a geometrically nonlinear system.

3 Extended Lagrangian and Triality Theory

The triality theory in nonconvex problems was originally proposed by the author (Gao, 1997, 1999, 2000) in static finite deformation theory and global optimization. In this section, we will generalize this interesting result into fully nonlinear dynamical systems. We assume that for a given fully nonlinear system, there exists a Gâteaux differentiable operator $\Lambda : \mathcal{U}_a \to \mathcal{E}_a$ such that the total potential of the system can be written as

$$P_\mu(u) = W_\mu(\Lambda(u)) - F_\mu(u), \tag{20}$$

where $W_\mu \in \check{\Gamma}_G(\mathcal{E}_a)$ is a convex canonical functional, while $F_\mu : \mathcal{U}_a \to R$ is a linear functional. Thus, the primal problem $(\mathcal{P})$ can be reformulated as the following.

Problem 3.1 (Primal Distributed-Parameter Control Problem) For a given primal feasible space $\mathcal{U}_k = \{u \in \mathcal{U}_a | \quad \partial_t u \in \mathcal{V}_a, \quad \Lambda(u) \in \mathcal{E}_a\}$ and the final state $(\bar{u}_c(x), \bar{v}_c(x))$, find the control field $\mu(x,t) \in \mathcal{M}$ such that the solution $\bar{u}(x,t)$ of the variational problem

$$(\mathcal{P}): \quad \Pi_\mu(u) = \int_0^{t_c} e^{\nu t}[K(\partial_t u) - W_\mu(\Lambda(u)) + F_\mu(u)]\,\mathrm{d}t \to \text{sta} \;\; \forall u \in \mathcal{U}_k \tag{21}$$

satisfying the controllability condition

$$(\bar{u}(x,t_c), \bar{u}_{,t}(x,t_c)) = (\bar{u}_c(x), \bar{v}_c(x)) \;\; \forall x \in \Omega.$$

It is easy to check that the criticality condition $D\Pi_\mu(\bar{u}) = 0$ leads to the the canonical governing equation

$$\rho(\bar{u}_{,tt} + \nu \bar{u}_{,t}) = DF_\mu(\bar{u}) - \Lambda_t^*(\bar{u}) DW_\mu(\Lambda(\bar{u})). \tag{22}$$

By the Legendre-Fenchel transformation, the conjugate of $W_\mu(\xi)$ is defined by

$$W_\mu^*(\xi^*) = \sup_{\xi \in \mathcal{E}} \{\langle \xi \,;\, \xi^* \rangle - W_\mu(\xi)\}.$$

Since $W_\mu : \mathcal{E}_a \to R$ is a convex canonical functional, $W_\mu^*(\xi^*)$ is well-defined on the range $\mathcal{E}_a^*$ of the duality mapping $DW_\mu^* : \mathcal{E}_a \to \mathcal{E}^*$, the canonical duality relation

$$\xi^* = DW_\mu(\xi) \Leftrightarrow \xi = DW_\mu^*(\xi^*) \Leftrightarrow W_\mu(\xi) + W_\mu^*(\xi) = \langle \xi \,;\, \xi^* \rangle$$

holds on $\mathcal{E}_a \times \mathcal{E}_a^*$. Moreover, we have $W_\mu^{**}(\xi) = W_\mu(\xi)$ for all $\xi \in \mathcal{E}_a$. Let $\mathcal{Z} = \mathcal{U} \times \mathcal{V}^* \times \mathcal{E}^*$ be the so-called *extended canonical phase space.*

Definition 3.1 Suppose that for a given problem $(\mathcal{P})$, there exists a Gâteaux differentiable operator $\Lambda : \mathcal{U} \to \mathcal{E}$ and canonical functionals $W_\mu \in \Gamma(\mathcal{E})$, $F_\mu \in \Gamma(\mathcal{U})$ such that $P_\mu(u) = W_\mu(\Lambda(u)) - F_\mu(u)$. Then

(1) the functional $H_\mu : \mathcal{Z} \to R$ defined by

$$H_\mu(u, p, \xi^*) = K^*(p) - W_\mu^*(\xi^*) + F_\mu(u) \in \Gamma(\mathcal{U}) \times \Gamma(\mathcal{V}^*) \times \Gamma(\mathcal{E}^*) \tag{23}$$

is called *extended canonical Hamiltonian density* associated with Π_μ;

(2) the functional $L_\mu : \mathcal{Z} \to R$ defined by

$$L_\mu(u, p, \xi^*) = \langle \partial_t u \,,\, p \rangle - \langle \Lambda(u) \,;\, \xi^* \rangle - H_\mu(u, p, \xi^*) \tag{24}$$

is called *extended Lagrangian density* of $(\mathcal{P})$ associated with Λ;

(3) the functional $\Xi_\mu : \mathcal{Z} \to R$ defined by

$$\Xi_\mu(u,p,\xi^*) = \int_0^{t_c} e^{\nu t} L_\mu(u,p,\xi^*)\,\mathrm{d}t \tag{25}$$

is called *extended Lagrangian form* of $(\mathcal{P})$. It is called *canonical Lagrangian form* if $\Xi_\mu \in \Gamma(\mathcal{U}) \times \Gamma(\mathcal{V}^*) \times \Gamma(\mathcal{E}^*)$. $\diamondsuit$

A point $(\bar{u},\bar{p},\bar{\xi}^*) \in \mathcal{Z}$ is said to be a critical point of Ξ_μ if Ξ_μ is Gâteaux-differentiable at $(\bar{u},\bar{p},\bar{\xi}^*)$ and $D\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) = 0$. It is easy to find out that the criticality condition $D\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) = 0$ leads to *canonical Lagrange equations*

$$D\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) = 0 \quad \Rightarrow \left\{ \begin{array}{l} \Lambda(\bar{u}) = D_{\xi^*} W_\mu^*(\bar{\xi}^*), \quad \partial_t \bar{u} = DK^*(\bar{p}), \\ \partial_t^* \bar{p} = \Lambda_t^*(\bar{u})\bar{\xi}^* - DF_\mu(\bar{u}). \end{array} \right. \tag{26}$$

By the fact that W_μ and F_μ are canonical functionals, we know that, by the Legendre duality theory, any critical point of Ξ_μ solves the variational problem $(\mathcal{P})$.

Since $F_\mu(u) : \mathcal{U}_a \to R$ is a linear functional, by the Riesz representation theory we know that there exists an element $\bar{u}^*(\mu) \in \mathcal{U}^*$ such that $F_\mu(u) = \langle u\ ,\ \bar{u}^*(\mu)\rangle$. Thus, the extended Lagrangian associated with $(\mathcal{P})$ can be written as

$$\Xi_\mu(u,p,\xi^*) = \int_0^{t_c} e^{\nu t}\left[\langle \partial_t u\ ,\ p\rangle - \langle \Lambda(u)\ ;\ \xi^*\rangle - K^*(p) + W^*(\xi^*) + \langle u\ ,\ \bar{u}^*(\mu)\rangle\right]\mathrm{d}t. \tag{27}$$

Note that $\Xi_\mu : \mathcal{V}_a^* \times \mathcal{E}_a^* \to R$ is a saddle functional for any given $u \in \mathcal{U}_a$, we have always the equality

$$\inf_{\xi^* \in \mathcal{E}_a^*} \sup_{p \in \mathcal{V}_a^*} \Xi_\mu(u,p,\xi^*) = \sup_{p \in \mathcal{V}_a^*} \inf_{\xi^* \in \mathcal{E}_a^*} \Xi_\mu(u,p,\xi^*) \ \ \forall u \in \mathcal{U}_a. \tag{28}$$

However, for any given $(p,\xi^*) \in \mathcal{V}_a^* \times \mathcal{E}_a^*$, the convexity of $\Xi_\mu(\cdot,p,\xi^*) \to R$ depends on the operator Λ. Let $\mathcal{L}_c \subset \mathcal{Z}_a = \mathcal{U}_a \times \mathcal{V}_a^* \times \mathcal{E}_a^*$ be a critical point set of Ξ_μ, i.e.

$$\mathcal{L}_c = \{(\bar{u},\bar{p},\bar{\xi}^*) \in \mathcal{Z}_a |\ \ \delta\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*;u,p,\xi^*) = 0 \ \ \forall (u,p,\xi^*) \in \mathcal{Z}_a\}.$$

For any given critical point $(\bar{u},\bar{p},\bar{\xi}^*) \in \mathcal{L}_c$, we let $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^* \subset \mathcal{Z}_a$ be its *neighborhood* such that $(\bar{u},\bar{p},\bar{\xi}^*)$ is the only critical point on $\mathcal{Z}_r$. The following triality theorem should play an important role in the stability analysis of nonlinear dynamical systems.

Theorem 3.1 (Triality Theorem) *Suppose that for a given control field $\mu(x,t)$ such that $(\bar{u},\bar{p},\bar{\xi}^*) \in \mathcal{L}_c$ is a critical point of Ξ_μ, and $\mathcal{Z}_r$ is a neighborhood of $(\bar{u},\bar{p},\bar{\xi}^*)$.*

If $\langle \Lambda(u)\ ;\ \bar{\xi}^*\rangle$ is concave on $\mathcal{U}_r$, then on $\mathcal{Z}_r$,

$$\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) = \min_u \max_p \min_{\xi^*} \Xi_\mu(u,p,\xi^*) = \max_p \min_u \min_{\xi^*} \Xi_\mu(u,p,\xi^*). \tag{29}$$

However, if $\langle \Lambda(u)\ ;\ \bar{\xi}^*\rangle$ is convex on $\mathcal{U}_r$, then on $\mathcal{Z}_r$ we have either

$$\begin{aligned} \Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) &= \min_u \max_p \min_{\xi^*} \Xi_\mu(u,p,\xi^*) = \min_p \max_u \min_{\xi^*} \Xi_\mu(u,p,\xi^*) \\ &= \min_{\xi^*,u} \max_p \Xi_\mu(u,p,\xi^*) = \min_{p,\xi^*} \max_u \Xi_\mu(u,p,\xi^*). \end{aligned} \tag{30}$$

or

$$\begin{aligned}\Xi_\mu(\bar{u},\bar{p},\bar{\xi}^*) &= \max_u \min_{\xi^*} \max_p \Xi_\mu(u,p,\xi^*) = \max_p \min_{\xi^*} \max_u \Xi_\mu(u,p,\xi^*) \\ &= \min_{\xi^*} \max_{u,p} \Xi_\mu(u,p,\xi^*) = \max_{u,p} \min_{\xi^*} \Xi_\mu(u,p,\xi^*). \end{aligned} \tag{31}$$

Proof Since $W^*_\mu \in \check{\Gamma}(\mathcal{E}^*_a)$, $K^* \in \check{\Gamma}(\mathcal{V}^*_a)$, if $\langle \Lambda(u)\ ;\ \bar{\xi}^* \rangle$ is concave on $\mathcal{U}_r$, then for a given $\bar{\xi}^*$, $\Xi_\mu \in \check{\Gamma}(\mathcal{U}_r) \times \hat{\Gamma}(\mathcal{V}^*_a)$ is a saddle functional. Thus the equality (29) follows from the saddle-Lagrangian duality theorem (cf. e.g., Gao, 2000d). However, if $\langle \Lambda(u)\ ;\ \bar{\xi}^* \rangle$ is convex on $\mathcal{U}_r$, then for any given $\xi^* \in \mathcal{E}^*_r$, the extended Lagrangian $\Xi_\mu \in \hat{\Gamma}(\mathcal{U}_r) \times \hat{\Gamma}(\mathcal{V}^*_a)$ is a *super-critical functional* (see Gao, 2000d). By the *super-Lagrangian duality theorem* proved in Gao (2000d), we have either (30) or (31). □

4 Dual Action and Tri-Duality in Dissipative Systems

The goal of this section is to develop a dual approach for solving the distributed parameter control problem $(\mathcal{P})$. For any given $u \in \mathcal{U}_k$, the extended Lagrangian density $\Xi_\mu(u,p,\xi^*)$ is a saddle functional on $\mathcal{V}^* \times \mathcal{E}^*$, and we have

$$\Pi_\mu(u) = \sup_{p \in \mathcal{V}^*} \inf_{\xi^* \in \mathcal{E}^*} \Xi_\mu(u,p,\xi^*) \quad \forall u \in \mathcal{U}_k. \tag{32}$$

On the other hand, the dual action $\Pi^d_\mu : \mathcal{V}^*_a \times \mathcal{E}^*_a \to R$ can be defined by

$$\begin{aligned}\Pi^d_\mu(p,\xi^*) &= \operatorname{sta}\{\Xi_\mu(u,p,\xi^*) |\ \forall u \in \mathcal{U}_a\} \\ &= F^\Lambda_\mu(p,\xi^*) - \int_0^{t_c} [K^*(p) - W^*_\mu(\xi^*)]\,\mathrm{d}t, \quad \forall (p,\xi^*) \in \mathcal{V}^*_a \times \mathcal{E}^*_a, \end{aligned} \tag{33}$$

where $F^\Lambda_\mu(p,\xi^*)$ is the so-called Λ-dual functional of $F_\mu(u)$ defined by

$$F^\Lambda_\mu(p,\xi^*) = \operatorname*{sta}_{u \in \mathcal{U}_a} \int_0^{t_c} e^{\nu t}[\langle \partial_t u\ ,\ p \rangle - \langle \Lambda(u)\ ;\ \xi^* \rangle + F_\mu(u)]\,\mathrm{d}t. \tag{34}$$

Since $F_\mu(u) = \langle u\ ,\ \bar{u}^*(\mu) \rangle$ is a linear functional, for any given $(p,\xi^*) \in \mathcal{V}^*_a \times \mathcal{E}^*_a$ and the applied control $\mu \in \mathcal{M}$, the solution $\bar{u}$ of this stationary problem (34) satisfies the balance equation

$$\partial^*_t p - \Lambda^*_t(\bar{u})\xi^* + \bar{u}^*(\mu) = 0 \quad \text{in } \Omega_t. \tag{35}$$

For geometrically linear conservative systems, where Λ is a linear operator, we have

$$F^\Lambda_\mu(p,\xi^*) = up|_{t=0}^{t=t_c}, \quad s.t.\ \ \Lambda^*\xi^* + p_{,t} = \bar{u}^*(\mu). \tag{36}$$

In this case,

$$\Pi^d_\mu(p,\xi^*) = up|_{t=0}^{t=t_c} + \int_0^{t_c} e^{\nu t}[W^*_\mu(\xi^*) - K^*(p)]\,\mathrm{d}t \tag{37}$$

is the classical complementary action in linear engineering dynamical systems (see Tabarrok and Rimrott, 1994) defined on the dual feasible space

$$\mathcal{T}_s = \{(p,\xi^*) \in \mathcal{V}_a \times \mathcal{E}^*_a |\ \ p_{,t} + \Lambda^*\xi^* = \bar{u}^*(\mu)\}.$$

In fully nonlinear systems, we let $\mathcal{T}_s \subset \mathcal{V}_a^* \times \mathcal{E}_a^*$ be a subspace such that for any given $(p, \xi^*) \in \mathcal{T}_s$, the critical point $\bar{u}$ can be determined by (35) as $\bar{u} = \bar{u}(p, \xi^*)$ and the dual action Π_μ^d is well defined by (33). Thus, by the operator decomposition $\Lambda = \Lambda_t + \Lambda_c$, we have

$$F_\mu^\Lambda(p, \xi^*) = e^{\nu t} u p|_{t=0}^{t=t_c} + \int_0^{t_c} e^{\nu t} G^d(p, \xi^*)\, \mathrm{d}t, \quad s.t. \;\; \partial_t^* p = \Lambda_t^*(\bar{u})\xi^* - u^*(\mu), \tag{38}$$

where $G^d(p, \xi^*) = \langle -\Lambda_c(\bar{u}) \; ; \; \xi^* \rangle$ is the so-called pure complementary gap functional. Then, the problem dual to the primal control problem $(\mathcal{P})$ can be proposed as the following.

Problem 4.1 (Dual Distributed-Parameter Control Problem) For a given dual feasible space $\mathcal{T}_s$ and the final state $(u_c(x), v_c(x))$, find the control field $\mu(x, t) \in \mathcal{M}$ such that the dual solution $(\bar{p}(x, t), \bar{\xi}^*(x, t))$ of the dual variational problem

$$(\mathcal{P}^d): \quad \Pi_\mu^d(p, \xi^*) \;\; \to \text{sta} \;\; \forall (p, \xi^*) \in \mathcal{T}_s \tag{39}$$

and the associated state $\bar{u}(x, t)$ satisfying the controllability condition

$$(\bar{u}(x, t_c), \rho^{-1}\bar{p}(x, t_c)) = (u_c(x), v_c(x)) \;\; \forall x \in \Omega. \tag{40}$$

Lemma 4.1 *Let $\Xi_\mu(u, p, \xi^*)$ be a given extended Lagrangian associated with $(\mathcal{P})$ and $\Pi_\mu^d(p, \xi^*)$ the dual action defined by (33). Suppose that $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^*$ is an open subset of $\mathcal{Z}_a$ and $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{Z}_r$ is a critical point of Ξ_μ on $\mathcal{Z}_r$, Π_μ is Gâteaux differentiable at $\bar{u}$, and Π_μ^d is Gâteaux differentiable at $(\bar{p}, \bar{\xi}^*)$. Then $D\Pi_\mu(\bar{u}) = 0$, $D\Pi_\mu^d(\bar{p}, \bar{\xi}^*) = 0$, and*

$$\Pi_\mu(\bar{u}) = \Xi_\mu(\bar{u}, \bar{p}, \bar{\xi}^*) = \Pi_\mu^d(\bar{p}, \bar{\xi}^*). \tag{41}$$

The proof of this lemma was given by the author in parametrical variational analysis (Gao, 1998).

Lemma 4.1 shows that the critical points of the extended Lagrangian are also the critical points for both the primal and dual variational problems.

Theorem 4.1 (Tri-Duality Theorem) *Suppose that for a given control field $\mu(x, t)$ such that $(\bar{u}, \bar{p}, \bar{\xi}^*) \in \mathcal{L}_c$ is a critical point of Ξ_μ and $\mathcal{Z}_r = \mathcal{U}_r \times \mathcal{V}_r^* \times \mathcal{E}_r^*$ is a neighborhood of $(\bar{u}, \bar{p}, \bar{\xi}^*)$ such that $\mathcal{V}_r^* \times \mathcal{E}_r^* \subset \mathcal{T}_s$. If $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ is concave on $\mathcal{U}_r$, then*

$$\Pi_\mu(\bar{u}) = \min_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_r^*} \min_{\xi^* \in \mathcal{E}_r^*} \Pi_\mu^d(p, \xi^*). \tag{42}$$

However, if $\langle \Lambda(u) \; ; \; \bar{\xi}^* \rangle$ is convex on $\mathcal{U}_r$, then

$$\Pi_\mu(\bar{u}) = \min_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \min_{(p, \xi^*) \in \mathcal{T}_s} \Pi_\mu^d(p, \xi^*); \tag{43}$$

$$\Pi_\mu(\bar{u}) = \max_{u \in \mathcal{U}_r} \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\xi}^*) = \max_{p \in \mathcal{V}_r^*} \min_{\xi^* \in \mathcal{E}_r^*} \Pi_\mu^d(p, \xi^*). \tag{44}$$

Proof This theorem can be proved by combining Lemma 4.1 and the triality theorem. □

5 Feedback Control Against Chaos in Dissipative Duffing System

As we have shown in the first section of this paper that the governing equations for shear/damping control of large deformed nonlinear beam structure are eventually equivalent to the well-known Duffing system. As a typical example, let us consider the very simple nonconvex dynamical problem over the time domain $I = (0, t_c)$

$$\Pi_\mu(u) = \int_I e^{\nu t}[\rho u'^2 - \frac{1}{2}a(\frac{1}{2}u^2 - \lambda)^2 + \mu u]\,\mathrm{d}t \quad \to \text{sta} \ \ \forall u \in \mathcal{U}_k. \tag{45}$$

For initial-value problem of this one-dimensional dynamical system, the kinematically admissible space $\mathcal{U}_k$ can simply be given as

$$\mathcal{U}_k = \{u \in \mathcal{L}^4(0, t_c)| \ \ u' \in \mathcal{L}^2(0, t_c), \ \ u(0) = u_0, \ \ u'(0) = v_0\}.$$

The criticality condition of Π_μ leads to the dissipative Duffing equation

$$\rho(u'' + \nu u') = au(\lambda - \frac{1}{2}u^2) + \mu(t), \ \ \forall t \in I, \ \ u \in \mathcal{U}_k. \tag{46}$$

In terms of the nonlinear canonical measure $\xi = \Lambda(u) = \frac{1}{2}u^2$, the energy density $W_\mu(\xi)$ and its conjugate $W^*_\mu(\varsigma)$ are convex functions:

$$W_\mu(\xi) = \frac{1}{2}a(\xi - \lambda)^2, \quad W^*_\mu(\varsigma) = \frac{1}{2a}\varsigma^2 + \lambda\varsigma.$$

The extended Lagrangian for this nonconvex system is

$$\Xi_\mu(u, p, \varsigma) = \int_I e^{\nu t}\left(pu' - \varsigma(\frac{1}{2}u^2 - \lambda) - \frac{1}{2\rho}p^2 + \frac{1}{2a}\varsigma^2 + \mu u\right)\mathrm{d}t. \tag{47}$$

The criticality condition $D_u\Xi_\mu(\bar{u}, p, \varsigma) = 0$ leads to the equilibrium equation

$$p' + \nu p + \bar{u}\varsigma = \mu \ \ \forall t \in I.$$

Clearly, the critical point $\bar{u} = (\mu - p' - \nu p)/\varsigma$ is well-defined for any nonzero ς. Thus, the dual feasible space can be defined as

$$\mathcal{T}_s = \left\{(p, \varsigma) \in \mathcal{C}^1(I) \left| \begin{array}{l} p(0) = \rho v_0, \ \ -\lambda a \le \varsigma(t) < +\infty, \\ \varsigma(t) \neq 0 \ \ \forall t \in I, \ \ \varsigma(0) = a(\frac{1}{2}u_0^2 - \lambda) \end{array}\right.\right\}.$$

Substituting $\bar{u} = (\mu - p' - \nu p)/\varsigma$ into Ξ^d_μ, the dual action is obtained as

$$\begin{aligned} &\Pi^d_\mu(p, \varsigma) = \underset{u\in\mathcal{U}_a}{\text{sta}}\ \Xi_\mu(u, p, \varsigma) \\ = \quad & e^{\nu t_c}p(t_c)u(t_c) - \rho v_0 u_0 + \int_I e^{\nu t}[\frac{1}{2a}\varsigma^2 + \lambda\varsigma + \frac{(p' + \nu p - \mu)^2}{2\varsigma} - \frac{1}{2\rho}p^2]\,\mathrm{d}t, \end{aligned} \tag{48}$$

which is well defined on $\mathcal{T}_s$. The criticality condition for Π^d_μ leads to the *dual Duffing system* in the time domain $I \subset R$

$$\left(\frac{1}{\varsigma}(p' + \nu p - \mu)\right)' + \frac{1}{\rho}p = 0, \tag{49}$$

$$\varsigma^2 \left(\frac{1}{a} \varsigma + \lambda \right) = \frac{1}{2} (\mu - p' - \nu p)^2. \tag{50}$$

This system consists of the so-called *differential-algebraic equations* (DAE's), which arise naturally in many applications (cf. Brenan *et al*, 1996). Although the numerical solution of these types of systems has been the subject of intense research activity in the past few years, the solvability of each problem depends mainly on the so-called *index* of the system. Clearly, the algebraic equation (50) has zero solution $\varsigma = 0$ if and only if $g = (\mu - p' - \nu p) = 0$. Otherwise, for any nonzero $g(t) = \mu(t) - p'(t) - \nu p(t)$, the algebraic equation (50) has at most three real roots $\varsigma_i(t)$ $(i = 1, 2, 3)$, each of them leads to the state solution $u_i(t) = (\mu(t) - p_i'(t) - \nu p_i(t))/\varsigma_i(t)$.

Theorem 5.1 (Stability and Bifurcation Criteria) *For a given parameter $\lambda > 0$, initial data (u_0, v_0) and the input control $\mu(t)$, if at a certain time period $I_s \subset I = (0, t_c)$,*

$$\lambda_p(t) = \frac{3}{2} \left(\frac{\mu(t) - p'(t) - \nu p(t)}{a} \right)^{2/3} > \lambda, \quad t \in I_s \tag{51}$$

then the Duffing system possesses only one solution set $(\bar{u}(t), \bar{p}(t), \bar{\varsigma}(t))$ satisfying $\bar{\varsigma}(t) > 0 \ \ \forall t \in I_s$, and over the period I_s,

$$\Pi_\mu(\bar{u}) = \min \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\varsigma}) = \min \Pi_\mu^d(p, \varsigma), \tag{52}$$

$$\Pi_\mu(\bar{u}) = \max \Pi_\mu(u) \quad \textit{iff} \quad \Pi_\mu^d(\bar{p}, \bar{\varsigma}) = \max_p \min_\varsigma \Pi_\mu^d(p, \varsigma). \tag{53}$$

However, if at a certain time period $I_b \subset I = (0, t_c)$ we have $\lambda_p(t) < \lambda$, then, the system possesses three sets of different solutions $(\bar{u}_i, \bar{p}_i(t), \bar{\varsigma}_i(t))$, $i = 1, 2, 3$. In the case that the three solutions $\varsigma_i(t)$ are in the following ordering

$$-a\lambda \le \bar{\varsigma}_3(t) \le \bar{\varsigma}_2(t) \le 0 \le \bar{\varsigma}_1(t) \ \ \forall t \in I_b, \tag{54}$$

then over the period I_b, the solution set $(\bar{u}_1(t), \bar{p}_1(t), \bar{\varsigma}_1(t))$ satisfies either (52) or (53); while the solution sets $(\bar{u}_i(t), \bar{p}_i(t), \bar{\varsigma}_i(t))$ $(i = 2, 3)$ satisfy

$$\Pi_\mu(\bar{u}_i) = \min_u \Pi_\mu(u) = \max_p \min_\varsigma \Pi_\mu^d(p, \varsigma) = \Pi_\mu^d(\bar{p}_i, \bar{\varsigma}_i), \quad i = 2, 3. \tag{55}$$

This theorem can be proved by combining the theorem given by Gao (2000d, Theorem 3.4.4) and the triality theorem.

Remark 5.1 By Theorem 3.4.4 proved by the author (Gao, 2000d), for any given continuous function $g(t)$, if $\bar{\varsigma}_i(t)$ $(i = 1, 2, 3)$ are the three solutions of the dual Euler-Lagrange equation (50) in the order of (54), then the associated $\bar{u}_1(t)$ is a global minimizer of the total potential

$$P_\mu(u) = \int_I e^{\nu t} \left[\frac{1}{2} a (\frac{1}{2} u^2 - \lambda)^2 - g(t) u \right] \mathrm{d}t,$$

while $\bar{u}_2(t)$ is a local minimizer of P_μ and $\bar{u}_3(t)$ is a local maximizer of P_μ.

In *algebraic geometry*, the dual Euler-Lagrange equation (50) is the so-called *singular algebraic curve* in (ς, g)-space, i.e. $\varsigma = 0$ is on the curve (see Silverman & Tate, 1992,

p. 99). With a change of variables, the singular cubic curve (50) can be given by the well-known *Weierstrass equation*

$$y^2 = x^3 + \alpha x^2 + \beta x + \gamma,$$

where $\alpha,\ \beta,\ \gamma \in R$ are constants. If we let $\mathcal{C}_{ns}$ be a set consisting of non-singular points on the curve, then $\mathcal{C}_{ns}$ is an Abelian group. This fact in algebraic geometry is very important in understanding the stability of the nonconvex dynamical systems. Actually, from Figure 5.1 we can see clearly that for a given input control, if $\lambda_p(t) < \lambda$, the cubic algebraic equation (50) possesses three different real solutions for $\varsigma(t)$. The two negative solutions $\bar{\varsigma}(t)$ are the sources that lead to the chaotic motion of the system. Thus, the inequality (51) provides a *bifurcation (or chaotic) criterion* for the Duffing system. Figure 5.1 also shows that if the continuous function $g(t) = \mu(t) - p'(t) - \nu p(t)$ is one-signed on certain time interval $I_b \subset I = (0, t_c)$, each root $\bar{\varsigma}(t)$ of (50) is also one-signed on I_i.

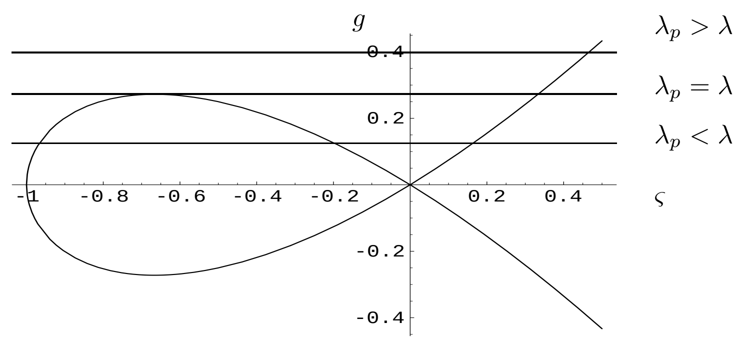


Figure 5.1. Invariant set of dual solutions and bifurcation criterion for Duffing equation (50).

Theoretically speaking, for the given same data, the Duffing equation (46) and its dual system (49-50) should have the same solution set. Numerically, the primal and dual Duffing problems give quite different results (see Figure 5.2 (a)). For the given data $a = 1, \lambda = 1.5, u_0 = 2, v_0 = 1.4$ and $\nu = 0$, Figures 5.2 and 5.3 show the numerical primal (solid line) and dual (dashed line) solutions. From the dual trajectories in the dual phase space ς-p-$p_{,t}$ (Figure 5.3 (c-d)) we can see that at the point $\varsigma_3(t) = -a\lambda$, if the function $g(t) = \mu(t) - p_{,t}(t) - \nu p(t)$ changes its sign, the state $u(t)$ crosses the t-axis and falls down to the another potential well in the phase space $\mathcal{Z} = \mathcal{U} \times \mathcal{V}^*$. The bifurcation is then occurred.

For the forced vibration with linear damping, the numerical results are extremely sensitive to the parameters. Figure 5.4 shows that the trajectories are chaotic in phase spaces q-p (Figure 5.4 (b)) and ς-p-g (Figure 5.4 (d)). However, trajectory in the dual phase space ς-g is an invariant (see Figure 5.4 (c)), which depends only on the parameters λ, a and the amplitude of the force $g(t)$.

As it is known that the nonconvex dynamical systems are very sensitive to both the parameters and numerical methods used. For the given periodic driving force $\mu(t) = 1.5\cos(2.75t)$ and $\nu = 0.1$, Figure 5.4 shows that different numerical solvers in MATLAB produce very different "chaotic results". However, solutions in dual phase space ς-g form an invariant set (Figure 5.4 (c)). This important fact shows that the triality theorem will play an important role in stability and bifurcation analysis of chaotic systems.

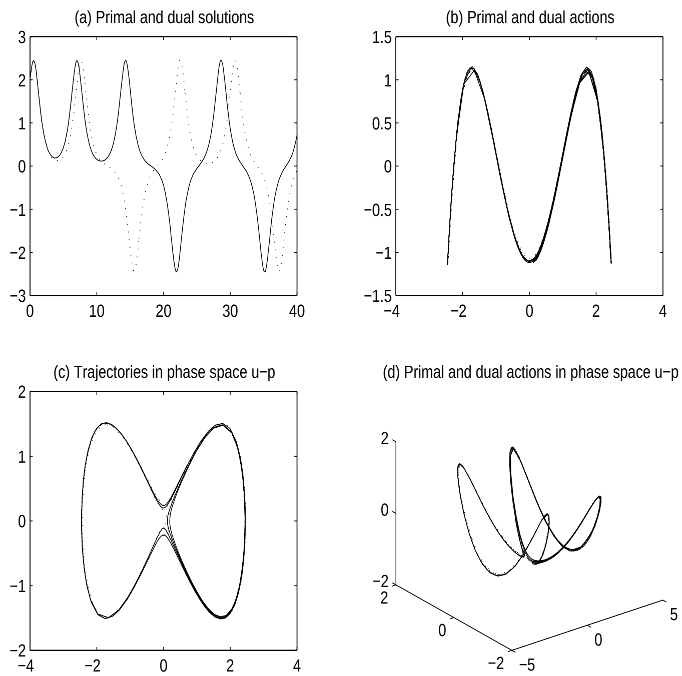


Figure 5.2. Primal and dual solutions in primal phase space.

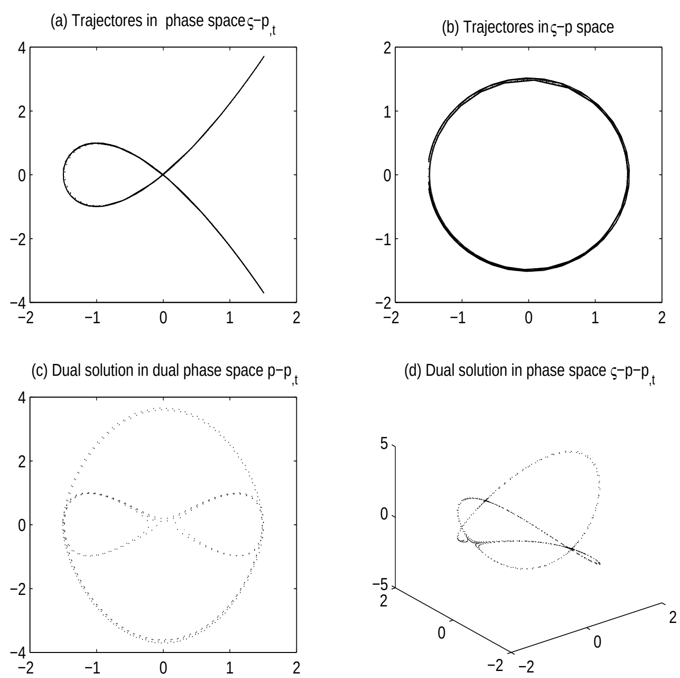
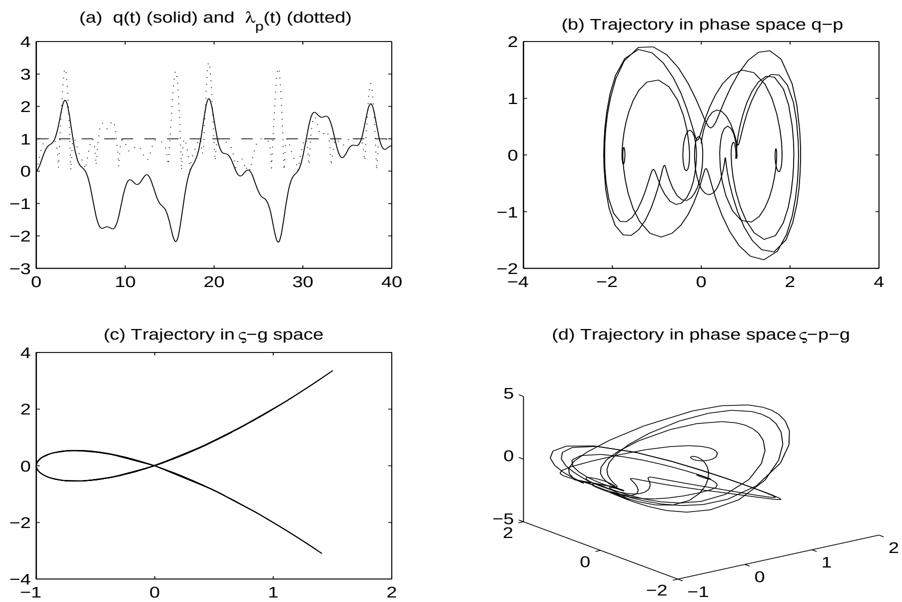
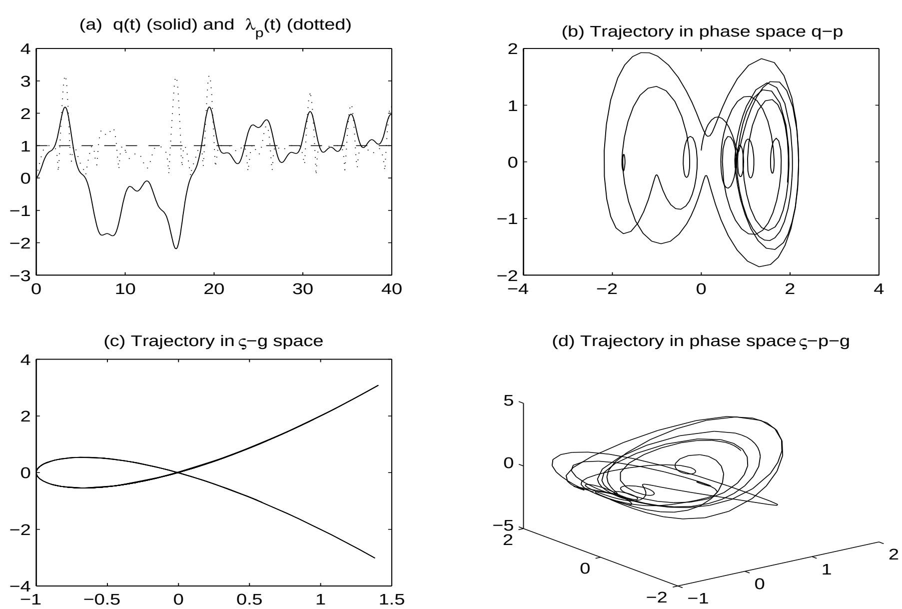


Figure 5.3. Duffing solutions in dual phase spaces.



(1) Numerical results computed by "ode23"



(2) Numerical results computed by "ode15s"

Figure 5.4. Chaos and invariant set: numerical results by two differential numerical methods in MATLAB.

Based on the canonical dual transformation method and theorems developed in this paper, the dual feedback control against the chaotic vibration of the Duffing system can be suggested as the following.

1. Periodic vibration on the whole phase plane.

Choosing the control parameters μ and ν such that the function $g(t) = \mu - p'(t) - \nu p(t)$ changes its sign at the point $\bar{\varsigma}_3(t) = -a\lambda$.

2. Unilateral vibrations on half phase planes (either $u(t) > 0$ or $u(t) < 0$).

There are two methods: (1) choosing the control parameters μ and ν such that the function $g(t) = \mu - p'(t) - \nu p(t)$ does not change its sign at the point $\bar{\varsigma}_3(t) = -a\lambda$; (2) choosing μ and ν such that

$$\lambda_p(t) = \frac{3}{2}\left(\frac{\mu(t) - p'(t) - \nu p(t)}{a}\right)^{2/3} > \lambda \ \ \forall t \in I. \tag{56}$$

By the bifurcation criterion (Theorem 5.1) we know that if $\lambda_p > \lambda$, the total potential of this dissipative Duffing equation is convex and the system is stable.

6 Concluding Remarks

The concept of duality is one of the most successful ideas in modern mathematics and science. The inner beauty of duality theory owes much to the fact that many different natural phenomena can be put in a unified trio-canonical framework (see Gao, 2000d, 2001). By the fact that the canonical physical variables appear always in pairs, the canonical dual transformation method can be used to solve many problems in natural systems. The associated extended Lagrange duality and triality theories have profound computational impacts. For any given nonlinear problem, as long as there exists a geometrical operator Λ such that the trio-canonical forms can be characterized correctly, the canonical dual transformation method and the associated triality principles can be used to establish nice theories and to develop powerful alternative algorithms for robust feedback control of chaotic systems. Actually, it has been shown that in global optimization many difficult nonconvex minimization problems in n-dimensional space can be converted into certain canonical dual problems (either convex minimization or concave maximization) in ONE-dimensional space, therefore, a class of problems have been solved completely, including the well-known quadratic minimization over a sphere (Gao, 2004), polynomial minimization (Gao, 2005), and quadratic programming with box constraints (Gao, 2006). In general n-dimensional distributed parameter systems, the dual algebraic equation (50) will be a tensor equation and the stability of the nonconvex system will depend on the eigenvalues of symmetrical canonical stress tensor field $\varsigma(x, t)$ (see Gao, 2001). The triality theory can be used for studying the controllability, observability and stability of distributed parameter control problems.

Acknowledgement

This research was supported by NSF grant CCF-0514768. Comments from two anonymous referees are gratefully acknowledged.

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